A Least-Squares Finite Element formulation of the Geometrically-Nonlinear Elasticity Equations

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Outline

- Introduction
  - First-Order System Least Squares (FOSLS) methodology
- Elasticity
  - Elasticity notation, definitions and FOSLS formulation
  - Theory overview
  - Computational results: Smooth solutions
  - Regularity, what (not) to expect
- FOSLS in Nonsmooth Domains
  - Weighted spaces and the weighted-norm approach
  - Theory and error bounds
  - Computational results: Nonsmooth solutions
- Concluding remarks and questions

Basic Least-Squares Methodology

Consider solving the first-order system $L\mathbf{u} = \mathbf{f}$ by minimizing

$$G(\mathbf{u}; f) = \|L\mathbf{u} - \mathbf{f}\|^2$$

over all $\mathbf{u} \in \mathcal{V}$. This leads to the weak problem: find $\mathbf{u} \in \mathcal{V}$ such that

$$(L\mathbf{u}, L\mathbf{v}) = (\mathbf{f}, L\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{V}.$$

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$\underline{\mathcal{V} \text{ ellipticitiy}}$: If $G(\mathbf{u}; 0)^{\frac{1}{2}} = \|L\mathbf{u}\|$ is equivalent to the norm on $\mathcal{V}$:

$$c_0 \|\mathbf{u}\|_{\mathcal{V}} \leq \|L\mathbf{u}\| \leq c_1 \|\mathbf{u}\|_{\mathcal{V}},$$

then the weak problem is well-posed in $\mathcal{V}$.

Discrete Problem

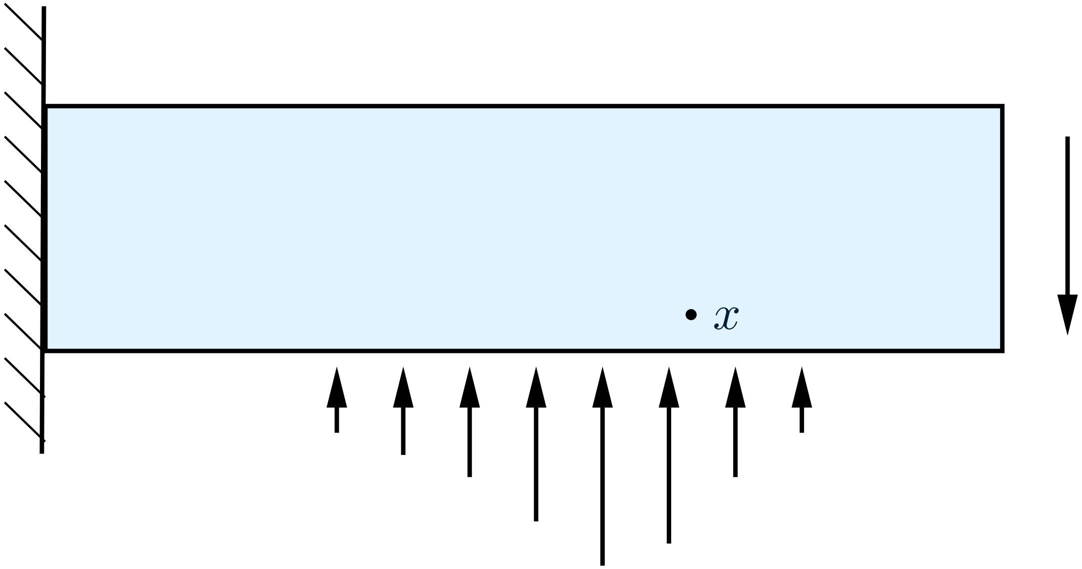
Consider the FE space $\mathcal{V}^h \subseteq \mathcal{V}$. The discrete weak problem is: find $\mathbf{u}^h \in \mathcal{V}^h$ such that

$$(L\mathbf{u}^h, L\mathbf{v}^h) = (\mathbf{f}, L\mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathcal{V}^h.$$

When $\mathcal{V}^h \subseteq H^1(\Omega)$ and $\|\cdot\|_{\mathcal{V}} = \|\cdot\|_1$, H^1 elliptic functionals result in:

- Existence and uniqueness of $\mathbf{u}^h \in \mathcal{V}^h$
- Optimal order FE discretization accuracy (given regularity of $L\mathbf{u} = \mathbf{f}$)
- $O(h^{-2})$ conditioned linear systems solved with standard multigrid methods
- Local functional values a free H^1 $a\ posterori$ error bound

