WEIGHTED-NORM FIRST-ORDER SYSTEM LEAST SQUARES (FOSLS) FOR PROBLEMS WITH CORNER SINGULARITIES

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Abstract.
A weighted-norm least-squares method is considered for the numerical approximation of solutions that have singularities at the boundary. While many methods suffer from a global loss of accuracy due to boundary singularities, the least-squares method can be particularly sensitive to a loss of regularity. The method we describe here requires only a rough lower bound on the power of the singularity and can be applied to a wide range of elliptic equations. Optimal order discretization accuracy is achieved in weighted H^1, and functional norms and L^2 accuracy is retained for boundary value problems with a dominant div/curl operator. Our analysis, including interpolation bounds and several Poincaré type inequalities, are carried out in appropriately weighted Sobolev spaces. Numerical results confirm the error bounds predicted in the analysis.

1. Introduction. In this paper, we develop a method for treating div/curl systems with reduced regularity. While motivated by first-order systems that arise from second-order elliptic boundary value problems, div/curl systems appear in many contexts, for example, in Maxwell's equations and in the vorticity from of Stokes equations. These problems have the fortunate property of a guaranteed smooth solution as long as the data and domain are smooth. However, many problems of interest are posed in non-smooth domains and, as a consequence, lose this property at a finite number of points on the boundary in two dimensions or along curves on the boundary in three dimensions. In the present paper, we study two-dimensional problems that have non-smooth solutions at *irregular boundary points*, that is, points that are corners of polygonal domains, locations of changing boundary condition type, or both. Similar behavior occurs in problems with discontinuous material coefficients and the methods presented here can easily be extended to that situation.

Standard solution techniques that attempt to approximately solve a div/curl system with reduced regularity using H^1-conforming finite elements will, in general, fail to converge. This phenomenon can be explained by noting that the Sobolev space $(H^1)^2$ is a closed subspace of $H(div) \cap H(curl)$ (see section 2), which implies either that $(H^1)^2 = H(div) \cap H(curl)$, the case of full regularity, or $(H^1)^2$ is a proper subspace. In this case, the co-dimension is finite and is spanned by so-called singular functions. In the presence of reduced regularity, the solution will, in general, not be in $(H^1)^2$. A standard finite element method using H^1-conforming elements will converge to the element of $(H^1)^2$ closest to the true solution in the $H(div) \cap H(curl)$ norm. Local mesh refinement will not alter this outcome.

If a basis for the singular functions is known, it can be incorporated directly into the finite element space (c.f.[10, 16, 24]). In [3, 4], this approach is applied to div/curl systems and shown to restore optimal convergence throughout the domain at a minimal additional cost. For some two-dimensional problems, the singular basis functions are known and can be included in the finite element space. For the other

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problems, or in three dimensions, the exact character of the singular functions is less well understood which makes this approach more difficult to implement.

As a different type of remedy for this so-called pollution effect, least-squares methods based on inverse norms can be effective for problems with irregular boundary points, discontinuous coefficients, and data in H^{-1}. For example, in [5, 8, 11–13, 19], the functional is posed in terms of H^{-1}-norms rather than L^2-norms, resulting in optimal L^2-approximations to the solution. A more recent approach, called FOSLL*, uses an inverse norm induced by the equations, and is shown in [14, 22] to be more efficient than the H^{-1}-norm methods.

Graded mesh refinement in weighted Sobolev spaces has been shown to be effective in restoring optimal convergence, in L^2- and H^1-norms, in the context of a Galerkin formulation of second-order elliptic problems with reduced regularity (c.f. [2], [1]). However, in that context, the solution is in H^1. Convergence would occur, although more slowly, without weighting and mesh refinement.

In [15], a weighted norm and a sequence of graded meshes is used in an $H(div)$ least-squares functional, arising from a second-order elliptic problem, to restore optimal convergence for the primal variable, in both L^2- and H^1-norms, if the flux variable is approximated in a finite element space satisfying the grid decomposition property, for example, Raviart-Thomas elements (c.f.[6]). The flux variable converges in a weighted $H(div)$-norm.

In this paper we examine div/curl systems that lack regularity within the first-order system least-squares (FOSLS) framework. The basic FOSLS approach is to recast the original system as an appropriate first-order system and apply an L^2 minimization principle over the residual of the equations. If possible, this reformulation is done by minimizing a functional whose quadratic part is equivalent to the product H^1-norm, indicating that the process is similar to solving a weakly coupled system of Poisson-like equations. This equivalence also guarantees optimal H^1-accuracy for standard discretizations. For div/curl systems with reduced regularity, as briefly mentioned above, the L^2 based functional fails to be H^1-equivalent and, as a consequence, standard discretizations suffer from the pollution effect. Here, a weighted-norm least-squares method is developed that restores optimal convergence using H^1-conforming finite elements without graded mesh refinement. It replaces the L^2-norms in the FOSLS functional with weighted L^2-norms, making the functional norm equivalent to a weighted H^1-norm. With an appropriate weighting function, this method recovers optimal order accuracy in the weighted L^2- and H^1-norms, and retains optimal L^2 convergence even near the singularity. Our method requires only the power of the singularity (not the actual singular solution) to be known *a priori* and, in practice, can be used with only a rough estimate of the power of the singularity, which can be adaptively determined if unknown (c.f.[4]).

The method developed here has some similarity to [15], but considers an $H(div) \cap H(curl)$ functional and considers more general boundary conditions. Most importantly, our analysis admits a more aggressive weighting, resulting in optimal order accuracy in the weighted norms without mesh refinement. In addition, we prove several Poincaré type inequalities in weighted Sobolev spaces under a variety of boundary conditions that, in addition to being necessary for our main result, may be of independent interest.

We use Poisson's equation on a domain with a reentrant corner as a model problem and as the formal setting for analysis. The resulting div/curl system is the focus. The analysis in this paper is restricted to two dimensions. However, the approach suggests a natural generalization to problems with reduced regularity due to discontinuous coefficients and to problems in three dimensions.

This paper is organized as follows. Section 2 contains notation and preliminary discussion. In section 3, the weighted FOSLS functionals are described. Section 4 contains Poincaré inequalities and regularity results in weighted Sobolev spaces. Error bounds for the weighted FOSLS functional are presented in section 5 and section 6 contains computational results.

2. Singular solutions and preliminaries. For vector function $\mathbf{u} = (u_1, u_2)^t$, let the divergence and curl of $\mathbf{u}$ be defined in the standard way: $\nabla \cdot \mathbf{u} = \partial_x u_1 + \partial_y u_2$ and $\nabla \times \mathbf{u} = \partial_x u_2 - \partial_y u_1$. Further, define the formal adjoint of the curl operator by

$$\nabla^{\perp} q = \begin{pmatrix} \partial_y q \\ -\partial_x q \end{pmatrix}.$$

We use standard notation for Sobolev spaces $H^k(\Omega)^d$, corresponding inner product $(\cdot,\cdot)_{k,\Omega}$, and norm $\|\cdot\|_{k,\Omega}$, for $k \geq 0$. We drop subscript Ω and superscript d when the domain and dimension are clear by context. Since $H^0(\Omega)$ coincides with $L^2(\Omega)$ we often denote $\|\cdot\|_0$ by $\|\cdot\|$. Define the subspaces of $L^2(\Omega)$ induced by the divergence and curl of $\mathbf{u}$ by

$$H(div) = \{\mathbf{u} \in L^2(\Omega) : \|\nabla \cdot \mathbf{u}\| < \infty\},$$
$$H(curl) = \{\mathbf{u} \in L^2(\Omega) : \|\nabla \times \mathbf{u}\| < \infty\}.$$

We also make use of the following general inequalities for nonnegative a and b:

$$|a|^2 + |b|^2 \leq |a+b|^2 \leq 2(|a|^2 + |b|^2). \tag{2.1}$$

Consider the function $f(r,\theta) = r^a$ in two dimensional polar coordinates. Assume that the origin lies on the boundary of domain

$$\Omega_w = \{(r,\theta) : 0 < r < R,\ 0 < \theta < \omega < 2\pi\},$$

as pictured in figure 2.1. By a direct computation it is clear that $f \in H^k(\Omega)$ only for $k < a+1$.

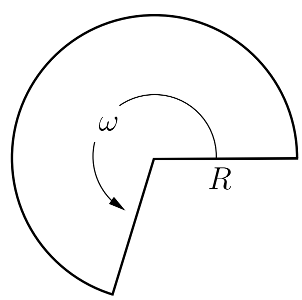


FIG. 2.1. *Simple wedge-shaped domain, Ω_w.*

Now, consider Poisson's equation on a domain in $\mathbb{R}^2$ with a corner of interior angle ω. It is well known that, for the case of Dirichlet or Neumann boundary conditions, the solutions of this boundary value problem may include those with radial part of the form $p \sim r^{\frac{\pi}{\omega}}$ in a local polar coordinate system centered at the corner. Thus, for the case of reentrant corners, $\omega > \pi$, the solution fails to be in $H^2(\Omega)$ and we say that the problem has a singularity (or singular solution). For problems with Dirichlet and Neumann boundary conditions meeting at the corner, solutions may have components of the form $p \sim r^{\frac{\pi}{(2\omega)}}$. Thus, for mixed boundary conditions, singularities may occur at corners with $\omega > \pi/2$. We now explore this issue in more detail.

