A COMPOSITION PROCEDURE FOR DIGITALLY SYNTHESIZED MUSIC ON LOGARITHMIC SCALES OF THE HARMONIC SERIES

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ABSTRACT

Discrete spectral frequencies within pitched sounds occur in a pattern known to music theorists as the harmonic series. Their simple-ratio intervals do not conform to the musical semitones of twelve-tone equal temperament (12Tet). New procedures for composing digitally synthesized music have been developed applying these intervals to fundamental frequencies. Digital synthesis provides for exact frequency specification. In 12Tet, each semitone is divided into 100 cents for tuning, dividing the octave into 1,200 fine increments. Some user interfaces for synthesis correlate to the 1,200 cents per octave of 12Tet, as opposed to frequencies in cycles per second. Octave frequencies being a ratio of two, with 12Tet dividing the octave into twelve equal intervals, the value of any one thus being the twelfth root of two, a logarithmic equation can be applied to calculate values in cents for the simple-ratio intervals of the harmonic series. Synthesis applications that assume 12Tet can be adjusted accordingly. New music applying these intervals has been composed and performed.

1. INTRODUCTION

This concerns application of what are essentially musical scales to the composition of computer music. Intervals within the scales match the harmonic series. Their exact pitches are virtually impossible for ordinary acoustic instruments, but are easily realized through digital synthesis. As such, works composed applying these scales represent a style of computer music composition.

To adjust tuning through interfaces of digital synthesis applications that assume 12Tet, and to graphically represent music applying these adjustments for performers of acoustic parts, the simple-ratio intervals of the scale must be converted into the 1,200 cent-per-octave standard of twelve-tone equal temperament (12Tet) using a logarithmic equation.

2. CHALLENGES

The technical and artistic challenges behind creation of these new scales were fourfold: 1) Create true “overtone” scales, intervals of which are in the actual simple ratios of the harmonic series; 2) Create means to realize these scales as a material component of musical composition by mathematically translating the simple-ratio frequency intervals of the scales into adjusted musical intervals of standard 12Tet so that: a) as necessary, commercial synthesis applications could be adjusted according to musically standardized user interfaces; and, b) pitches could be graphically represented for vocal or acoustic performers according to standard music notation; 3) Develop a series of scale transpositions sharing common tones by which modulation from one to another may be effected; and, 4) Compose, perform and record new music according to technical and aesthetic implications of applying the new scale and tonal system.

2.1. No True Overtone Scale

Certain music of the 20th century was composed applying a Lydian-Mixolydian scale [1], as shown in Figure 1.

Figure 1. Lydian-Mixolydian scale.

When this scale is applied in jazz improvisation, it is sometimes called an “overtone” scale, or an “acoustic” scale, even though its standard 12Tet pitches only approximate frequency intervals in the fourth octave of the harmonic series [2]. It is still not a true overtone scale because its intervals do not actually conform to those of the observable harmonic series.

Designing and building nonstandard acoustic instruments that would play a true overtone scale according to actual intervals of the harmonic series and then training musicians to play them would be impractical at best. Writing code or creating object-based digital synthesis patches for such intervals could be distracting and time consuming for those approaching the computer music studio as classically trained composers or performers. Commercial synthesis applications bring their own problems, as detailed below.
2.2. Musical Standards vs. Acoustic Reality

The 12Tet standard is assumed for many commercial synthesis applications, and for standard musical notation. The actual intervals of the harmonic series conform neither to pitch relationships of 12Tet nor to standard musical notation or notation applications, inasmuch as they assume and refer to 12Tet.

2.2.1. Synthesis interface standards

Many digital synthesis applications can produce finely specified pitch relations; however, where user interfaces translate into a standard cents division of the 12Tet semitone, making correct pitch adjustments so intervals conform to simple ratios of the harmonic series requires a logarithmic formula.

2.2.2. Music notation standards

Musicians can match, play or sing in concert with pitch-adjusted musical tones that do not conform to 12Tet; however, the standard music notation they read and the ways they read it assume 12Tet. There is no standard for indicating the exact measurements in cents of pitch adjustments that deviate from 12Tet, yet some means of indication must be applied for the sake of musicians who would perform pitch-adjusted musical materials according to a notated musical score.