Elasticity Notation and Definitions



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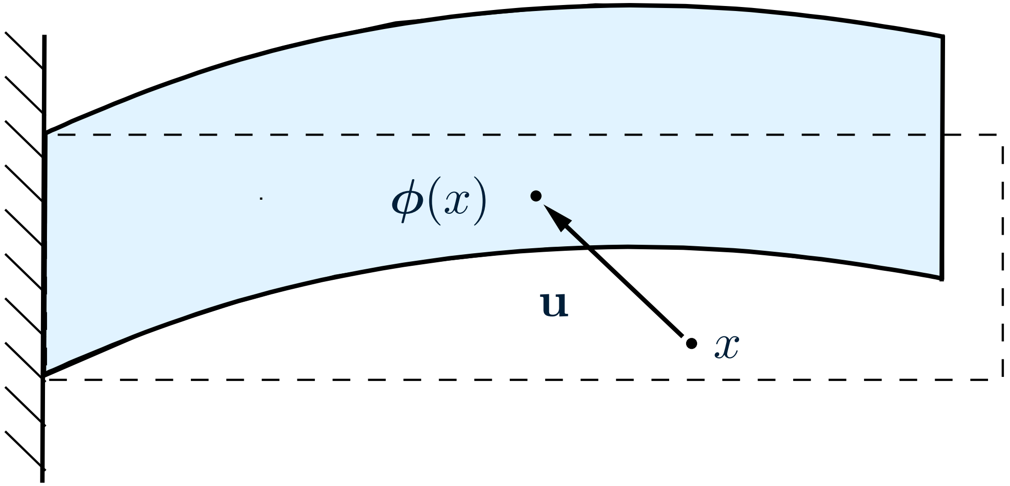

$\mathbf{u}$	Displacement
ϕ	Deformation
E	Strain Tensor
Σ	Stress Tensor

$$\phi(x) = x + \mathbf{u}(x)$$

$$\begin{aligned} E &= \tfrac{1}{2}(I - \nabla\phi^t\nabla\phi) \\ &= \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^t + \nabla\mathbf{u}^t\nabla\mathbf{u}) \end{aligned}$$

$$\Sigma = \lambda tr(E)I + 2\mu E + o(E)$$

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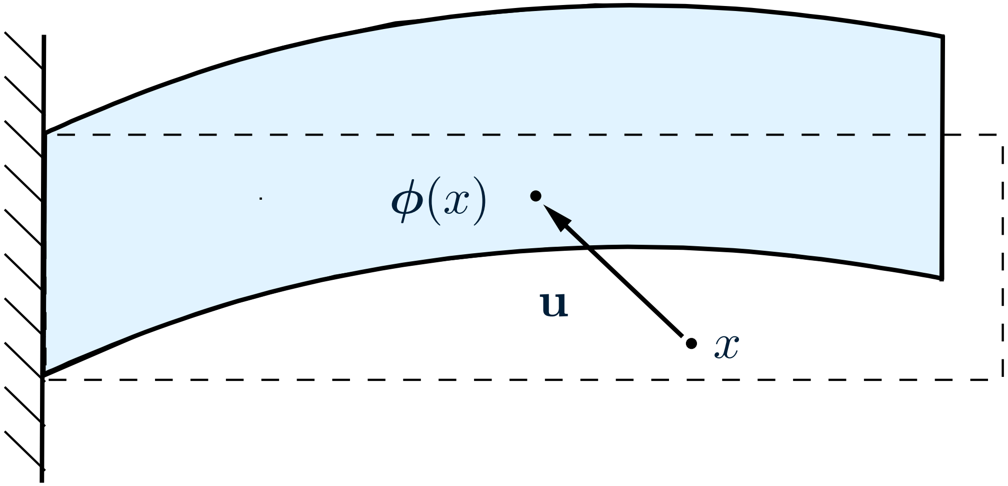

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Nonlinear Elasticity Equations

$$\begin{cases} \nabla \cdot [(I + \nabla \mathbf{u})\Sigma(\nabla \mathbf{u})] = \mathbf{f} & \text{in} \quad \Omega \\ \mathbf{n} \cdot (I + \nabla \mathbf{u})\Sigma(\nabla \mathbf{u}) = \mathbf{g} & \text{on} \quad \Gamma_T \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \Gamma_D \end{cases}$$

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The equilibrium equation may be written as

$$\underbrace{\mu \Delta \mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}}_{linear} + \underbrace{\nabla \cdot P_3(\nabla \mathbf{u})}_{nonlinear} = \mathbf{f}$$

- Accurate for deformations of “large-displacement and small-strain”
- May scale with $\mu = 1$, and let λ determine incompressibility