Define the power of the singularity to be $\alpha = \pi/\omega$ for Dirichlet or Neumann boundary conditions and $\alpha = \pi/(2\omega)$ for mixed boundary conditions. The solution to Poisson's equation may be written as

$$p(r,\theta) = p_0(r,\theta) + s(r,\theta),$$

where $p_0(r,\theta) \in H^2(\Omega)$ and $s(r,\theta) \in H^{1+m}(\Omega)$ for $m < \alpha$. The singular part of the solution has the form

$$s(r,\theta) = r^{\alpha}(\kappa_1 \sin(\alpha\theta) + \kappa_2 \cos(\alpha\theta)),$$

where the values of κ_1 and κ_2 depend on boundary conditions (see [17, 18]).

For the FOSLS formulation of this problem, we may similarly decompose unknown $\mathbf{u} = \nabla p$ as

$$\mathbf{u}(r,\theta) = \mathbf{u}_0(r,\theta) + \nabla s(r,\theta),$$

where $\nabla s(r,\theta)$ has the form

$$\nabla s(r,\theta) = \alpha r^{\alpha-1} \begin{pmatrix} \kappa_1 \sin(\alpha-1)\theta + \kappa_2 \cos(\alpha-1)\theta \\ \kappa_1 \cos(\alpha-1)\theta - \kappa_2 \sin(\alpha-1)\theta \end{pmatrix}.$$

Thus, the unknown $\mathbf{u}(r,\theta)$ is in $H^k(\Omega)$ only for $k < \alpha$.

For example, consider Poisson's equation posed on the simple domain in figure 2.1. Let the solution to this boundary value problem in polar coordinates be $p = \chi(r) r^{\frac{2}{3}} \sin(2\theta/3)$, where $\chi(r)$ is a smooth transition function that is 1 on a platform near the origin and vanishes at the boundaries not adjacent to the origin. Then, $p = 0$ on $\partial\Omega$ and

$$\begin{aligned} \Delta p &= \frac{1}{r}\partial_r(r\partial_r p) + \frac{1}{r^2}\partial^2_{\theta\theta} p \\ &= \left(r^{\frac{2}{3}}\chi''(r) + \tfrac{7}{3} r^{\frac{-1}{3}}\chi'(r)\right)\sin(2\theta/3), \end{aligned}$$

and, thus, it is clear that $\Delta p \in L^2(\Omega)$, but $p \notin H^2(\Omega)$. We say this problem fails to provide *full lifting* of the data (from $L^2(\Omega)$ to $H^2(\Omega)$ e.g.). The solution, $\mathbf{u} = \nabla p$, is, thus, not in $H^1(\Omega)$.

3. Weighted-norm least squares. As before, let Ω be a domain with a corner of interior angle ω at the origin, and we may, without loss of generality, further assume $diam(\Omega) \leq 1$. For $f \in L^2(\Omega)$, let p satisfy

$$\begin{cases} -\Delta p = f, & \text{in} \quad \Omega, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot\nabla p = 0, & \text{on} \quad \Gamma_N, \end{cases} \tag{3.1}$$

where $\mathbf{n}$ is the outward unit normal to Ω and $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$. When this problem is H^2 regular, the normal FOSLS methodology is to introduce the new unknown, $\mathbf{u} = \nabla p$, and rewrite system (3.1) as

$$\begin{cases} \mathbf{u} - \nabla p = 0, & \text{in} \quad \Omega, \\ -\nabla \cdot \mathbf{u} = f, & \text{in} \quad \Omega, \\ \nabla \times \mathbf{u} = 0, & \text{in} \quad \Omega, \\ \boldsymbol{\tau} \cdot \mathbf{u} = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \mathbf{u} = 0, & \text{on} \quad \Gamma_N, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n} \cdot \nabla p = 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{3.2}$$

Here, $\boldsymbol{\tau}$ is the counter-clockwise unit tangental vector to Ω. Since this system can be posed completely in terms of $\mathbf{u}$, we may decouple the equations in (3.2), solve for $\mathbf{u}$ first, and then recover p from $\mathbf{u}$. To this end, define the two L^2-norm functionals,

$$\begin{aligned} G(\mathbf{u}; f) &= \|\nabla \cdot \mathbf{u} + f\|^2 + \|\nabla \times \mathbf{u}\|^2, \\ G_2(p; \mathbf{u}) &= \|\mathbf{u} - \nabla p\|^2, \end{aligned}$$

and the spaces,

$$\begin{aligned} \mathcal{V} &= \{\mathbf{v} \in H^1(\Omega) : \ \boldsymbol{\tau} \cdot \mathbf{v} = 0 \ \text{ on } \ \Gamma_D, \ \ \mathbf{n} \cdot \mathbf{v} = 0 \ \text{ on } \ \Gamma_N\}, \\ \mathcal{W} &= \{q \in H^1(\Omega) : \ q = 0 \ \text{ on } \ \Gamma_D\}. \end{aligned}$$

Thus, the two-stage solution process is to minimize $G(\mathbf{v}; f)$ over $\mathcal{V}$ and then, given the minimizer, $\mathbf{u}$, minimize G_2 over $\mathcal{W}$:

$$\begin{aligned} &(1) \quad G(\mathbf{u}; f) = \inf_{\mathbf{v} \in \mathcal{V}} G(\mathbf{v}; f), \\ &(2) \quad G_2(p; \mathbf{u}) = \inf_{q \in \mathcal{W}} G_2(q; \mathbf{u}). \end{aligned} \tag{3.3}$$

The goal of the FOSLS methodology is, generally, to formulate functionals whose quadratic part is equivalent to the H^1 norm whenever possible. The second stage functional is H^1 equivalent and the solution we seek is always in H^1. The first stage functional, however, is not always H^1 equivalent. For domains with reentrant corners, there is no H^1 sequence of functions that converges to the solution in the $H(div) \cap H(curl)$ norm. To illustrate, consider the example above where $p = \chi(r) r^{\frac{2}{3}} \sin(2\theta/3)$ and $\mathbf{u} = \nabla p$. A simple computation reveals that $\nabla \cdot \mathbf{u}, \nabla \times \mathbf{u} \in L^2(\Omega)$, but $\mathbf{u} \notin H^1(\Omega)$.

Define the weighted functional by

$$G_w(\mathbf{u}; f) = \|w(\nabla \cdot \mathbf{u} + f)\|^2 + \|w \nabla \times \mathbf{u}\|^2, \tag{3.4}$$

where the weight function has the form $w = r^\beta$ for some $\beta > 0$.

Define the weighted Sobolev norm, $\|\cdot\|_{k,\beta}$, on Ω in terms of the standard L^2 norm, $\|\cdot\|_0$, by

$$\|q\|_{k,\beta} = \Big(\sum_{|j| \le k} \|r^{\beta - k + |j|} D^j q\|_0^2\Big)^{1/2}, \tag{3.5}$$

where D^j is the standard distributional derivative of order j. Similarly, define the weighted seminorm by

$$|q|_{k,\beta} = (\sum_{|j|=k} \|r^{\beta-k+|j|} D^j q\|_0^2)^{1/2} \tag{3.6}$$

and the associated weighted Sobolev space by

$$H^k_\beta(\Omega) = \{q : \|q\|_{k,\beta} < \infty\}. \tag{3.7}$$

Define the div/curl operator, L, and vector $\mathbf{f}$ by $L = \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$. We may now write the weighted functional from (3.4) as

$$G_w(\mathbf{u}; f) = \|L\mathbf{u} - \mathbf{f}\|^2_{0,\beta}.$$

The weighted-norm least-squares minimization problem for the first-stage solution is then: find $\mathbf{u} \in \mathcal{V}$ such that

$$G_w(\mathbf{u}; f) = \inf_{\mathbf{v}\in\mathcal{V}} G_w(\mathbf{v}; f).$$

The second-stage solution for p remains as described above. We seek values of β that make $H^1(\Omega)$ dense in $H(div) \cap H(curl)$ in the weighted functional norm and result in the most accurate discretizations possible.

For the discrete problem, we may choose any finite dimensional subset of H^1 over which to minimize the weighted functional. Let $\mathcal{P}^h$ denote the space of C^0 piecewise polynomial (or tensor product) elements on triangles (or quadrilaterals) of meshsize h and $\mathcal{V}^h$ the subspace of $\mathcal{P}^h$ that satisfies the appropriate boundary conditions on Ω:

$$\mathcal{V}^h = \{\mathbf{v}^h \in \mathcal{P}^h : \boldsymbol{\tau}\cdot\mathbf{v}^h = 0 \text{ on } \Gamma_D, \;\; \mathbf{n}\cdot\mathbf{v}^h = 0 \text{ on } \Gamma_N\}.$$

The discrete weighted-norm least-squares minimization problem is, then, to minimize the discrete functional: find $\mathbf{u}^h \in \mathcal{V}^h$ such that

$$G_w(\mathbf{u}^h; f) = \min_{\mathbf{v}^h\in\mathcal{V}^h} G_w(\mathbf{v}^h; f). \tag{3.8}$$

By unweighting the equations near the singularity, the functional is freed from trying to approximate the solution (which is not in $H^1(\Omega)$) in the H^1 sense near the singularity. But, away from the singularity, the weighted functional retains the same character as the normal non-weighted functional. We now consider the choice of weight parameter β and its relation to weighted and nonweighted *a priori* error bounds on the approximated solution.

4. Poincaré Bounds and Regularity Estimates. In this section, we establish several theoretical results in weighted Sobolev spaces and error bounds for the weighted-norm method.

Here, we establish several Poincaré bounds in the domain Ω_w. We first prove a result for the scalar pure Neumann and pure Dirichlet problems, and then for the scalar mixed boundary condition problem. These results lead to a Poincaré inequality for the vector case.

