A CURL-FREE VECTOR FIELD THAT IS NOT A GRADIENT

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Math 225

Recall our main theorem about vector fields.

**Theorem.** Let \( R \) be an open region in \( \mathbb{R}^2 \) and let \( \mathbf{F} \) be a \( C^1 \) vector field on \( R \). The following statements about \( \mathbf{F} \) are equivalent:

1. There is a differentiable function \( f : R \to \mathbb{R} \) such that \( \nabla f = \mathbf{F} \).
2. If \( C \) is a piecewise \( C^1 \) path in \( R \), then \( \int_C \mathbf{F} \cdot d\mathbf{x} \) depends only on the endpoints of \( C \).
3. \( \oint_C \mathbf{F} \cdot d\mathbf{x} = 0 \) for every piecewise \( C^1 \) simple, closed curve in \( R \).

Furthermore, statements (1)–(3) imply

4. \( \text{curl} \mathbf{F} = 0 \),

and (4) implies (1)–(3) when \( R \) is simply connected (so all four are equivalent when \( R \) is simply connected).

The purpose of this handout is to explore what happens when \( R \) is not simply connected, that is, when there exist non-gradient vector fields with zero curl.

**AN IMPORTANT EXAMPLE**

Consider the vector field

\[
\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}
\]

defined on \( R = \mathbb{R}^2 \setminus \{(0,0)\} \), that is, on all of \( \mathbb{R}^2 \) except the origin. Letting \( P = -y/(x^2+y^2) \) and \( Q = x/(x^2+y^2) \), it is a simple matter to show that \( \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \), and so \( \text{curl} \mathbf{F} = 0 \).

We can show that (1)–(3) are false for \( \mathbf{F} \) by finding a simple, closed curve \( C \) for which \( \oint_C \mathbf{F} \cdot d\mathbf{x} \neq 0 \). Let \( C \) be the unit circle parameterized counterclockwise by \( x = \cos t \), \( y = \sin t \), \( 0 \leq t \leq 2\pi \). We have

\[
\oint_C \mathbf{F} \cdot d\mathbf{x} = \oint_C \left( \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \right)
\]

\[
= \int_0^{2\pi} \frac{-\sin t}{1} (-\sin t) \, dt + \frac{\cos t}{1} \cos t \, dt = \int_0^{2\pi} \, dt = 2\pi.
\]

It follows from the theorem that \( \mathbf{F} \) is not the gradient of any function defined on \( R \).
Exercise 1. Show that if \( C \) is any circle centered at the origin oriented counterclockwise, then \( \oint_C \mathbf{F} \cdot d\mathbf{x} = 2\pi \), that is, the radius of the circle does not affect the value of the integral.

Exercise 2. Let \( D \) be a disk centered at the origin. Since \( \text{curl} \mathbf{F} = 0 \), it would be very easy to use Green’s Theorem to conclude that \( \oint_C \mathbf{F} \cdot d\mathbf{x} = \iint_D \text{curl} \mathbf{F} \, dA = \iint_D 0 \, dA = 0 \), where the circle \( C \) is the boundary of \( D \), in contrast to the previous exercise. Why is the reasoning here incorrect?

Now suppose that \( \hat{R} \) is a simply connected subset of \( R \). To be concrete, suppose \( \hat{R} \) is all of \( \mathbb{R}^2 \) except the negative \( x \)-axis. Then the theorem implies that \( \mathbf{F} \) is the gradient of some function \( \hat{f}: \hat{R} \to \mathbb{R} \). Suppose further that \( \hat{R} \) is all of \( \mathbb{R}^2 \) except the positive \( x \)-axis. Then \( \mathbf{F} \) is the gradient of some function \( \tilde{f}: \tilde{R} \to \mathbb{R} \). Then the functions \( \hat{f} \) and \( \tilde{f} \) have the same gradient, namely \( \mathbf{F} \), on their common domain \( \hat{R} \cap \tilde{R} \). You might guess that if two functions have the same gradient, then they must differ by a constant. That’s true, but only if you restrict yourself to a connected set. In this case the common domain \( \hat{R} \cap \tilde{R} \) consists of two disjoint connected subsets, namely the upper and lower half planes (\( y > 0 \) on one and \( y < 0 \) on the other). Evidently the two functions differ by different constants on these two half planes, so you can’t “fix” the situation by adding a constant to one of them.