2.3. Means of Modulation

The formal dynamics of many kinds of music depend on some kind of exposition, development and recapitulation of sonic materials. This can involve modulation from one series of intra-related pitch classes (scale) to another, and some kind of return to the initial series. In many cases such modulation depends on what pitch classes and motifs are shared between a given series and its successor.

As analogous to such types of music, it was found desirable to develop transpositions of any new scales that would produce a network of pitches shared among all transpositions. This would depend on intervals between the tonics of each scale transposition. This would facilitate modulation as a compositional procedure.

2.4. Application to Composition

Provided that simple-ratio intervals of the harmonic series are mathematically translated into 12Tet pitches adjusted by certain numbers of cents, that the means of synthesis and graphic notation are adjusted accordingly, and that theoretical pitch relationships are developed to provide for musical modulation, it would remain to compose new music applying those materials, and to hear it realized in performance.

3. REALIZATION OF SCALE AND MUSIC

Developing frequency relationships of the harmonic series into musical scales begins with regarding the fundamental frequency as tonic, against which other pitches of the series are arrayed intervallically. To represent intervals musically, they are translated to values in cents using a logarithmic formula. Once represented musically, they can be applied in composition using digital synthesis resources and representative musical notation.

3.1. A True Overtone Scale

Taking the frequency of C1 (32.703 Hz) as a fundamental, the frequencies of tones having the same relationships as harmonics in the harmonic series over that fundamental can be calculated accordingly.

So, a true overtone scale based on the harmonic series can be expressed as a list of frequencies, or as a series of ratios by which those frequencies are related:

<table>
<thead>
<tr>
<th>HARMONIC</th>
<th>FREQUENCY Hz</th>
<th>RATIO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (C1)</td>
<td>32.703</td>
<td>1/1</td>
</tr>
<tr>
<td>2 (C2)</td>
<td>65.406</td>
<td>2/1</td>
</tr>
<tr>
<td>3</td>
<td>98.109</td>
<td>3/2</td>
</tr>
<tr>
<td>4 (C3)</td>
<td>130.81</td>
<td>4/3</td>
</tr>
<tr>
<td>5</td>
<td>163.52</td>
<td>5/4</td>
</tr>
<tr>
<td>6</td>
<td>196.22</td>
<td>6/5</td>
</tr>
<tr>
<td>7</td>
<td>228.92</td>
<td>7/6</td>
</tr>
<tr>
<td>8 (C4)</td>
<td>261.62</td>
<td>8/7</td>
</tr>
<tr>
<td>9</td>
<td>294.33</td>
<td>9/8</td>
</tr>
<tr>
<td>10</td>
<td>327.03</td>
<td>10/9</td>
</tr>
<tr>
<td>11</td>
<td>359.73</td>
<td>11/10</td>
</tr>
<tr>
<td>12</td>
<td>392.44</td>
<td>12/11</td>
</tr>
<tr>
<td>13</td>
<td>425.14</td>
<td>13/12</td>
</tr>
<tr>
<td>14</td>
<td>457.84</td>
<td>14/13</td>
</tr>
<tr>
<td>15</td>
<td>490.55</td>
<td>15/14</td>
</tr>
<tr>
<td>16 (C5)</td>
<td>523.25</td>
<td>16/15</td>
</tr>
</tbody>
</table>

Table 1. Overtone scale as list of frequencies and ratios

3.2. Translation into Musical Terms

There is a way to mathematically translate simple-ratio intervals of the harmonic series into musical intervals of the 1,200 cents-per-octave 12Tet standard.