Lemma 4.1 *Take $\Omega = \Omega_w$ and let $\beta > 0$, $\epsilon > 0$ and $\gamma = \beta - 3/2 - \epsilon$. Further, assume that $\gamma \neq -2$. For functions $q \in H^1_\beta(\Omega)$, with $r^{\beta-\epsilon}\nabla q, r^{\gamma+1}\nabla q \in L^2(\Omega)$, that can be chosen to satisfy*

$$\iint_\Omega r^\gamma \, q(r,\theta) \, r \, dr \, d\theta = 0, \tag{4.1}$$

we have the bound:

$$\|q\|_{0,\beta-1} \leq C(\epsilon) \|\nabla q\|_{0,\beta-\epsilon}, \tag{4.2}$$

where $C(\epsilon)$ depends on ϵ, β, and Ω and $C(\epsilon) \to \infty$ as $\epsilon \to 0$.

Proof. For any two points (r,θ) and (r_0,θ_0) in Ω, write $q(r,\theta)$ as

$$q(r,\theta) = q(r_0,\theta_0) + \int_{r_0}^{r} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, d\hat{r} + \int_{\theta_0}^{\theta} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\hat{\theta}.$$

Multiplying both sides of the equation by $r_0^{\gamma+1}$, integrating with respect to r_0 and θ_0 over Ω, and Fubini's theorem yield

$$\begin{aligned}
&\frac{R^{\gamma+2}\omega}{\gamma+2} q(r,\theta) \\
&= \int_0^\omega \int_0^R \int_{r_0}^{r} r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, d\hat{r} \, dr_0 \, d\theta_0 + \int_0^\omega \int_0^R \int_{\theta_0}^{\theta} r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\hat{\theta} \, dr_0 \, d\theta_0 \\
&= \omega \int_0^r \int_0^{\hat{r}} r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, dr_0 \, d\hat{r} - \omega \int_r^R \int_{\hat{r}}^R r_0^{\gamma+1} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, dr_0 \, d\hat{r} \\
&\quad + \int_0^R \int_0^\theta \int_0^{\hat{\theta}} r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\theta_0 \, d\hat{\theta} \, dr_0 - \int_0^R \int_\theta^\omega \int_{\hat{\theta}}^\omega r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\theta_0 \, d\hat{\theta} \, dr_0 \\
&= \frac{\omega}{\gamma+2} \int_0^R \hat{r}^{\gamma+2} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, d\hat{r} - \frac{\omega R^{\gamma+2}}{\gamma+2} \int_r^R \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \, d\hat{r} \\
&\quad + \int_0^R \int_0^\omega \hat{\theta} \, r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\hat{\theta} \, dr_0 - \omega \int_0^R \int_0^\omega r_0^{\gamma+1} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \, d\hat{\theta} \, dr_0.
\end{aligned}$$

Note that above q is, by assumption, sufficiently smooth to apply Fubini's theorem. By the triangle inequality we have that

$$\begin{aligned}
&\left| \frac{R^{\gamma+2}\omega}{\gamma+2} q(r,\theta) \right| \\
&\quad \leq \frac{\omega}{\gamma+2} \int_0^R \hat{r}^{\gamma+2} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} + \frac{\omega R^{\gamma+2}}{\gamma+2} \int_r^R \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \\
&\quad\quad + 2\omega \int_0^R \int_0^\omega r_0^{\gamma+1} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right| \, d\hat{\theta} \, dr_0.
\end{aligned}$$

Now, squaring each side and using $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we get,

$$
\begin{aligned}
|q(r,\theta)|^2 \le \frac{3}{R^{2(\gamma+2)}} \left(\int_0^R \hat{r}^{\gamma+2} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 + 3 \left(\int_r^R \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 \\
+ \frac{12(\gamma+2)^2}{R^{2(\gamma+2)}} \left(\int_0^R \int_0^\omega r_0^{\gamma+1} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right| \, d\hat{\theta}\, dr_0 \right)^2 .
\end{aligned}
\tag{4.3}
$$

Multiply each side of (4.3) by $r^{2\beta-1}$ and integrate with respect to r and θ over Ω. We consider each of the terms on the resulting right-hand side separately. First,

$$
\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1} \left(\int_0^R \hat{r}^{\gamma+2} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 dr\, d\theta \\
&\le R \int_0^\omega \int_0^R \int_0^R r^{2\beta-1}\, \hat{r}^{2\gamma+4} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 d\hat{r}\, dr\, d\theta \\
&= \frac{R^{2\beta+1}}{2\beta} \int_0^\omega \int_0^R \hat{r}^{2\gamma+3} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 \hat{r}\, d\hat{r}\, d\theta \\
&\le \frac{R^{2\beta+1}}{2\beta} \|\nabla q\|^2_{0,\gamma+\frac{3}{2}} .
\end{aligned}
$$

We now consider the second term in (4.3). Since by the Schwarz inequality,

$$
\begin{aligned}
&\left(\int_r^R \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 \\
&= \left(\int_r^R \hat{r}^{(\epsilon-1)/2} \hat{r}^{(1-\epsilon)/2} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 \\
&\le \int_r^R \hat{r}^{\epsilon-1}\, d\hat{r} \int_r^R \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 \, d\hat{r} \\
&= \frac{R^\epsilon}{\epsilon} \int_r^R \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 \, d\hat{r},
\end{aligned}
$$

we can apply Fubini's theorem to bound the second term:

$$
\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1} \left(\int_r^R \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right| \, d\hat{r} \right)^2 dr\, d\theta \\
&\le \frac{R^\epsilon}{\epsilon} \int_0^\omega \int_0^R \int_r^R r^{2\beta-1}\, \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 d\hat{r}\, dr\, d\theta \\
&= \frac{R^\epsilon}{\epsilon} \int_0^\omega \int_0^R \int_0^{\hat{r}} r^{2\beta-1}\, \hat{r}^{1-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 dr\, d\hat{r}\, d\theta \\
&= \frac{R^\epsilon}{2\epsilon\beta} \int_0^\omega \int_0^R \hat{r}^{2\beta-\epsilon} \left| \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}} \right|^2 \hat{r}\, d\hat{r}\, d\theta \\
&= \frac{R^\epsilon}{2\epsilon\beta} \|\nabla q\|^2_{0,\beta-\frac{\epsilon}{2}} \\
&\le \frac{R^\epsilon}{2\epsilon\beta} \|\nabla q\|^2_{0,\beta-\epsilon},
\end{aligned}
$$

The third term can be bounded similarly:

$$\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-1} \left(\int_0^R \int_0^\omega r_0^{\gamma+1} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right| \, d\hat{\theta} \, dr_0 \right)^2 dr \, d\theta \\
\le\; & \left(\int_0^R \int_0^\omega r^{2\beta-1} \, dr \, d\theta \right) R\omega \int_0^\omega \int_0^R r_0^{2\gamma+2} \left| \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right|^2 dr_0 \, d\hat{\theta} \\
=\; & \frac{R^{2\beta+1}\omega^2}{2\beta} \int_0^\omega \int_0^R r_0^{2\gamma+3} \left| \frac{1}{r_0} \frac{\partial q(r_0,\hat{\theta})}{\partial \hat{\theta}} \right|^2 r_0 \, dr_0 \, d\hat{\theta} \\
=\; & \frac{R^{2\beta+1}\omega^2}{2\beta} \|\nabla q\|^2_{0,\gamma+\frac{3}{2}}.
\end{aligned}$$

Putting the three terms together and substituting $\gamma = \beta - 3/2 - \epsilon$ we may now write the bound

$$\begin{aligned}
&\int_0^\omega \int_0^R r^{2\beta-2} \left| q(r,\theta) \right|^2 r \, dr \, d\theta \\
&\quad \le \left(\frac{3R^{-2\epsilon}}{2\beta} + \frac{3R^{\epsilon}}{2\epsilon\beta} + \frac{6(\beta + \frac{1}{2} - \epsilon)^2 R^{-2\epsilon}\omega^2}{\beta} \right) \|\nabla q\|^2_{0,\beta-\epsilon},
\end{aligned}$$

and the lemma follows by taking the square root of both sides. ■

In what follows, let χ be a smooth function of r where $\chi = 1$ for $r < \eta$ and $\chi = 0$ for $r > 2\eta$. We take η to be sufficiently small to ensure that $supp(\chi) \subset \Omega$.

Lemma 4.2 *Take $\Omega = \Omega_w$ and let q be a scalar function in $H^1_\beta(\Omega)$, where $\beta > 0$. The following bound holds for χ as defined above:*

$$\|\chi q\|_{0,\beta-1} \le \frac{1}{\beta} \|\nabla(\chi q)\|_{0,\beta}. \tag{4.4}$$

Proof. Hardy's inequality for $f(t)$ defined for $t > 0$ with $\lim_{t\to 0} f(t) = 0$ gives (see [20]):

$$\int_0^\infty \frac{f^2}{t^2} \, dt \le 4 \int_0^\infty \left| f' \right|^2 \, dt. \tag{4.5}$$

The lemma follows after a change of variables, $t = r^{-2\beta}$, a substitution $f(r) = \chi q(r,\theta)$ for fixed θ, and an integration on both sides with respect to θ. ■

Lemma 4.3 *Take $\Omega = \Omega_w$ and let either $q \in H^1_\beta(\Omega)$ with $q = 0$ on $\partial\Omega$ or $q \in H^1_\beta(\Omega)/\mathbb{R}$ with $n \cdot \nabla q = 0$ on $\partial\Omega$. Then*

$$\|q\|_{0,\beta-1} \le C \|\nabla q\|_{0,\beta} \tag{4.6}$$

for $\beta > 0$ where C depends only on Ω, and β.