Newton-FOSLS

- Pose in terms of $\mathbf{U} = \nabla \mathbf{u}$ (First-Order System)
    - $\mathbf{\Phi} = \mathbf{I} + \mathbf{U}$ is the Jacobian of the mapping
    - $det(\mathbf{\Phi}) \approx 1$, $\|\mathbf{\Phi}\|_{\ell^2}^2 \approx 2$
- 2-Stage Solution Algorithm
    - Solve for $\mathbf{U}$
    - Recover $\mathbf{u}$ from: $\nabla \mathbf{u} = \mathbf{U}$
- Linearize around a current approximation by Newton's method
    - Initial guess: $\mathbf{U}_0 = 0$ ... the reference configuration
    - General Newton Step: $\nabla \cdot A_n \mathbf{U} = \mathcal{F}_n$
    - 1^{st} step is linear elasticity

Problem Modification

- Problem: System matrix $A_n = A(\Phi_n)$ is generally indefinite
- Can modify A_n without changing the solution

Choose B to be such that $\nabla \cdot B = \nabla \times$,

$$\implies \nabla \cdot B \nabla p = 0 \text{ for any } p,$$

Choose $\tilde{A}_n = A_n + B$ then,

$$\nabla \cdot \tilde{A}_n \mathbf{U} = \nabla \cdot A_n \mathbf{U}$$

- Choose B to shift the spectrum of A_n positive...

Properties of $\tilde{A}_n$

- The geometrically-nonlinear model assumes $o(E) \approx 0$. Thus we may also impose an assumption on the size of $E = \frac{1}{2}(\Phi^t\Phi - I)$.
- The ℓ^2 norm of the strains and the full spectrum of $\tilde{A}_n$ can be written in terms of $\delta = det(\Phi)$, $\sigma = \|\Phi\|_{\ell^2}^2$.
- We show $\tilde{A}_n$ positive definite under the small strains assumption:

$$\|\Phi^t\Phi - I\|_{\ell^2} < \frac{\sqrt{2}}{\lambda + 3} \implies \tilde{A}_n \text{ is positive definite.}$$

Newton-FOSLS

- Define the FOSLS functional for each step:

$$G(\mathbf{U};\mathbf{U}_n,\mathcal{F}_n) = \|\nabla\cdot\tilde{A}_n\mathbf{U} - \mathcal{F}_n\|_0^2 + \|\nabla\times\mathbf{U}\|_0^2,$$

for $\mathbf{U}$ in the space

$$\mathcal{V} = \{\mathbf{V}\in H^1(\Omega)^4 : \boldsymbol{\tau}\cdot\mathbf{V} = \mathbf{0} \text{ on } \Gamma_D\}.$$

- Solve by minimization principle: Choose $\mathbf{U}\in\mathcal{V}$ so that

$$G(\mathbf{U};\mathbf{U}_n,\mathcal{F}_n) = \inf_{\mathbf{V}\in\mathcal{V}} G(\mathbf{V};\mathbf{U}_n,\mathcal{F}_n).$$

- We show a well-posed Newton minimization, converging by an inexact nested iteration-Newton-FOSLS-multigrid process (cf. Codd, Manteuffel and McCormick).