Remembering that the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers:

$$\log xy = \log x + \log y,$$

it follows that

$$\log x^n = n \log x.$$
Setting this equation aside for the moment, it is known—given any octave of exactly 2:1—that when the value of an equal semitone in 12Tet is denoted by \( a \), it provides that

\[ a^{12} = 2, \]  

(3)

thus, \( a \) is the twelfth root of two, or

\[ a = (2)^{1/12} = 1.05946. \]  

(4)

This interval is the tempered semitone. With the cent defined as \( \frac{1}{100} \) semitone and \( \varepsilon \) denoting the ratio for it, such that

\[ \varepsilon^{100} = (2)^{1/12}, \]  

(5)

it follows that

\[ \varepsilon = (2)^{1/1200}. \]  

(6)

As an interval of \( n \) cents would be given by the ratio

\[ \varepsilon^n = 2^{n/1200}, \]

finding the number of cents \( n \) in any interval of frequency ratio \( R \) requires that

\[ 2^{n/1200} = R. \]  

(7)

Taking the logarithm of each side gives

\[ \log (2^{n/1200}) = \log R, \]

and by applying Equation (2) it follows that

\[ \frac{n}{1200} \log 2 = \log R. \]

The number of the cents in the interval is then given by

\[ n = 1200 \frac{\log R}{\log 2}, \]

so that

\[ n = 3986 \log R. \]  

(8)

By applying Equation (8) the number of cents in any frequency ratio \( n \) can be calculated. For example, the number of cents in a just fifth, which has a frequency ratio \( R \) of \( \frac{3}{2} \), per Table 1, can be calculated so:

\[ \log R = \log \left( \frac{3}{2} \right) = \log 3 - \log 2 = 0.477 - 0.301 = 0.176, \]

and consequently,

\[ n = 3986 \times 0.176 \]

\[ = 702 \text{ cents}. \]

So, while an equally tempered fifth, at seven semitones, has a value of 700 cents, a just fifth with a frequency ratio of \( \frac{3}{2} \) has a value of 702 cents (rounded to the nearest cent) [3].

The just fifth from Table 1 can be shown in standard music notation by representing the C\(_2\) as having no deviation from 12Tet—indicated by a zero—and the G\(_2\) has having a deviation of plus two cents. These indications can be added to individual noteheads in the manner of musical articulations:

Figure 2. Just fifth with cents deviation from 12Tet.

Thus, the entire scale of intervals from Table 1 can be calculated and then represented in standard music notation, according to each tone’s deviation in cents from 12Tet:

Figure 3. Intervals from Table 1 as musical scale.

This is a true “overtone” scale, accurately reflecting the intervallic relationships of the harmonic series. Compare the fourth octave of the series as represented in the treble staff of Figure 3 with the Lydian-Mixolydian “overtone” scale in Figure 1.

With cents deviation from 12Tet indicated for each note of the scale, synthesis applications that assume 12Tet and cents can be adjusted or automated to produce a scale accurate to the intervallic relationships of the harmonic series. Moreover, adjusted pitches can be accurately represented in a musical score for performers expected to match or otherwise perform in relation to them.

It should be noted, in terms of style characteristics, that the catchall term “microtonal” would be applied to this scale, and to music composed with it. “Microtonal” refers to音乐
to the broad category of any and all music not conforming to 12Tet. There are a wide variety of approaches and procedures that fit into this category, and as wide a variety of interests and reasoning behind them [4]. The digital generation of music has provided especially fruitful means of experimentation and generation of new pieces that fall into this broad category [5].

3.3. Means of Modulation

Figure 4 shows the scale transposed to a fundamental of $E_1$, at 41.25 Hz, the lowest open string on an ordinary contrabass and the lowest E on a piano. Note that the $E_1$ of 12Tet based on $A_4$ at 440 Hz is 41.203 Hz. The $E_1$ given here is unadjusted for equal temperament, as though tuned from A 440 using only the just fifths produced by harmonics on the strings of a contrabass; therefore, it is 41.25 Hz.

Figure 4. The scale transposed to E.

Note that the fourth octave of the series contains eight tones. In Figure 5 these eight tones have been transposed down three octaves to serve as tonics for eight transpositions of the series, each conceived as a diatonic set:

Figure 5. Fourth octave 23ba for tonics of transpositions.

A separate transposition of the basic diatonic set can then be created with each of these pitches serving as tonic. Figure 6 shows the diatonic set on F# +4. Note that the relationships between the tones within the set (Figure 6) are the same as those in the E set (Figure 4), with values of deviation from 12Tet adjusted by a constant of plus four cents:

Figure 6. Transposition to F#+4.