Proof. First, if $q = 0$ on $\partial\Omega$, write $q = q(r,\theta)$ as

$$q(r,\theta) = \int_0^\theta \frac{\partial q(r,\hat{\theta})}{\partial \hat{\theta}} \, d\hat{\theta}.$$

Square both sides and multiply by $r^{2\beta-1}$:

$$r^{2\beta-1}\left|q(r,\theta)\right|^2 = r^{2\beta-1}\left|\int_0^\theta \frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\,d\hat\theta\right|^2$$
$$\le r^{2\beta+1}\omega\int_0^\omega \left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2\,d\hat\theta.$$

Integrate both sides with respect to r and θ over Ω:

$$\int_0^\omega\int_0^R r^{2\beta-2}\left|q(r,\theta)\right|^2\; r\,dr\,d\theta \le \omega\int_0^\omega\int_0^{R_0} r^{2\beta+1}\int_0^\omega\left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2\,d\hat\theta\,dr\,d\theta$$
$$\le \omega^2\int_0^\omega\int_0^{R_0} r^{2\beta}\left|\frac{1}{r}\frac{\partial q(r,\hat\theta)}{\partial\hat\theta}\right|^2\; r\,dr\,d\hat\theta.$$

The lemma follows since the right side is bounded by $C\|\nabla q\|_{0,\beta}^2$.

Now, if $q \in H^1_\beta(\Omega)/\mathbb{R}$ then it may be chosen to satisfy

$$\iint_\Omega r^\gamma\, q\, r\,dr\,d\theta = 0 \tag{4.7}$$

for γ chosen as in lemma 4.1. By the triangle inequality and lemma 4.2 we get

$$\|q\|_{0,\beta-1} \le \|\chi\, q\|_{0,\beta-1} + \|(1-\chi)q\|_{0,\beta-1}$$
$$\le C\|\nabla(\chi q)\|_{0,\beta} + \left(\int_0^\omega\int_\eta^R \left(\frac{r}{\eta}\right)^2 r^{2\beta-2}(1-\chi)^2\, q^2\, r\,dr\,d\theta\right)^{\frac{1}{2}}$$
$$\le C\left(\|\nabla(q)\|_{0,\beta} + \|q\|_{0,\beta}\right).$$

Apply lemma 4.1 with $\epsilon = 1$ to the $\|q\|_{0,\beta}$ term on the right side and the lemma follows. ■

For the problem with mixed boundary conditions, consider Ω_w partitioned into subdomains $\Omega_0 = \{(r,\theta) : r \le \frac{1}{2}R_0, 0 \le \theta \le \omega\}$ and $\Omega_1 = \Omega\backslash\Omega_0$, as shown in figure 4.1.

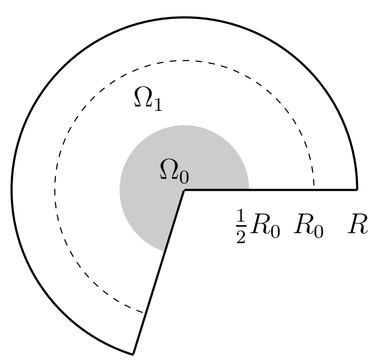


FIG. 4.1. *Wedge-shaped domain, Ω_w, partitioned into subdomains Ω_0 and Ω_1.*

Lemma 4.4 *Consider domain $\Omega = \Omega_w$, as pictured in figure 4.1, and let $q \in H^1_\beta(\Omega)$ vanish on the line segment of $\partial\Omega$ corresponding to $\theta = 0$ and $r < R_0$. Then, there is a constant, C, dependent only on Ω, β, and R_0, such that*

$$\|q\|_{0,\beta-1} \leq C\|\nabla q\|_{0,\beta} \tag{4.8}$$

for $\beta > 0$.

Proof. For points (r, θ) in Ω_0 we may derive the bound,

$$\|q\|_{0,\beta-1,\Omega_0} \leq C\|\nabla q\|_{0,\beta,\Omega_0}, \tag{4.9}$$

completely analogous to the proof of lemma 4.3. Now, consider points (r, θ) in Ω_1. We may write $q = q(r, \theta)$ as

$$q(r,\theta) = \int_{\tilde{r}}^{r} \frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\, d\hat{r} + \int_0^{\theta} \frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\, d\hat{\theta},$$

where the point $(\tilde{r}, 0)$ is on the part of $\partial\Omega_1$ where q vanishes. By the Schwarz inequality, the triangle inequality and inequality (2.1) we have the bound

$$|q(r,\theta)|^2 \leq 2(R - \frac{1}{2}R_0) \int_{\tilde{r}}^{r} \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r} + 2\omega \int_0^{\theta} \left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\right|^2 d\hat{\theta}. \tag{4.10}$$

We now expand the limits in the integrals, multiply each side by $r^{2\beta-1}$, integrate with respect to r over $(\frac{1}{2}R_0, R)$, integrate with respect to θ over $(0, \omega)$, and integrate with respect to $\tilde{r}$ over $(\frac{1}{2}R_0, R_0)$, by Fubini's theorem:

$$\begin{aligned}
&(\frac{1}{2}R_0) \int_0^{\omega} \int_{\frac{1}{2}R_0}^{R} r^{2\beta-1} |q(r,\theta)|^2 \, dr\, d\theta \\
&\quad \leq 2(R - \frac{1}{2}R_0) \int_{\frac{1}{2}R_0}^{R_0}\int_0^{\omega}\int_{\frac{1}{2}R_0}^{R} \int_{\frac{1}{2}R_0}^{R} r^{2\beta-1} \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r}\, dr\, d\theta\, d\tilde{r} \\
&\qquad + 2\omega \int_{\frac{1}{2}R_0}^{R_0}\int_0^{\omega}\int_{\frac{1}{2}R_0}^{R} \int_0^{\theta} \left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial \hat{\theta}}\right|^2 d\hat{\theta}\, dr\, d\theta\, d\tilde{r}.
\end{aligned} \tag{4.11}$$

We use the inequalities,

$$\frac{1}{2}R_0 \leq \tilde{r} \leq R_0, \quad \tilde{r} \leq \hat{r} \leq r \leq R,$$

to derive the following simple bounds:

$$1 \leq \left(\frac{2\hat{r}}{R_0}\right), \quad r \leq \left(\frac{2R}{R_0}\right)\hat{r}, \quad r \leq \left(\frac{2R}{R_0}\right)\tilde{r}. \tag{4.12}$$

By applying the bounds in (4.12) and Fubini's theorem, we may now write (4.11) as

$$
\begin{aligned}
&(\frac{1}{2}R_0)\int_0^\omega \int_{\frac{1}{2}R_0}^R r^{2\beta-2}|q(r,\theta)|^2 r\ dr\, d\theta \\
&\quad\leq (R-\frac{1}{2}R_0)^2 R_0 \int_0^\omega \int_{\frac{1}{2}R_0}^R \left(\frac{2\hat{r}}{R_0}\right)^2 \left(\frac{2R}{R_0}\right)^{2\beta-1} \hat{r}^{2\beta-1} \left|\frac{\partial q(\hat{r},\theta)}{\partial \hat{r}}\right|^2 d\hat{r}\, d\theta \\
&\qquad + 2\omega^2 (R-\frac{1}{2}R_0)\int_{\frac{1}{2}R_0}^{R_0}\int_0^\omega \left(\frac{2R}{R_0}\right)^{2\beta-1} \tilde{r}^{2\beta-1}\left|\frac{\partial q(\tilde{r},\hat{\theta})}{\partial\hat{\theta}}\right|^2 d\hat{\theta}\, d\tilde{r} \\
&\quad\leq 2^{2\beta+1}(R-\frac{1}{2}R_0)^2 R_0^{-2\beta}R^{2\beta-1}\int_0^\omega\int_{\frac{1}{2}R_0}^R \hat{r}^{2\beta}\left|\frac{\partial q(\hat{r},\theta)}{\partial\hat{r}}\right|^2 \hat{r}\ d\hat{r}\, d\theta \\
&\qquad + 2^{2\beta}\omega^2(R-\frac{1}{2}R_0)\left(\frac{R}{R_0}\right)^{2\beta-1}\int_{\frac{1}{2}R_0}^R\int_0^\omega \tilde{r}^{2\beta}\left|\frac{1}{\tilde{r}}\frac{\partial q(\tilde{r},\hat{\theta})}{\partial\hat{\theta}}\right|^2 \tilde{r}\ d\hat{\theta}\, d\tilde{r},
\end{aligned}
$$

which directly implies

$$\|q\|_{0,\beta-1,\Omega_1} \leq C\|\nabla q\|_{0,\beta,\Omega_1}, \tag{4.13}$$

where

$$C = 2^{2\beta+1}(R-\frac{1}{2}R_0)R^{2\beta-1}R_0^{-2\beta-1}\left(2(R-\frac{1}{2}R_0)+\frac{\omega^2}{R_0}\right).$$

Combining inequalities (4.9) and (4.13) completes the lemma. ∎

We now consider a similar Poincaré inequality for the vector case. Again, consider $\Omega = \Omega_w$, where $\partial\Omega$ is partitioned into Dirichlet and Neumann boundaries, Γ_D and Γ_N respectively. The following lemma is valid for the pure Dirichlet and Neumann cases and for the mixed boundary condition cases when Γ_D includes a part of the boundary adjacent to the origin and $\omega \neq \frac{3\pi}{2}$.

Lemma 4.5 *Take $\Omega = \Omega_w$ and let $\mathbf{u} \in H^1_\beta(\Omega)^2$ satisfy $\boldsymbol{\tau}\cdot\mathbf{u} = 0$ on Γ_D and $\mathbf{n}\cdot\mathbf{u} = 0$ on Γ_N. Assume that for the mixed boundary condition case that $\omega \neq 3\pi/2$. Then there is a constant, C, dependent only on Ω, β and the length of the segments of Γ_D and Γ_N adjacent to the origin, such that*

$$\|\mathbf{u}\|_{0,\beta-1} \leq C\|\nabla\mathbf{u}\|_{0,\beta} \tag{4.14}$$

for $\beta > 0$.