What can these functions be? Given a vector field with zero curl, you have a method for finding a function whose gradient is the vector field. It’s a bit messy to apply with this vector field, so I’ll just tell you what the answer is: it’s \( \theta \), the angle of polar coordinates, plus an arbitrary constant, of course.

Exercise 3. Verify this by implicit differentiation. To do this, first show that \( x \sin \theta = y \cos \theta \) (can you do this without dividing by anything that might be zero?). Then apply \( \frac{\partial}{\partial x} \) to both sides of this, treating \( y \) as a constant and \( \theta \) as a function of \( x \) and \( y \).

Now you may be thinking that there is a contradiction here: first we decided that \( \mathbf{F} \) is not the gradient of any function on \( R \), and then we showed that it is the gradient of \( \theta \)? The resolution of the apparent contradiction is that \( \theta \) is not a well-defined function on \( R \), even though we often treat it as such. If you start with \( \theta = 0 \) on the positive \( x \)-axis and then go counterclockwise around the origin, \( \theta \) increases. When you get close to the positive \( x \)-axis in the fourth quadrant, \( \theta \) is near \( 2\pi \). There is no way to make it a continuous, let alone differentiable, function on \( R \). What about on \( \hat{R} \) and \( \tilde{R} \)? On \( \hat{R} \) you can take \( \theta \) to have values between \( -\pi \) and \( \pi \). On \( \tilde{R} \) you can take \( \theta \) to have values between \( 0 \) and \( 2\pi \). More generally, if \( \hat{R} \) is an arbitrary simply connected subset of \( R \), then \( \mathbf{F} \) is the gradient of some function \( \hat{f}: \hat{R} \to \mathbb{R} \). By adding a constant, you can get \( \hat{f} + C \) to agree with some suitable definition of \( \theta \) on \( \hat{R} \), but it may not agree with any of the “usual” definitions of \( \theta \) on all of \( R \).

Exercise 4. Suppose that \( \hat{R} \) is a connected, narrow spiral-shaped subregion of \( R \) that contains \((1,0)\) and \((5,0)\), but that to get from \((1,0)\) to \((5,0)\), staying in the region, you have to cross the negative \( x \)-axis. If \( \tilde{f}(1,0) = 0 \), then \( \tilde{f} \) agrees with the usual definition of \( \theta \) near \((1,0)\). What is \( \tilde{f}(5,0) \) in this case? Notice that \( \tilde{f} \) is well-defined, but that it takes on a range of values larger than \( 2\pi \)!
Exercise 5. Show that the following vector field, also with domain \( R \), is the gradient of a function defined on all of \( R \):
\[
\frac{x}{x^2 + y^2} \mathbf{i} + \frac{y}{x^2 + y^2} \mathbf{j}
\]

**The general case on** \( R = \mathbb{R}^2 \setminus \{(0,0)\} \)

Now suppose that \( \mathbf{G} \) is an arbitrary vector field defined on \( R = \mathbb{R}^2 \setminus \{(0,0)\} \), that \( \text{curl} \mathbf{G} = 0 \), and that \( \mathbf{G} \) is not the gradient of any function. By the theorem, there must be a simple, closed curve \( C \) such that \( \oint_C \mathbf{G} \cdot d\mathbf{x} \neq 0 \). Let’s find one.

Suppose that \( C \) is a simple, closed curve in \( R \). If \( C \) does not go around the origin, then \( C \) is contained in some simply connected subregion \( \tilde{R} \) of \( R \). Then the theorem implies that \( \oint_C \mathbf{G} \cdot d\mathbf{x} = 0 \). (You can also conclude this by applying Green’s Theorem to the region bounded by \( C \).) Thus, if \( C \) is a simple, closed curve for which \( \oint_C \mathbf{G} \cdot d\mathbf{x} \neq 0 \), then \( C \) must go around the origin.