Figure 7 shows the diatonic set on G# -14. All the transposed tones are adjusted to their constant of minus fourteen cents. In addition, arrows pointing from some notes indicate letter names of the tonic(s) of any other diatonic set(s) containing the same pitch class.

Figure 7. G#-14 indicating sets with common tones.

Note the presence of D# -12 and G# -14 from the G# set (Figure 7) in the E diatonic set (Figure 4), and A# -10 from the G# set (Figure 7) in the F# set (Figure 5). Most of the tones in each of the eight diatonic sets are common to at least one other set, and all eight sets have at least one tone in common with every other set, allowing for some kind of common-tone transpositions across the sets.

3.4. Composition with the Scale

This scale and system of common-tone relationships suggest procedural possibilities for composition. Polyphony, counterpoint, and harmony have been created applying the scale and/or its interrelated transpositions.

3.4.1. Polyphonic procedures

One polyphonic procedure with this scale applies tones from the diatonic set in melodic or other tonal events, while the tonic is continuously or intermittently sounded below. When material comprises one or more tones
common to another transposition, the tonic changes to that of the transposition, thereby effecting modulation to a new set. Events continue with tones applied from the new set.

Such a texture is analogous to 12th century organum e.g. of Léonin, wherein the tenor comprises a long-breathed chant melody beneath comparatively florid melodic material in the duplum [6]. Tonics of diatonic sets in series are analogous to the tenor. Melodic material above is analogous to the duplum.

An example can be taken from the electroacoustic part of this author’s multimedia piece The Madman’s Diary [7]. Figure 8 shows a musical reduction of mm. 49b-96 of the score (6 mm. before 1C to 1G). The modulations and organum-like texture become clear in the reduction.

Figure 8. Polyphonic procedure in The Madman’s Diary.

3.4.2. Contrapuntal procedures

Contrapuntal procedures were derived from pitch relationships within the diatonic set. Intervals within the scale were categorized according to a sonically significant system of “perfect” and “imperfect” consonance, “dissonance,” and “empty” consonance.

Criteria for categorization were as follows:

Octaves/unisons (1200¢/0¢) and just fifths/fourths (702¢/498¢) were discarded as “empty” consonances. Certain thirds and sixths were considered “perfect,” vis à vis octaves, fifths and unisons of 16th century practice. These include just major thirds and minor sixths (386¢/814¢), just minor thirds and major sixths (316¢/884¢), a minor third and major sixth 10¢ from equal temperament (290¢/910¢) and a major third and minor sixth thirteen cents from just intonation (373¢/827¢).

“Imperfect” consonances include all other thirds and sixths from 247¢ to 455¢ and 745¢ to 953¢ (technically, the extremes of these are large major seconds, small fourths, large fifths and small minor sevenths), as well as seconds of 9:8 (204¢) and larger and their inversions (sevenths), the compound form of the 182¢ second (a ninth) and its inversion (a seventh), and the “just” tritone (first in the harmonic series) of 583¢/617¢.

Dissonances include the simple 182¢ second, all seconds simple and compound smaller than 182¢ with their inversions, and all tritones other then the “just.”

These categorizations provide a workable balance of perfect and imperfect consonance and dissonance, divided not only according to ratios and relative consonance and dissonance, but also according to historically dynamic precedents related to consonance and dissonance.

Intervals in these categories have been applied as analogous to 16th century counterpoint. Figure 9 shows another musical reduction from The Madman’s Diary. Four-voice counterpoint in the electronic part mm. 358-370 (4B ff.) is represented. The music of the vocal line has been omitted for clarity. As with Figure 8, the contrapuntal texture becomes clear in the reduction.

Figure 9. Contrapuntal procedure in The Madman’s Diary.

3.4.3. Harmonic procedures

Tertian chords have been constructed by superimposing the consonant intervals deemed “perfect” and “imperfect.” Chords with all perfect consonances are here called perfect, those with any combination termed imperfect. There are fifteen possible triadic combinations, eight possible tetrads and one pentad.