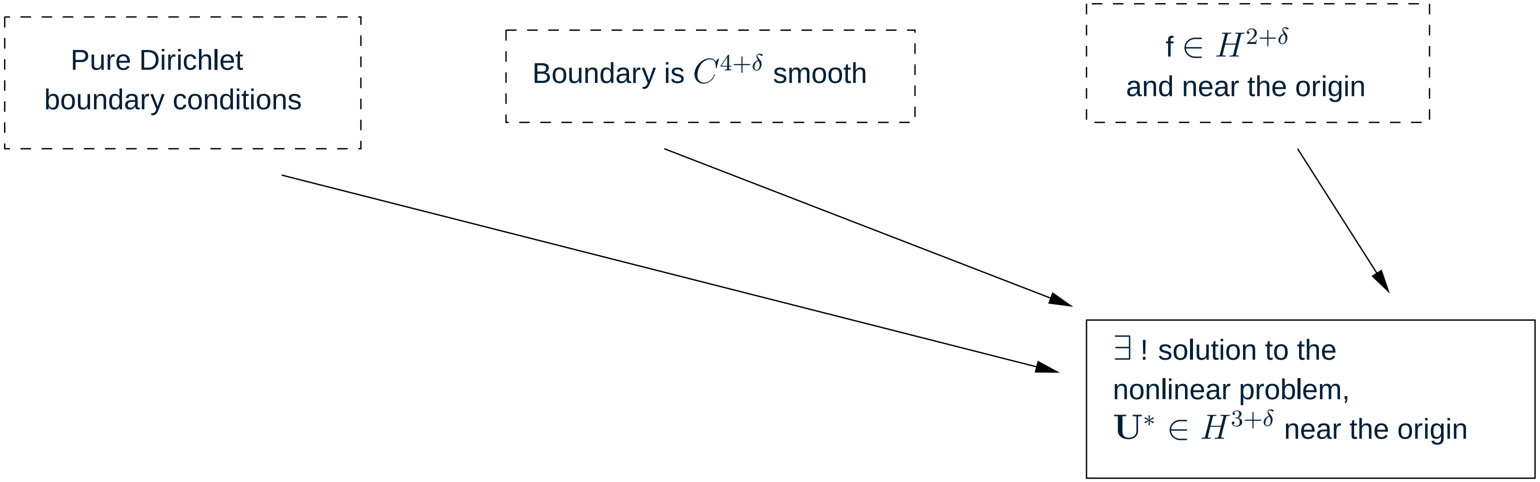
Elasticity Theory

Pure Dirichlet
boundary conditions

Boundary is $C^{4+\delta}$ smooth

f $\in H^{2+\delta}$
and near the origin

Elasticity Theory



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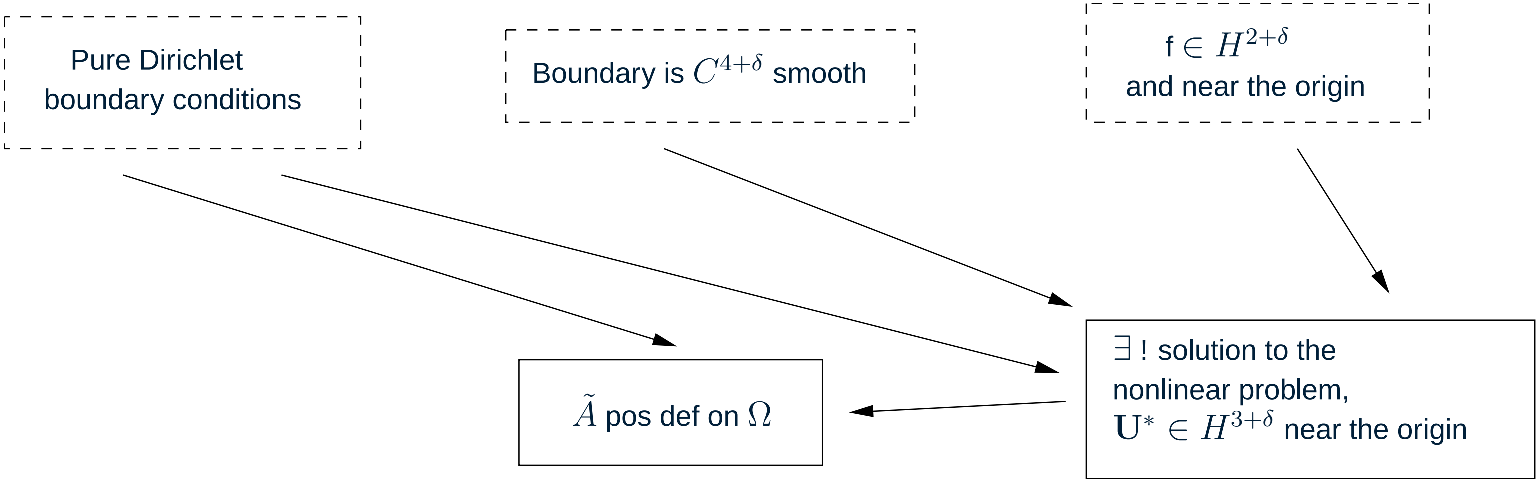