Suppose that \( C_1 \) and \( C_2 \) are two simple, closed curves in \( R \) that go around the origin counterclockwise. Furthermore, assume that they don’t intersect and that \( C_1 \) is the outer curve. (Draw a picture!) Let \( \Omega \) be the annular region between \( C_1 \) and \( C_2 \). Then \( \partial \Omega \) is \( C_1 \cup C_2 \), which is given its correct orientation by having \( C_1 \) go counterclockwise and \( C_2 \) go clockwise. By Green’s Theorem we have
\[
0 = \iint_{\Omega} \text{curl} \mathbf{G} \, dA = \oint_{\partial \Omega} \mathbf{G} \cdot d\mathbf{x} = \oint_{C_1} \mathbf{G} \cdot d\mathbf{x} - \oint_{C_2} \mathbf{G} \cdot d\mathbf{x},
\]
and so
\[
\oint_{C_1} \mathbf{G} \cdot d\mathbf{x} = \oint_{C_2} \mathbf{G} \cdot d\mathbf{x}.
\]

Exercise 6. Show that this holds even if \( C_1 \) and \( C_2 \) intersect.

Thus every simple, closed curve \( C \) that goes around the origin counterclockwise gives the same non-zero value for the integral \( \oint_C \mathbf{G} \cdot d\mathbf{x} \). (This should remind you of Exercise 1.) This number is a property of the vector field \( \mathbf{G} \). Denote this value by \( c \).

Define a new vector field \( \tilde{\mathbf{G}} \) by modifying \( \mathbf{G} \):
\[
\tilde{\mathbf{G}} = \mathbf{G} - \frac{c}{2\pi} \mathbf{F},
\]
where \( \mathbf{F} \) is the vector field in the previous section.

I claim that \( \tilde{\mathbf{G}} \) is the gradient of some function on \( R \). To see this, suppose \( C \) is a simple, closed curve in \( R \). If \( C \) doesn’t go around the origin, then, as observed above, \( \oint_C \mathbf{G} \cdot d\mathbf{x} = 0 \). For the same reason, \( \oint_C \mathbf{F} \cdot d\mathbf{x} = 0 \) (in fact \( \mathbf{F} \) is a special case of \( \mathbf{G} \)). Thus, \( \oint_C \tilde{\mathbf{G}} \cdot d\mathbf{x} = 0 \).

Now suppose \( C \) goes around the origin counterclockwise. Then
\[
\oint_C \tilde{\mathbf{G}} \cdot d\mathbf{x} = \oint_C \mathbf{G} \cdot d\mathbf{x} - \frac{c}{2\pi} \oint_C \mathbf{F} \cdot d\mathbf{x} = c - \frac{c}{2\pi} (2\pi) = 0.
\]
Since \( \oint_C \mathbf{G} \cdot d\mathbf{x} = 0 \) for every simple, closed curve \( C \) in \( R \), the theorem implies that \( \mathbf{G} \) is the gradient of some function \( \tilde{g} \): \( R \to \mathbb{R} \). Recall that \( \mathbf{F} \) is almost the gradient of \( \theta \), except that \( \theta \) isn’t a well-defined function on \( R \). Similarly, \( \mathbf{G} \) is almost the gradient of \( \tilde{g} + c\theta \). On a simply connected subregion of \( R \), \( \mathbf{G} \) will be the gradient of \( \tilde{g} + c\theta \), where \( \theta \) is some extension of the usual \( \theta \) to the subregion.

Note that this says that every vector field \( \mathbf{G} \) on \( R \) that is curl-free and not a gradient can be made into a gradient simply by adding a constant multiple of the vector field \( \mathbf{F} \) that is the main example,

\[
\mathbf{F} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}.
\]

Thus the vector field \( \mathbf{F} \) is, in some sense, the only example, at least on \( R \).

**The Case of More Holes**

Now suppose that \( R \subset \mathbb{R}^2 \) is a region with finitely many holes. A hole can be more than just a point; can be the result of removing a simply connected set. In general, \( R = R_0 \setminus (H_1 \cup \cdots \cup H_n) \) in which \( R_0 \) is a simply connected region, each \( H_i \) is a non-empty, closed set that is either a point or the closure of a simply connected region, and the sets \( H_1, \ldots, H_n \) are disjoint subsets of \( R_0 \).

For each index \( i \), let \( P_i(x_i, y_i) \) be a point in \( H_i \), and consider the vector field

\[
\mathbf{F}_i = \frac{-(y - y_i)}{(x - x_i)^2 + (y - y_i)^2} \mathbf{i} + \frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2} \mathbf{j}.
\]

This is similar to the main example \( \mathbf{F} \), but it is centered at \( P_i \) instead of the origin. If \( \mathbf{r} \) denotes the position vector of a point \( X \) from the origin and \( r \) denotes the distance from \( X \) to the origin, note that \( \mathbf{F} \) can be written as \( \mathbf{F} = \mathbf{r}_\perp / r^2 \). In the same way, let \( \mathbf{r}_i = X - P_i \) be the position vector of \( X \) relative to \( P_i \), and let \( r = \|X - P_i\| = ||\mathbf{r}_i|| \) be the distance from \( P_i \) to \( X \). Then we have \( \mathbf{F}_i = (\mathbf{r}_i)_\perp / r_i^2 \).