Perfect Imperfect Tetrads Pentad
Triads Triads (Imperfect) (Imperfect)

Figure 10. Chords and consonances.
The perfect and imperfect chords for the E diatonic set are shown in Figure 10. Those containing the tonic (E) are in the first line, those remaining that contain F# +4 are in the second line, and those left that contain G# -14 are in the third line. This accounts for all possible combinations. Such systematic inventory is not meant to suggest primacy for the tones by which chords are accounted for.

This set of relationships is the same at all transpositions. Numbers showing deviation from 12Tet are omitted in Figures 10 and 11 for clarity. Allowance of the 182¢ second as a consonance only if inverted to a seventh or compounded to a 9th—in this case involving the F# and G#—is indicated by the asterisks in Figure 10.

The progression shown in Figure 11 is gradual. At each change of bass the chord above is held, and at each change of chord only one member of the chord is changed. This is in keeping with the stylistically minimalist character of the texture in that passage.

4. EXTENSION OF THE SCALE

The eight-tone scale, series of transpositions with common tones, and contrapuntal and harmonic procedures described above were all developed and applied in pieces composed between 1996 and 1999 [8]. In 2000 a theoretical and practical extension was added to the system [9].

4.1. Octave 5 of the Harmonic Series

The harmonic series theoretically extends indefinitely. While the fourth octave of the harmonic series contains intervallic relationships that can form the eight-tone scale described above, the fifth octave contains intervals that would form a sixteen-tone scale.

Extending the series of harmonics, frequencies and ratios by which the harmonics are related in Table 1 for another octave results in the following:

<table>
<thead>
<tr>
<th>HARMONIC</th>
<th>FREQUENCY (Hz)</th>
<th>RATIO</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 (C₃)</td>
<td>523.25</td>
<td>1/1</td>
</tr>
<tr>
<td>17</td>
<td>555.95</td>
<td>17/16</td>
</tr>
<tr>
<td>18</td>
<td>588.65</td>
<td>18/17</td>
</tr>
<tr>
<td>19</td>
<td>621.36</td>
<td>19/18</td>
</tr>
<tr>
<td>20</td>
<td>654.06</td>
<td>20/19</td>
</tr>
<tr>
<td>21</td>
<td>686.76</td>
<td>21/20</td>
</tr>
<tr>
<td>22</td>
<td>719.47</td>
<td>22/21</td>
</tr>
<tr>
<td>23</td>
<td>752.17</td>
<td>23/22</td>
</tr>
<tr>
<td>24</td>
<td>784.87</td>
<td>24/23</td>
</tr>
<tr>
<td>25</td>
<td>817.58</td>
<td>25/24</td>
</tr>
<tr>
<td>26</td>
<td>850.28</td>
<td>26/25</td>
</tr>
<tr>
<td>27</td>
<td>882.98</td>
<td>27/26</td>
</tr>
<tr>
<td>28</td>
<td>915.68</td>
<td>28/27</td>
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<tr>
<td>29</td>
<td>948.39</td>
<td>29/28</td>
</tr>
<tr>
<td>30</td>
<td>981.09</td>
<td>30/29</td>
</tr>
<tr>
<td>31</td>
<td>1013.8</td>
<td>31/30</td>
</tr>
<tr>
<td>32 (C₄)</td>
<td>1046.5</td>
<td>32/31</td>
</tr>
</tbody>
</table>

Table 2. Extension of the overtone scale through octave 5.

As these intervals continue in simple ratios, their calculation as tones in 12Tet can be made according to the same formula: $n = 3986 \log R$.

Transposed to E, as before, and notated with tunings in cents, a more distinctive sixteen-tone microtonal scale results. Figure 12 shows this scale placed in the mid-treble range.

Figure 12. 16-tone scale from octave 5.

A cycle of eight sixteen-tone scales with tones in common was constructed, in the same manner as the eight-tone scale, with tonics in the same relation as the eight-tone scale. This allowed for modulation in the same manner. Contrapuntal and harmonic procedures could also be developed, though such systems would be considerably more complex than those developed for the eight-tone scale. For this reason, music composed with the sixteen-tone scale has followed the organum-like polyphonic procedures outlined above, without a thorough set of contrapuntal or harmonic procedures to match.