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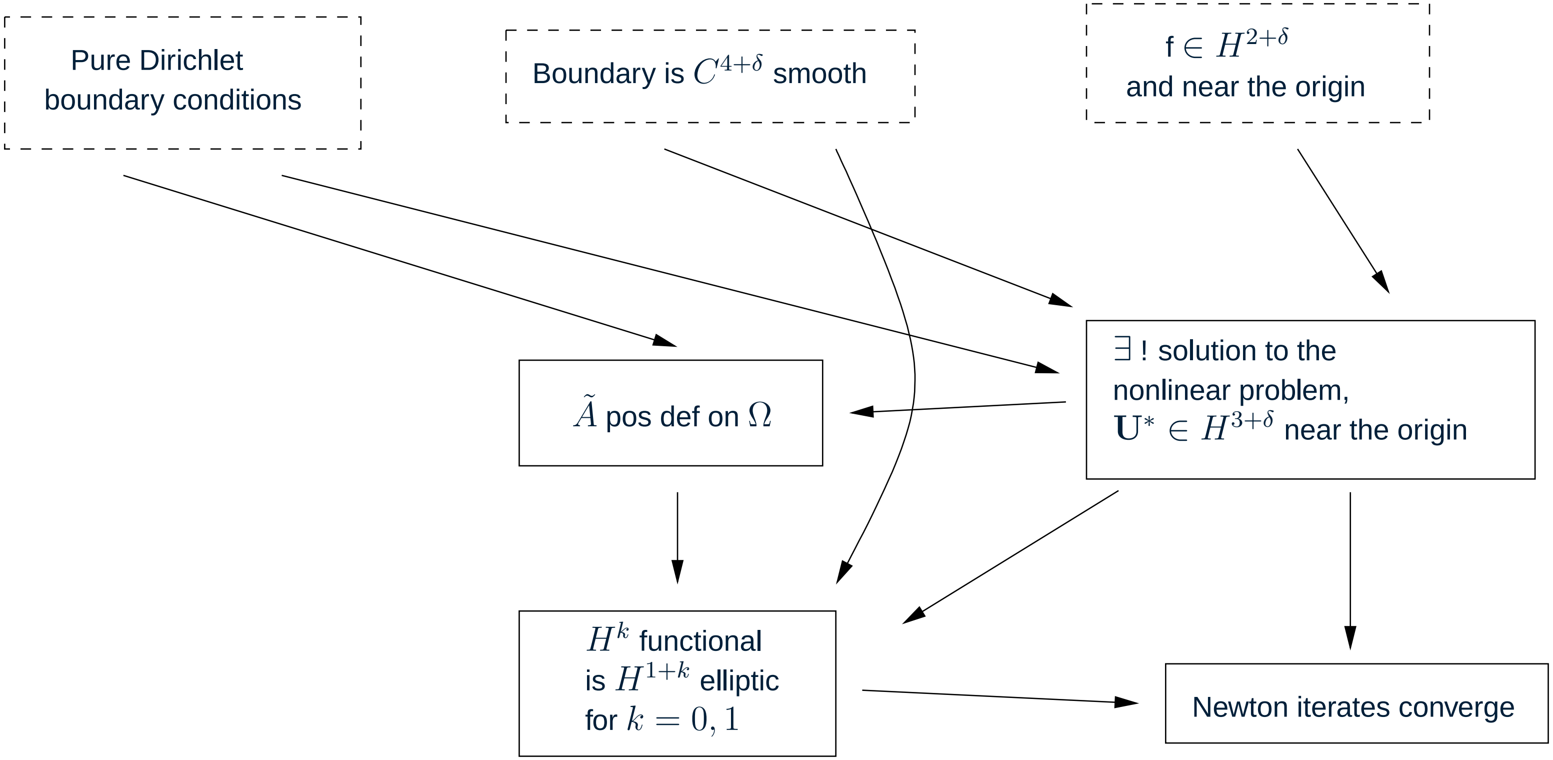

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H^k functional is H^{1+k} elliptic for $k = 0, 1$