Proof. First, consider the case when $\boldsymbol{\tau}\cdot\mathbf{u} = 0$ on $\partial\Omega$. Denote the part of $\partial\Omega$ aligned with $\theta = 0$ as Γ_1 and the part of $\partial\Omega$ aligned with $\theta = \omega$ as Γ_2. Thus, $u_1 = 0$ on Γ_1 and $\tau_x u_1 + \tau_y u_2 = 0$ on Γ_2. Since u_1 and $\tau_x u_1 + \tau_y u_2$ satisfy the conditions in lemma 4.4, we may use

$$\|u_1\|_{0,\beta-1} \leq C\|\nabla u_1\|_{0,\beta}$$

and

$$\|\tau_x u_1 + \tau_y u_2\|_{0,\beta-1} \leq C\|\nabla(\tau_x u_1 + \tau_y u_2)\|_{0,\beta}.$$

Further, take $\tau_y \neq 0$, since $\tau_y = 0$ corresponds to either $\omega = \pi$, for which the result holds trivially since the boundary is smooth, or $\omega = 2\pi$, which we do not consider. Now,

$$\begin{aligned}
\|\mathbf{u}\|^2_{0,\beta-1} &= \|u_1\|^2_{0,\beta-1} + \|u_2\|^2_{0,\beta-1} \\
&= \|u_1\|^2_{0,\beta-1} + \frac{1}{\tau_y^2}\|\tau_x u_1 - \tau_x u_1 + \tau_y u_2\|^2_{0,\beta-1} \\
&\leq (1 + 2\frac{\tau_x^2}{\tau_y^2})\|u_1\|^2_{0,\beta-1} + \frac{2}{\tau_y^2}\|\tau_x u_1 + \tau_y u_2\|^2_{0,\beta-1} \\
&\leq C(\|\nabla u_1\|^2_{0,\beta} + \|\nabla(\tau_x u_1 + \tau_y u_2)\|^2_{0,\beta}) \\
&\leq C\|\nabla \mathbf{u}\|^2_{0,\beta}.
\end{aligned}$$

The case when $\mathbf{n}\cdot\mathbf{u} = 0$ on $\partial\Omega$ is analogous since $u_2 = 0$ on Γ_1 and $n_x u_1 + n_y u_2 = 0$ on Γ_2. Also, when $\omega \neq \frac{\pi}{2}, \frac{3\pi}{2}$, the case for mixed boundary conditions follows similarly using the result of lemma 4.4. The case for mixed boundary conditions when $\omega = \pi/2$, follows from appealing to symmetry in the the pure Dirichlet problem for $\omega = \pi$. ■

Remark 4.6 *Lemma 4.5 can be directly extended to more generally shaped domains. The proof of the scalar Poincaré bounds in lemmas 4.1, 4.2, 4.3 and 4.4 are simplified when the domain has the shape of Ω_w with only one irregular boundary point. Since we are primarily interested in a local result, proving lemma 4.5 in the simple domain is sufficient for our purposes.*

Consider the following scalar Poisson problem in Ω_w:

$$\begin{cases} \Delta p = f, & \text{in} \quad \Omega, \\ p = 0, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot\nabla p = 0, & \text{on} \quad \Gamma_N. \end{cases} \tag{4.15}$$

We refer to system (4.15) as the pure Dirichlet problem when $\partial\Omega = \Gamma_D$; the pure Neumann problem when $\partial\Omega = \Gamma_N$; and the mixed boundary condition problem when Γ_D includes the part of $\partial\Omega$ coinciding with one of either $\theta = 0$ or $\theta = \omega$, and $\Gamma_N = \partial\Omega\backslash\Gamma_D$ with $\Gamma_N \neq \emptyset$.

The following regularity results can be found in [23] and [21].

Lemma 4.7 *Assume $|1-\beta| < \pi/\omega$ for the pure Dirichlet problem, $0 < |1-\beta| < \pi/\omega$ for the pure Neumann problem and $|1-\beta| < \pi/2\omega$ for the mixed boundary condition problem. Then, for every $f \in H^0_\beta(\Omega)$, there exists a unique solution to (4.15), $p \in H^2_\beta(\Omega)$ for the pure Dirichlet and mixed boundary condition cases and $p \in H^2_\beta(\Omega)/\mathbb{R}$ for the pure Neumann problem. Moreover, there exists a constant, C, independent of p, such that*

$$\|p\|_{2,\beta} \leq C\|f\|_{0,\beta}. \tag{4.16}$$

Proof. See Chapter 1 of [23] for the Dirichlet and Neumann problems and Chapter 2 of [21] for the mixed boundary problem. ■

Define the subspace of functions in $H^1_\beta(\Omega)$ satisfying the appropriate boundary conditions by

$$\mathcal{V}_\beta = \{\mathbf{v} \in H^1_\beta(\Omega) : \ \boldsymbol{\tau}\cdot\mathbf{v} = 0 \ \text{ on } \ \Gamma_D, \ \ \mathbf{n}\cdot\mathbf{v} = 0 \ \text{ on } \ \Gamma_N\}.$$

We now prove a regularity result for functions in $\mathcal{V}_\beta$. Recall that the power of the singularity is defined as $\alpha = \pi/\omega$ for Dirichlet or Neumann boundary conditions and $\alpha = \pi/(2\omega)$ for mixed boundary conditions.

Lemma 4.8 *Consider domain $\Omega = \Omega_w$. Then there is a positive constant, C, independent of $\mathbf{u}$, such that, for $|1-\beta| < \alpha$, the following bound holds for all $\mathbf{u} \in \mathcal{V}_\beta$:*

$$\|\mathbf{u}\|_{1,\beta} \leq C\|L\mathbf{u}\|_{0,\beta}. \tag{4.17}$$

Proof. From lemma 4.7 we know that any $\mathbf{u} \in \mathcal{V}_\beta$ has the decomposition

$$\mathbf{u} = \nabla\phi + \nabla^\perp\psi, \tag{4.18}$$

where $\phi, \psi \in H^2_\beta(\Omega)$ satisfy

$$\begin{cases} \Delta\phi = \nabla\cdot\mathbf{u}, & \text{in} \quad \Omega, \\ \quad \phi = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{4.19}$$

and

$$\begin{cases} \quad \Delta\psi = \nabla\times\mathbf{u}, & \text{in} \quad \Omega, \\ \mathbf{n}\cdot\nabla\psi = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.20}$$

Then, by applying lemma 4.7 to problems (4.19) and (4.20) we have

$$\begin{aligned} \|\mathbf{u}\|_{1,\beta} &= \|\nabla\phi + \nabla^\perp\psi\|_{1,\beta} \leq \|\nabla\phi\|_{1,\beta} + \|\nabla^\perp\psi\|_{1,\beta} \\ &\leq \|\phi\|_{2,\beta} + \|\psi\|_{2,\beta} \leq C(\|\nabla\cdot\mathbf{u}\|_{0,\beta} + \|\nabla\times\mathbf{u}\|_{0,\beta}) \leq C\|L\mathbf{u}\|_{0,\beta}, \end{aligned}$$

which completes the proof. ■

5. Error Bounds. Let $\mathcal{T}^h = \cup_{i=1}^N \tau_i$ be a quasi-uniform triangulation of polygonal domain Ω. Let $\mathcal{I}^h$ represent standard interpolation onto a piecewise polynomial finite element space of degree k. From finite element theory, we have the following interpolation bounds.

Lemma 5.1 *Let Ω be a polygonal domain. There exists a constant, C, independent of v, such that, for all $v \in H^m(\Omega)$,*

$$\left(\sum_{\tau\in\mathcal{T}^h} \|v - \mathcal{I}^h v\|^2_{s,\tau}\right)^{1/2} \leq Ch^{m-s}|v|_m \tag{5.1}$$

for $0 \leq s \leq m$ and $1 < m$. Here, $\mathcal{I}^h$ denotes interpolation by a piecewise polynomial of degree $k = m-1$. (Note: here the norm $\|\cdot\|_{s,\tau}$ is the standard $H^s(\tau)$ norm.)

Proof. See [9] or [7]. ■

We now consider a weighted interpolation bound for functions on domains with a polygonal corner at the origin. Define the modified interpolation operator, $\mathcal{I}^h_0$, by

$$\mathcal{I}^h_0 u|_\tau \;=\; \begin{cases} \mathcal{I}^h u = \sum_{i=0}^n u(a_i)\phi_i \;, & \text{if } \overline{\tau} \text{ does not intersect the origin,} \\ \sum_{i=1}^n u(a_i)\phi_i \;, & \text{if } \overline{\tau} \text{ intersects the origin,} \end{cases}$$

where $\mathcal{I}^h$ is a standard polynomial interpolation operator, ϕ_i are basis functions corresponding to the $n+1$ nodal points, a_i, and a_0 is the origin, $(0,0)$. Thus, the modified interpolation has a value of zero at the origin and resembles $\mathcal{I}^h$ away from the origin.

Lemma 5.2 *Let Ω be a polygonal domain. There exists a constant, C, independent of u, such that, for all $u \in H^m_\beta(\Omega)$ satisfying equation (4.8),*

$$\left(\sum_{\tau \in \mathcal{T}^h} \|u - \mathcal{I}^h_0 u\|^2_{1,\beta,\tau} \right)^{1/2} \leq C h^{m-1} \|u\|_{m,\beta}, \tag{5.2}$$

for $1 < m$ and $\beta > 0$, where $\mathcal{I}^h_0$ is the modified interpolation operator onto piecewise polynomials of degree $k = m - 1$ defined above.