Pieces composed with the new scale included Psalm [10], the electronic soundtrack of a web-based multimedia work of the same title (presented at ICMC 2002), another web-based multimedia work entitled Spring [11], and a
tape piece called *Organum on Ash Grove* [12] (presented at ICMC 2008). The texture for *Psalm* was created with a series of long, low tones, each the fundamental of a diatonic set, that over time slowly describes a hymn tune (“Foundation”). Higher-pitched events comprising pitch classes of the sixteen-tone diatonic set of each fundamental were constructed above these low tones. Each different low tone has its own particular set of events applied each time that tone is sounded; thus, the form of the piece is modular: given sets of events associated with a tone are heard each time that tone returns in the underlying melody.

The texture of *Organum on Ash Grove* was composed according to the same procedure (with “Ash Grove” as the hymn tune) except that each of the eight sixteen-tone scales was divided into two different eight-tone scales comprising the odd-numbered scale degrees for one (the eight-tone scale of octave four) and the even-numbered scale degrees for the other (the tones added in octave five). *Psalm* and *Organum on Ash Grove* both have rapidly pulsing digitally generated sound-mass textures too complex to represent with musical notation.

### 4.2. Octave 6 of the Harmonic Series.

A more recent piece is *Body Periodic* [13], a tape piece. It was composed applying the next logical extension of the overall procedure: a thirty-two-tone scale corresponding to the theoretical sixth octave of the harmonic series. Figure 13 shows the notated scale, calculated by applying the logarithmic formula above, with C as the fundamental.

![Figure 13. 32-tone scale from octave 6.](image)

The texture of *Body Periodic* is essentially two-part counterpoint over a tonic drone, without modulation, in a strophic form having a pattern of additive variance. Melodic materials, though linear, are comparatively rapid and complex.

A reduction showing the basic contrapuntal structure is shown in Figure 14 (the strophe containing this structure can be most clearly heard from 1:10 to 2:16 of the piece). Much of the scale is treated as a series of free or non-harmonic tones in relation to a number of tonal triadic combinations, with the 386¢ just major third between C ±0 and E -14 functioning as tonic. There are additional elements to the piece—speech samples, non-tonal sounds—but the tonal element is most striking. Its florid, microtonal melodic materials woven into a very simple harmonic scheme makes it suggestive of some Persian or Indian music.

![Figure 14. Reduction from *Body Periodic*.](image)

The newest piece to apply the 32-tone scale, presented in conjunction with this paper, is *Organum on St. Denio* [14]. As the title suggests, its texture is like those of *Psalm* and *Organum on Ash Grove*, but with the 32-tone scale from *Body Periodic*. It combines the modular forms and rapid-pulse complexity of *Psalm* and *Organum on Ash Grove* with the linear/melodic materials of *Body Periodic*. A reduction of the first 41 seconds illustrates its organum-like texture and common-tone modulations:

(Continued, below)
These new scales, the translation of their simple-ratio intervals according to musical standards and graphic representation; the development of transpositions for modulation, of contrapuntal and harmonic procedures, and of distinctive microtonal intervallic relationships provide innovative systems for applying tonal materials in the composition of computer music. These systems comprise an interconnected group of diatonic sets with intervallic relationships found in the harmonic series; and thus, together with extensions of the basic scale, reflect the acoustic properties of pitched sound itself. Translation into 12Tet allows this system to be realized using commercial digital synthesis applications, and to be represented musically.

Procedural possibilities for composition have been successfully applied in musical works intended for both experimentation and public performance. These include the tape pieces Organum Duplum (1997), Klirrfarbenstrukturen (1997), Two Sketches (1998), and Picture Tube (1998); The Madman’s Diary for tenor, electronic music, video, strings and percussion (1999); soundtracks for the web-delivered multimedia pieces Psalm (2001), and Spring (2002); Accompanied Poetry of Rumi for reader, electronic music and chorus (2004); The Realm, for reader and electronic music (2006); Organum on Ash Grove (2007), Body Periodic (2008), and Organum on St. Denio (2009), all tape pieces.

As with any innovation in the creative arts, continual practice and experimentation with these materials and procedures will be necessary to more fully realize their stylistic and aesthetic implications.

6. REFERENCES