Newton iterates converge

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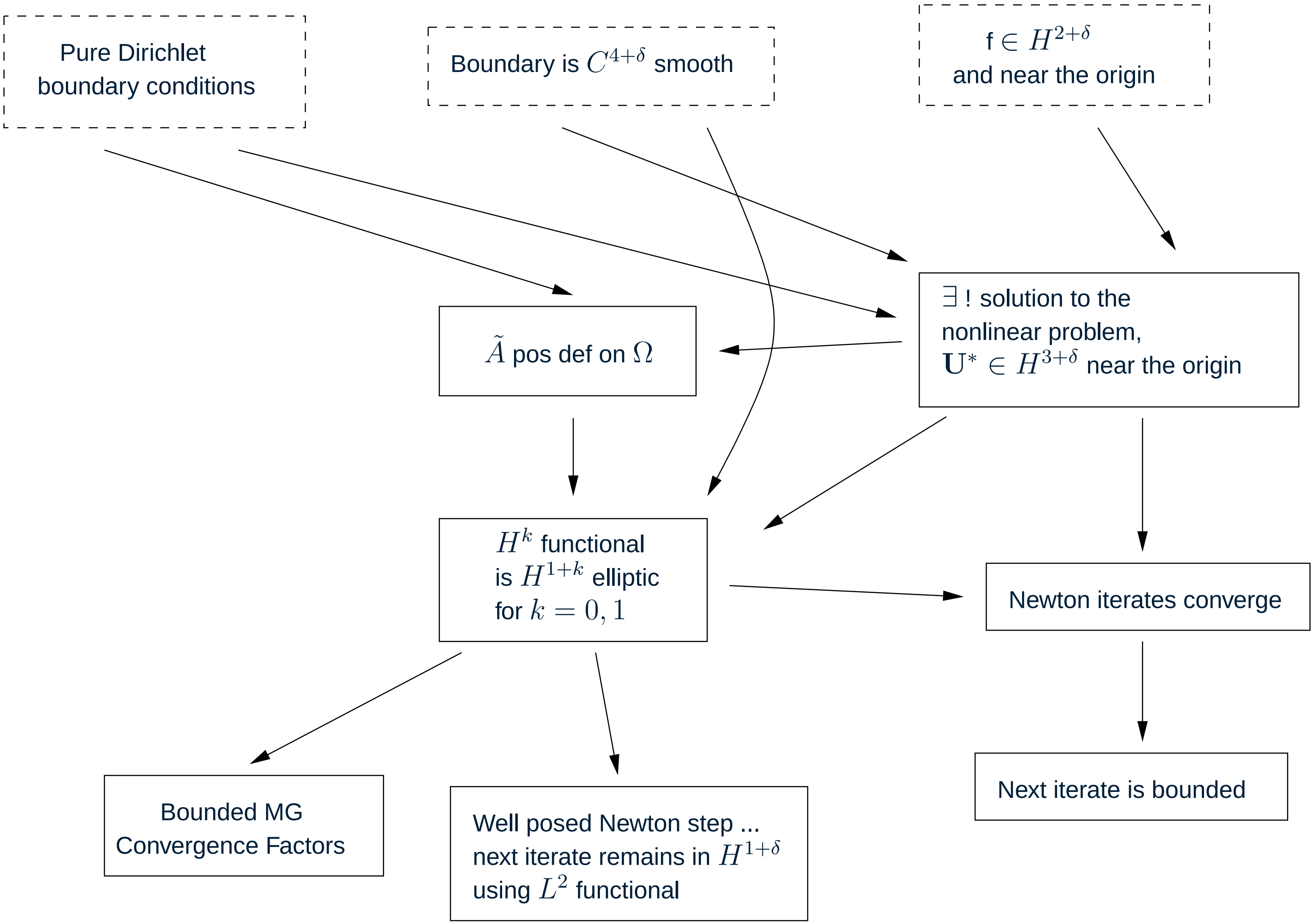

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Well posed Newton step ...
next iterate remains in $H^{1+\delta}$
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Next iterate is bounded