Proof. Define $K_0 = \{\tau \mid \bar{\tau} \cap (0,0) \neq \emptyset\}$ as the set of elements adjacent to the origin. On $\mathcal{T}^h \backslash K_0$, we have $h \leq r_{min} \leq r = \sqrt{x^2+y^2} \leq r_{max} \leq r_{min} + \sqrt{2}h$ with $r_{min} = \inf\{r|(x,y) \in \tau\}$ and $r_{max} = \sup\{r|(x,y) \in \tau\}$ in τ, and

$$\begin{aligned}
\|u - \mathcal{I}^h_0 u\|^2_{1,\beta,\tau} &= \|u - \mathcal{I}^h u\|^2_{1,\beta,\tau} \\
&= \int_\tau r^{2\beta} |\nabla(u - \mathcal{I}^h u)|^2 + r^{2(\beta-1)} |u - \mathcal{I}^h u|^2 d\tau \\
&\leq r^{2\beta}_{max} \int_\tau |\nabla(u - \mathcal{I}^h u)|^2 d\tau + r^{2\beta}_{max} r^{-2}_{min} \int_\tau |u - \mathcal{I}^h u|^2 d\tau \\
&\leq C r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,0,\tau} + C r^{2\beta}_{max} r^{-2}_{min} h^{2m} |u|^2_{m,0,\tau} \\
&= C r^{2\beta}_{max} h^{2(m-1)} (1 + r^{-2}_{min} h^2) |u|^2_{m,0,\tau} \leq C r^{2\beta}_{max} h^{2(m-1)} |u|^2_{m,0,\tau} \\
&\leq C h^{2(m-1)} r^{2\beta}_{max} r^{-2\beta}_{min} \int_\tau r^{2\beta} |D^m u|^2 d\tau \\
&\leq C h^{2(m-1)} \left(\frac{r_{min} + \sqrt{2}h}{r_{min}} \right)^{2\beta} \int_\tau r^{2\beta} |D^m u|^2 d\tau \\
&\leq C h^{2(m-1)} \int_\tau r^{2\beta} |D^m u|^2 d\tau.
\end{aligned}$$

We now consider the case for which $\tau \in K_0$. Let $\delta \in \mathcal{C}^\infty$ be a cut-off function defined by

$$\delta(r) = \begin{cases} 1, & \text{if} \quad r \leq h/3, \\ 0, & \text{if} \quad r > 2h/3, \end{cases}$$

with $|\delta^{(m)}| \leq ch^{-m}$, where $\delta^{(m)}$ is the m^{th} derivative of δ. By the triangle inequality,

$$\|u - \mathcal{I}^h_0 u\|_{1,\beta,\tau} \leq \|\delta u - \mathcal{I}^h_0(\delta u)\|_{1,\beta,\tau} + \|(1-\delta)u - \mathcal{I}^h_0((1-\delta)u)\|_{1,\beta,\tau}. \tag{5.3}$$

By the definition of δ we have $\mathcal{I}^h_0((1-\delta)u) = \mathcal{I}^h((1-\delta)u)$ and $\mathcal{I}^h_0(\delta u) = 0$. For the second term in (5.3), we apply lemmas 4.4 and 5.1, the properties in δ, and Fubini's theorem to obtain

$$
\begin{aligned}
&\|(1-\delta)u - \mathcal{I}_0^h((1-\delta)u)\|_{1,\beta,\tau}^2 = \|(1-\delta)u - \mathcal{I}^h((1-\delta)u)\|_{1,\beta,\tau}^2 \\
&\le C\|\nabla((1-\delta)u - \mathcal{I}^h((1-\delta)u))\|_{0,\beta,\tau}^2 \le Ch^{2\beta}\int_\tau |\nabla((1-\delta)u - \mathcal{I}^h((1-\delta)u))|^2 d\tau \\
&\le Ch^{2(\beta+m-1)}\int_\tau |D^m((1-\delta)u)|^2 d\tau \le Ch^{2(\beta+m-1)}\int_\tau \sum_{|j|\le m} |D^{m-j}(1-\delta)D^j u|^2 d\tau \\
&\le Ch^{2(\beta+m-1)}\left(\iint_{\frac{h}{3}}^{\frac{2h}{3}} \sum_{|j|\le m-1} |h^{|j|-m}D^j u|^2 d\tau + \iint_{\frac{h}{3}}^{r(\theta)} |(1-\delta)D^m u|^2 d\tau\right) \\
&\le Ch^{2(\beta+m-1)}\left(\sum_{|j|\le m-1} \iint_{\frac{h}{3}}^{\frac{2h}{3}} h^{-2\beta} r^{2(\beta+|j|-m)}|D^j u|^2 d\tau + \iint_{\frac{h}{3}}^{r(\theta)} (r/h)^{2\beta}|D^m u|^2 d\tau\right) \\
&= Ch^{2(m-1)}\sum_{|j|\le m}\int_\tau r^{2(\beta+|j|-m)}|D^j u|^2 d\tau = Ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.
\end{aligned}
$$

Using the properties of δ and Fubini's theorem result in a similar bound for the first term in (5.3):

$$
\begin{aligned}
&\|\delta u - \mathcal{I}_0^h(\delta u)\|_{1,\beta,\tau}^2 = \|\delta u\|_{1,\beta,\tau}^2 = \int_\tau r^{2\beta}|\nabla(\delta u)|^2 + r^{2(\beta-1)}|\delta u|^2 d\tau \\
&\le C\int_\tau r^{2\beta}(|\nabla\delta\cdot u|^2 + |\delta\nabla u|^2) + r^{2(\beta-1)}|\delta u|^2 d\tau \\
&\le C\iint_{\frac{h}{3}}^{\frac{2h}{3}} r^{2\beta}h^{-2}|u|^2 d\tau + C\iint_0^{\frac{2h}{3}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau \\
&\le C\iint_{\frac{h}{3}}^{\frac{2h}{3}} r^{2(\beta-1)}|u|^2 d\tau + C\iint_0^{\frac{2h}{3}} r^{2\beta}|\nabla u|^2 + r^{2(\beta-1)}|u|^2 d\tau \\
&\le C\int_\tau r^{2(m-1)}(r^{2(\beta-m+1)}|\nabla u|^2 + r^{2(\beta-m)}|u|^2) d\tau \le Ch^{2(m-1)}\|u\|_{m,\beta,\tau}^2.
\end{aligned}
$$

Thus we have

$$
\sum_{\tau\in\mathcal{T}_h} \|u - \mathcal{I}_0^h u\|_{1,\beta,\tau}^2 \le Ch^{2(m-1)}\sum_{\tau\in\mathcal{T}_h}\|u\|_{m,\beta,\tau}^2 \le Ch^{2(m-1)}\|u\|_{m,\beta}^2,
$$

and the lemma follows. ■

Lemma 5.3 *Assume equation* (4.14) *holds in* Ω. *Then, for all* $\mathbf{u}^h \in \mathcal{V}^h$,

$$
\|\mathbf{u}^h\|_{0,\beta} \le Ch^{-\eta}\|\mathbf{u}^h\|_{0,\beta+\eta} \tag{5.4}
$$

for $\beta > -1$ *and* $\eta > 0$.

Proof. Using lemma 4.5 and an inverse inequality, we may write

$$
\|\mathbf{u}^h\|_{0,\beta} \le C\|\nabla\mathbf{u}^h\|_{0,\beta+1} \le Ch^{-1}\|\mathbf{u}^h\|_{0,\beta+1},
$$

which establishes (5.4) for $\eta = 1$. Repeated application of this inequality thus validates (5.4) for any positive integer. Now consider

$$\begin{aligned}\|\mathbf{u}^h\|_{0,\beta}^2 &= \langle r^\beta \mathbf{u}^h, r^\beta \mathbf{u}^h\rangle = \langle r^{\beta-1/2}\mathbf{u}^h, r^{\beta+1/2}\mathbf{u}^h\rangle \\ &\le \|\mathbf{u}^h\|_{0,\beta-1/2}\|\mathbf{u}^h\|_{0,\beta+1/2} \le Ch^{-1}\|\mathbf{u}^h\|_{0,\beta+1/2}^2.\end{aligned}$$

Taking the square root establishes (5.4) for $\eta = 1/2$. Repeating these steps leads to (5.4) for all $\eta = \eta_n = k_n/2^{\ell_n}$ for any nonnegative integers k_n and ℓ_n. For any $\eta > 0$, choose a monotonically decreasing sequence, $\{\eta_n\}$, such that $\eta_n > \eta$ and $lim_{n\to\infty}|\eta_n - \eta| = 0$. Now, $g_n = (r^{\beta+\eta_n}\mathbf{u}^h)^2$ is a monotonically increasing function that converges to $g = (r^{\beta+\eta}\mathbf{u}^h)^2$ pointwise everywhere. Thus, by the Lebesgue monotone convergence theorem, we have

$$\|\mathbf{u}^h\|_{0,\beta+\eta}^2 = \int g dx = \lim_n \int g_n dx = \lim_n \|\mathbf{u}^h\|_{0,\beta+\eta_n}^2$$

and, therefore,

$$\begin{aligned}\|\mathbf{u}^h\|_{0,\beta} &= \lim_n \|\mathbf{u}^h\|_{0,\beta} \\ &\le \lim_n Ch^{-\eta_n}\|\mathbf{u}^h\|_{0,\beta+\eta_n} \\ &= (\lim_n Ch^{-\eta_n})(\lim_n \|\mathbf{u}^h\|_{0,\beta+\eta_n}) \\ &= Ch^{-\eta}\|\mathbf{u}^h\|_{0,\beta+\eta},\end{aligned}$$

which completes the proof. ∎

Define an *irregular boundary point* of polygonal domain Ω to be a point on $\partial\Omega$ where interior angle ω satisfies $\omega > \pi$ when Dirichlet or Neumann boundary conditions are applied on both sides of the point or $\omega > \pi/2$ when one Dirichlet boundary and a Neumann boundary meet at the corner. We now present error bounds for the numerical solution in weighted and unweighted norms.