If \( C \) is a simple, closed curve in \( R \) oriented counterclockwise, then \( \frac{1}{2\pi} \oint_C \mathbf{F}_i \cdot d\mathbf{x} \) is equal to \( 2\pi \) or \( 0 \) depending on whether \( C \) goes around \( P_i \) or not. More generally, suppose that \( C \) is piecewise \( C^1 \) loop in \( R \). It can be shown that \( \frac{1}{2\pi} \oint_C \mathbf{F}_i \cdot d\mathbf{x} \) is an integer. It is called the winding number of \( C \) around \( P_i \) because it counts how many times \( C \) goes around \( P_i \). Since \( C \) stays in \( R_i \), this integer also counts the number of times \( C \) goes around \( H_i \). The collection of integers, \( \frac{1}{2\pi} \oint_C \mathbf{F}_i \cdot d\mathbf{x} \) for \( 1 \leq i \leq n \), gives considerable information about how the curve \( C \) makes its way around all of the holes in the region.

**Exercise 7.** Let \( P_1(0,0) \), \( P_2(1,0) \), \( P_3(0,1) \), and let \( \mathbf{F}_1 \), \( \mathbf{F}_2 \), and \( \mathbf{F}_3 \) be the corresponding vector fields. Draw a closed curve \( C \) such that

\[
\frac{1}{2\pi} \oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 2, \quad \frac{1}{2\pi} \oint_C \mathbf{F}_2 \cdot d\mathbf{x} = 1, \quad \text{and} \quad \frac{1}{2\pi} \oint_C \mathbf{F}_3 \cdot d\mathbf{x} = -1.
\]

**Exercise 8.** Suppose \( P_1 \) is a point and \( \mathbf{F}_1 \) is the corresponding vector field. If \( C \) is a curve such that \( \frac{1}{2\pi} \oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 0 \), it can be shown that \( C \) can be morphed to a point (a curve that
doesn’t go anywhere) without passing through \( P_1 \) during the morphing process. (This is a substantial theorem. It says that no matter how crazy the curve \( C \) is, if \( \oint_C \mathbf{F}_1 \cdot d\mathbf{x} = 0 \), then \( C \) can be “unwound” from around \( P_1 \).) Let \( P_1(0,0) \) and \( P_2(1,0) \), and let \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \) be the corresponding vector fields. Draw a closed curve \( C \) such that

\[
\oint_C \mathbf{F}_1 \cdot d\mathbf{x} = \oint_C \mathbf{F}_2 \cdot d\mathbf{x} = 0
\]

for which it is intuitively clear that \( C \) cannot be morphed to a point without passing through at least one of the points.

Suppose \( \mathbf{G} \) is a vector field on \( R \) with \( \text{curl} \mathbf{G} = 0 \). For each \( i \), let \( C_i \) be a simple, closed curve in \( R \), oriented counterclockwise, that goes around the hole \( H_i \), but none of the other holes. Let \( c_i = \oint_{C_i} \mathbf{F}_i \cdot d\mathbf{x} \), and define a new vector field by

\[
\mathbf{\tilde{G}} = \mathbf{G} - \frac{1}{2\pi} \sum_{i=1}^{n} c_i \mathbf{F}_i.
\]

Then it can be shown that \( \mathbf{\tilde{G}} \) is the gradient of some function on \( R \). This implies that if \( \text{curl} \mathbf{G} = 0 \), then the only difference between \( \mathbf{G} \) and a gradient is some linear combination of the vector fields \( \mathbf{F}_1, \ldots, \mathbf{F}_n \).

Exercise 9. Let \( \mathbf{\tilde{G}} \) be the vector field in the paragraph above. Prove that \( \mathbf{\tilde{G}} \) is the gradient of some function on \( R \). Do this by supposing that \( C \) is a piecewise \( C^1 \), simple, closed curve in \( R \), and giving a careful explanation of why \( \oint_C \mathbf{\tilde{G}} \cdot d\mathbf{x} = 0 \).