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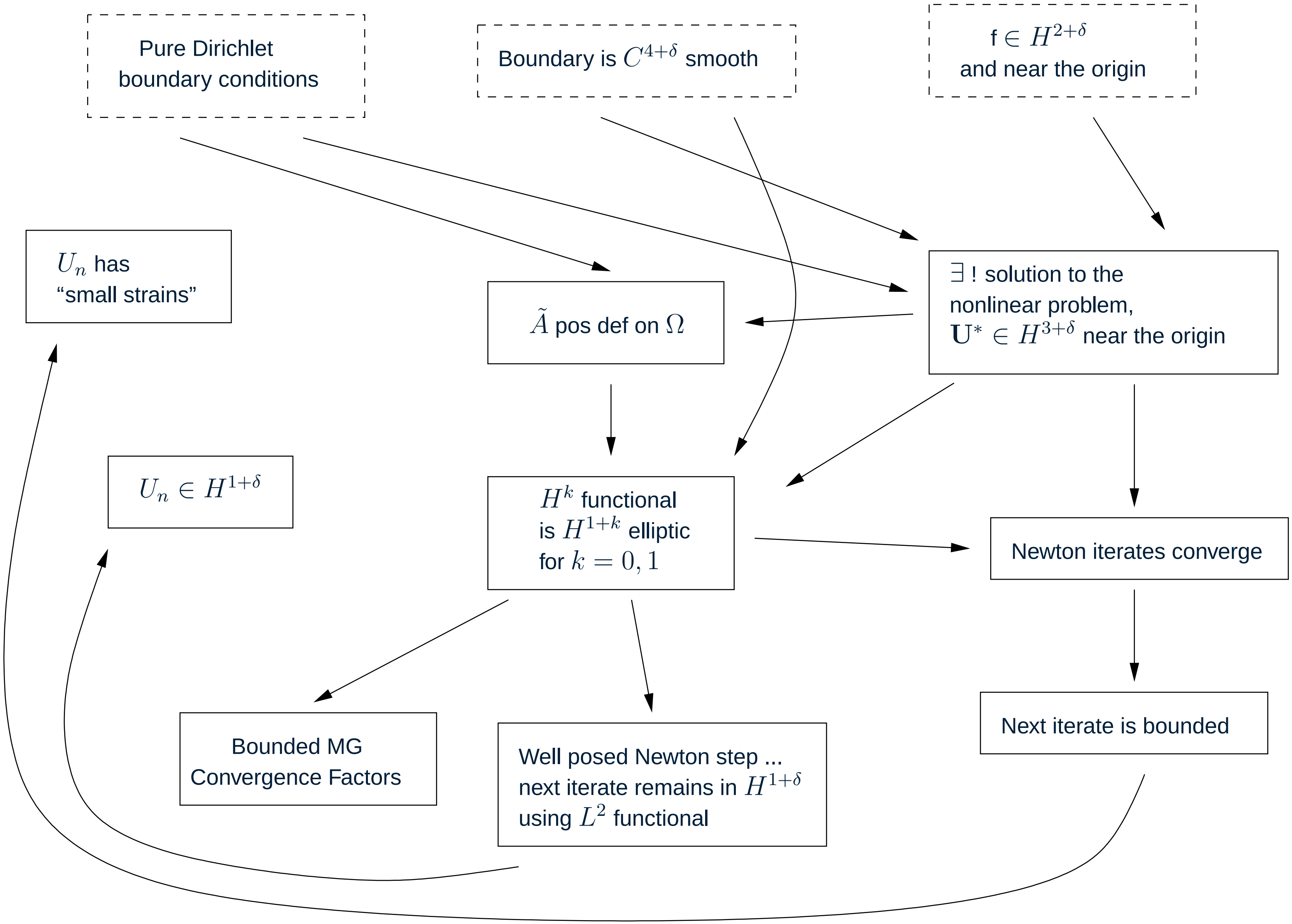

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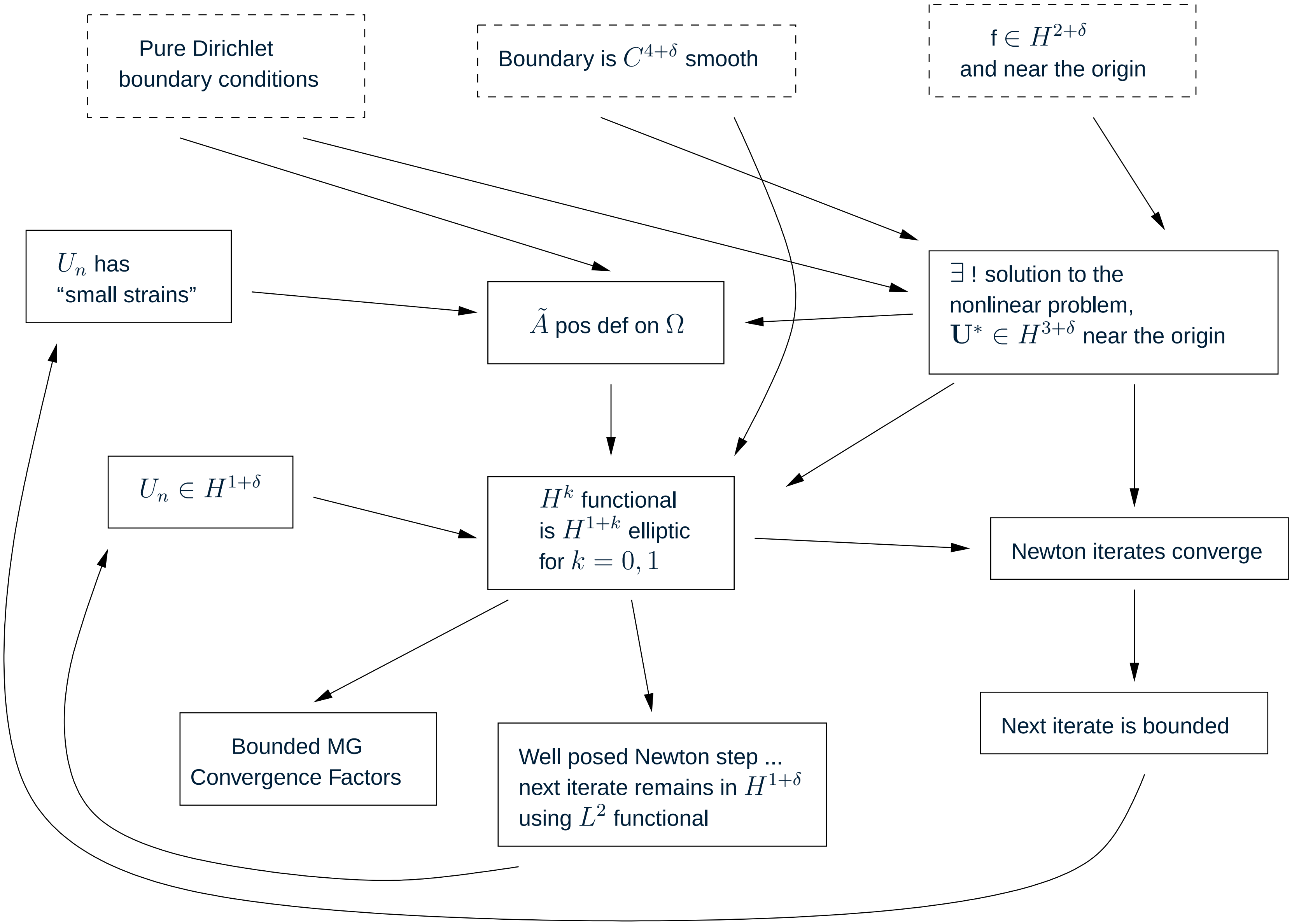