Theorem 5.4 *Let Ω be a polygonal domain with one irregular boundary point of interior angle ω and let $\mathbf{f} \in L^2(\Omega)$. Suppose $\mathbf{u} \in \mathcal{V}$ satisfy $L\mathbf{u} = \mathbf{f}$. If $\mathbf{u}^h \in \mathcal{V}^h$ is chosen to minimize the weighted functional,*

$$G_w(\mathbf{u}^h; \mathbf{f}) = \|L\mathbf{u}^h - \mathbf{f}\|_{0,\beta}^2 = \inf_{\mathbf{v}^h\in\mathcal{V}^h} \|L\mathbf{v}^h - \mathbf{f}\|_{0,\beta}^2,$$

for $|1-\beta| < \alpha$, then the approximation error, $\mathbf{u}-\mathbf{u}^h$, satisfies the following bounds:

$$\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{5.5}$$

$$G_w(\mathbf{u}-\mathbf{u}^h; \mathbf{0})^{1/2} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{5.6}$$

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta} \le Ch^{s+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{5.7}$$

$$\|\mathbf{u}-\mathbf{u}^h\|_{0} \le Ch^{s}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \tag{5.8}$$

where $s < \alpha$, for $\alpha+\beta \le k+1$ with k the degree of the piecewise polynomial elements in $\mathcal{V}^h$.

Proof. By lemmas 4.8 and 5.2, we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} &\le C\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} \le C\|L(\mathbf{u}-\mathcal{I}_0^h\mathbf{u})\|_{0,\beta} \\ &\le C\|\mathbf{u}-\mathcal{I}_0^h\mathbf{u}\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta},\end{aligned}$$

which establishes both (5.5) and (5.6) since we may write

$$\|L(\mathbf{u}-\mathbf{u}^h)\|_{0,\beta} = G_w(\mathbf{u}-\mathbf{u}^h;\mathbf{0})^{1/2}.$$

Note that lemmas 4.8 and 5.2 are satisfied for $|1-\beta|<\alpha$ and $\alpha+\beta\le 2$.

For the weighted L^2-norm, we write

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta}^2 = \sum_{\tau\in\mathcal{T}^h}\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta,\tau}^2.$$

The Cauchy inequality yields, for any $\epsilon\in(0,1)$,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta,\tau}^2 &= \int_\tau r^{2\beta}|\mathbf{u}-\mathbf{u}^h|^2 d\tau \le \left(\int_\tau 1 d\tau\right)^{1-\epsilon}\left(\int_\tau \left(r^{2\beta}|\mathbf{u}-\mathbf{u}^h|^2\right)^{\frac{1}{\epsilon}} d\tau\right)^{\epsilon} \\ &\le Ch^{2-2\epsilon}\left(\int_\tau \left(r^{\beta}|\mathbf{u}-\mathbf{u}^h|\right)^{\frac{2}{\epsilon}} d\tau\right)^{\frac{\epsilon}{2}\cdot 2} = Ch^{2-2\epsilon}\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|_{L^{\frac{2}{\epsilon}}(\tau)}^2.\end{aligned}$$

Since $H^1(\Omega)$ is continuously imbedded into $L^q(\Omega)$ for all $q\in[1,\infty)$,

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta}^2 &= \sum_{\tau\in\mathcal{T}^h}\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta,\tau}^2 \le Ch^{2-2\epsilon}\sum_{\tau\in\mathcal{T}^h}\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|_{L^{\frac{2}{\epsilon}}(\tau)}^2 \\ &\le Ch^{2-2\epsilon}\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|_{L^{\frac{2}{\epsilon}}(\Omega)}^2 \le Ch^{2-2\epsilon}\|r^\beta(\mathbf{u}-\mathbf{u}^h)\|_{H^1(\Omega)}^2 \\ &\le Ch^{2-2\epsilon}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta}^2.\end{aligned}$$

Thus, by (5.5), we have

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta} \le Ch^{1-\epsilon}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta} \le Ch^{s+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta},$$

where any $s<\alpha$.

We now consider the bound on $\|\mathbf{u}-\mathbf{u}^h\|_0$. Let $K_0=\{\tau \mid \bar\tau\cap(0,0)\ne\emptyset\}$ and $K_1=\mathcal{T}^h\backslash K_0$. First, we consider the case $\beta<1$. If $\tau\in K_0$, then $r\le Ch$ and $r^{1-\beta}\le Ch^{1-\beta}$. Thus, we have

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\tau}^2 \le Ch^{2(1-\beta)}\|\mathbf{u}-\mathbf{u}^h\|_{0,\beta-1,\tau}^2 \le Ch^{2(1-\beta)}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta,\tau}^2. \tag{5.9}$$

If $\tau\in K_1$, we use the technique above to get

$$\|\mathbf{u}-\mathbf{u}^h\|_{0,\tau}^2 \le Ch^{2(1-\epsilon)}\|\mathbf{u}-\mathbf{u}^h\|_{L^{\frac{2}{\epsilon}}(\tau)}^2.$$

Again, since H^1 is continuously imbedded into L^q for all $q\in[1,\infty)$, we have

$$\begin{aligned}&\sum_{\tau\in K_1}\|\mathbf{u}-\mathbf{u}^h\|_{0,\tau}^2 \le Ch^{2-2\epsilon}\sum_{\tau\in K_1}\|\mathbf{u}-\mathbf{u}^h\|_{L^{\frac{2}{\epsilon}}(\tau)}^2 \\ &\le Ch^{2-2\epsilon}\|\mathbf{u}-\mathbf{u}^h\|_{L^{\frac{2}{\epsilon}}(K_1)}^2 \le Ch^{2-2\epsilon}\|\mathbf{u}-\mathbf{u}^h\|_{H^1(K_1)}^2 \\ &= Ch^{2-2\epsilon}\int_{K_1} r^{-2\beta}r^{2\beta}\left(|\mathbf{u}-\mathbf{u}^h|^2+|\nabla(\mathbf{u}-\mathbf{u}^h)|^2\right)d\Omega \\ &\le Ch^{2(1-\beta-\epsilon)}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta,K_1}^2.\end{aligned} \tag{5.10}$$

Hence by (5.9), (5.10), and (5.5) we have

$$\|\mathbf{u}-\mathbf{u}^h\|_0^2 = \sum_{\tau\in K_0} \|\mathbf{u}-\mathbf{u}^h\|_{0,\tau}^2 + \sum_{\tau\in K_1} \|\mathbf{u}-\mathbf{u}^h\|_{0,\tau}^2 \le h^{2(1-\beta-\epsilon)}\|\mathbf{u}-\mathbf{u}^h\|_{1,\beta}^2$$
$$\le Ch^{2s}\ \|\mathbf{u}\|_{\alpha+\beta,\beta}^2,$$

where $s<\alpha$. The proof for $\beta \ge 1$ follows analogously. ■

Remark 5.5 *For the optimal finite element convergence of $O(h)$ with respect to the weighted functional and H^1 norms, we select $\beta = 2-\alpha$. But theorem 5.4 also requires that $\beta < 1+\alpha$. Thus, when $\alpha \in [1/2,1)$, we may use a weighting with $\beta = 2-\alpha$ and expect optimal rates, but when $\alpha \in (0,1/2)$, our theory only guarantees at best $O(2\alpha)$ convergence using $\beta = 1+\alpha$. Numerical results, however, indicate that values of β larger than the theory allows can be used to recover optimal rates. We explore this in the next section.*

6. Computational results. In this section, we present some numerical examples of the weighted-norm procedure to validate the error bounds in the previous section.

As a test problem, we minimize the weighted functional on the following L-shaped domain: $\Omega = (-0.5,0.5)^2 \setminus [0,0.5)\times(-0.5,0]$, which yields $\alpha = \pi/\omega = 2/3$. Function $\mathbf{f}$ is chosen so that the solution of this test problem is $\mathbf{u} = \nabla(\chi(r)r^{\frac{2}{3}}\sin(2\theta/3))$, where $\chi(r)=1$ for $r<1/8$, $\chi(r)=0$ for $r>3/8$, and $\chi(r)$ is C^2 smooth. Again, note that $\mathbf{f}\in L^2(\Omega)$ but $\mathbf{u}\notin H^1(\Omega)$.

Define the following measures of the accuracy of the computed solution, $\mathbf{u}^h$:

nonweighted functional norm	$G^{1/2} = (\|\nabla\cdot\mathbf{u}^h - f\|_0^2 + \|\nabla\times\mathbf{u}^h\|_0^2)^{1/2}$,
nonweighted L^2 norm of the error	$\epsilon^0 = \|\mathbf{u}-\mathbf{u}^h\|_0$,
nonweighted H^1 seminorm of the error	$\epsilon^1 = \|\mathbf{u}-\mathbf{u}^h\|_1$,
weighted functional norm	$G_w^{1/2} = G_w(\mathbf{u}^h;f)^{1/2}$,
weighted L^2 norm of the error	$\epsilon_w^0 = \|\mathbf{u}-\mathbf{u}^h\|_{0,\beta}$,
weighted H^1 seminorm of the error	$\epsilon_w^1 = \|\mathbf{u}-\mathbf{u}^h\|_{1,\beta}$.

Since $\alpha = 2/3$, we choose the optimal weight parameter, $\beta = 2-\alpha = 4/3$, for our computations. Table 6.1 summarizes discretization error and convergence rates for $\beta = 4/3$.

Asymptotic convergence rates in Ω are found to be approximately $O(h)$ for $G_w^{1/2}$ and ϵ_w^1, $O(h^2)$ for ϵ_w^0 and $O(h^{\frac{2}{3}})$ for ϵ_0. The approximation does not converge in either the ϵ_1 or $G^{1/2}$ measures since $\mathbf{u}\notin H^1(\Omega)$.

To distinguish between behavior near to and away from the singularity, we consider the error of the solution above on a partitioning of Ω. Define $\Omega_0 = \Omega\cap(\frac{3}{8},\frac{5}{8})^2$ and $\Omega_1 = \Omega\backslash\Omega_0$; see figure 6.1.

Table 6.2 summarizes the asymptotic discretization accuracy obtained at the finest mesh size in subdomains Ω_0 and Ω_1. Away from the singularity we observe optimal accuracy in all measures. As expected, near the singularity, the solution fails to converge in the nonweighted functional and H^1 norms. The nonweighted L^2 error achieves accuracy of approximately $O(h^{\frac{2}{3}})$ near the singularity.

h	$G_w^{1/2}$	Ratio	Rate	ϵ_w^1	Ratio	Rate
8^{-1}	5.52			3.81		
16^{-1}	4.34	1.27	0.35	1.47	2.59	1.37
32^{-1}	2.34	1.85	0.89	6.66e-01	2.21	1.14
64^{-1}	1.19	1.97	0.98	2.97e-01	2.24	1.16
128^{-1}	5.98e-01	1.99	0.99	1.41e-01	2.11	1.08
256^{-1}	3.00e-01	1.99	0.99	6.74e-02	2.09	1.06
512^{-1}	1.50e-01	2.00	1.00	3.31e-02	2.04	1.03

h	ϵ_w^0	Ratio	Rate	ϵ^0	Ratio	Rate
8^{-1}	3.08e-01			3.72e-01		
16^{-1}	1.35e-01	2.28	1.19	1.93e-01	1.93	0.95
32^{-1}	4.07e-02	3.32	1.73	8.93e-02	2.16	1.11
64^{-1}	1.11e-02	3.67	1.88	4.93e-02	1.81	0.86
128^{-1}	2.98e-03	3.72	1.90	3.00e-02	1.64	0.71
256^{-1}	7.84e-04	3.80	1.93	1.87e-02	1.60	0.68
512^{-1}	2.06e-04	3.81	1.93	1.18e-02	1.58	0.66

TABLE 6.1
Convergence of discretization error for weighted-norm FOSLS.

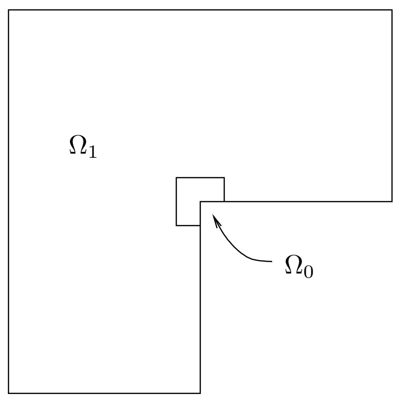
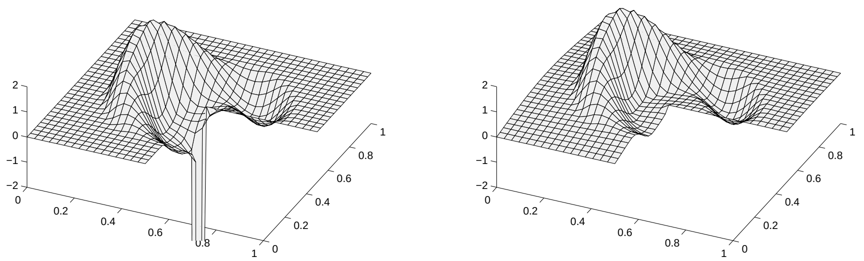


FIG. 6.1. *L-shaped domain Ω and subdomains Ω_0 and Ω_1.*

	$G_w^{1/2}$	$G^{1/2}$	ϵ_w^1	ϵ^1	ϵ_w^0	ϵ^0
Ω_1	$O(h)$	$O(h)$	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
Ω_0	$O(h)$	$O(1)$	$O(h)$	$O(1)$	$O(h^2)$	$O(h^{\frac{2}{3}})$
Ω	$O(h)$	$O(1)$	$O(h)$	$O(1)$	$O(h^2)$	$O(h^{\frac{2}{3}})$

TABLE 6.2
Accuracy in Ω_0, Ω_1, and Ω with $\beta = 2 - \alpha$.

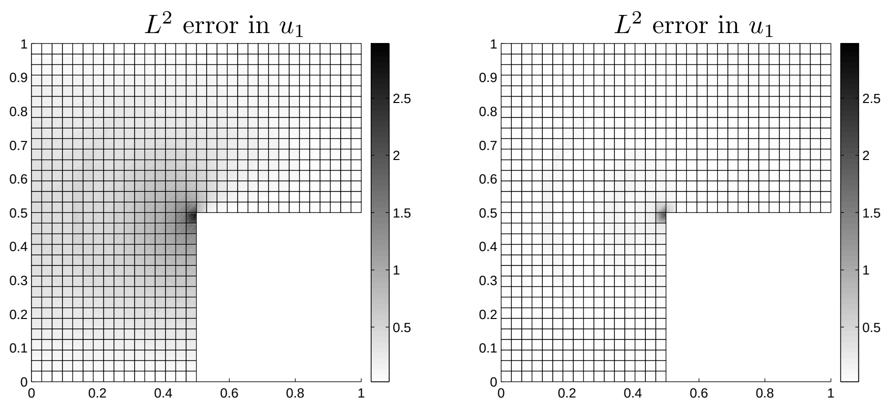
Figure 6.2 shows the first component of the exact solution, u_1, and the standard FOSLS approximation u_1^h. Figure 6.3 shows the error of the first component of the approximated solution for the standard FOSLS and the weighted-norm FOSLS methods. We see that the error in the approximation in standard FOSLS is highest near the



(a) Exact component u_1.

(b) Component u_1^h computed by standard FOSLS.

FIG. 6.2. *Exact solution component u_1 and solution component u_1^h approximated by standard FOSLS on the $h = 32^{-1}$ mesh.*



(a) Standard FOSLS.

(b) Weighted-norm FOSLS, $\beta = 2 - \alpha$.

FIG. 6.3. *Reduction of the pollution effect by the weighted-norm procedure. Each plot is the error of solution component u_1^h on the $h = 32^{-1}$ mesh.*

singularity, but remains large even away from the corner point. In the weighted-norm FOSLS implementation, the error remains large near the singularity, as we expect, but is now concentrated only near the corner point. The pollution effect is removed by the weighting procedure.

There are many boundary value problems not directly covered by the theory presented here that are of interest. For example, Poisson's equation with mixed boundary conditions on the domain used above has a value of $\alpha = 1/3$. To recover optimal convergence for this problem, the weighted-norm method requires a value of β larger than theorem 5.4 allows. In other elliptic equations (e.g., Stokes or the linear elasticity equations), the value of α is generally smaller than for Poisson's equation for the same domain and boundary condition type. In each of these cases, a larger β

value is necessary for optimal convergence. This leads us to consider using larger β than the theory allows.

Consider the same example problem as above on uniform mesh sizes of $h = 1/8, 1/16, ..., 1/512$, and values of β ranging from 1/3 to 23/6.

Figure 6.4 plots the convergence rate at the finest level for: the weighted functional norm, $G_w^{1/2}$; the weighted L^2 norm, ϵ_w^0; and the L^2 norm, ϵ^0. While the functional

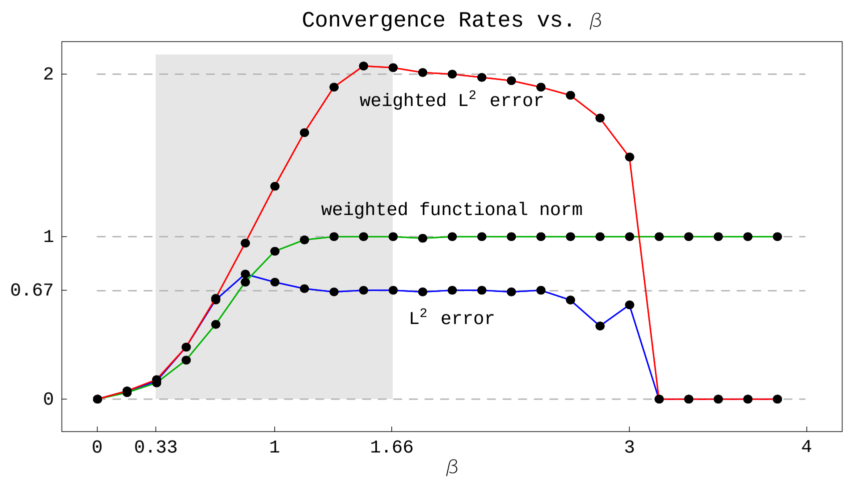


Fig. 6.4. *Convergence rates versus β. The shaded region indicates values of β for which the assumptions of theorem 5.4 are satisfied.*

norm retains optimal accuracy for large values of β, the solution fails to converge in the weighted and nonweighted L^2 measures for $\beta \gtrsim 3$. This indicates that, although the weighted-norm approach seems to be more robust than the theory allows, large values of β should still be used with caution.

The method presented here is applicable to a wide range of problems and provides an efficient alternative to more specialized techniques for treating singularities in boundary value problems. Further numerical results for other problems including systems and in three dimensions can be seen in a companion paper.

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