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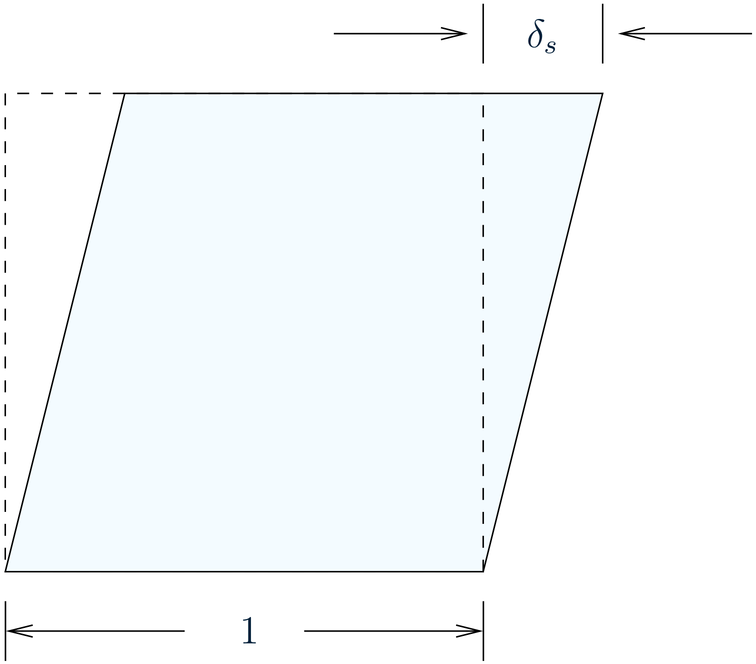
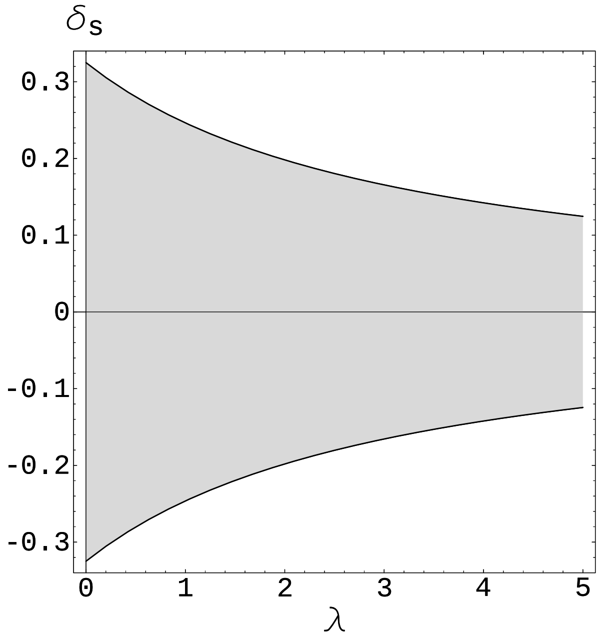
  - Most likely a sufficient but not necessary condition.
- The domain is sufficiently smooth...
  - We seek a method for LS problems in nonsmooth domains.
  - A necessary and sufficient condition for most problems.

Small Strains: Pure Shear and Pure Tensile Strain

Pure shear strain:

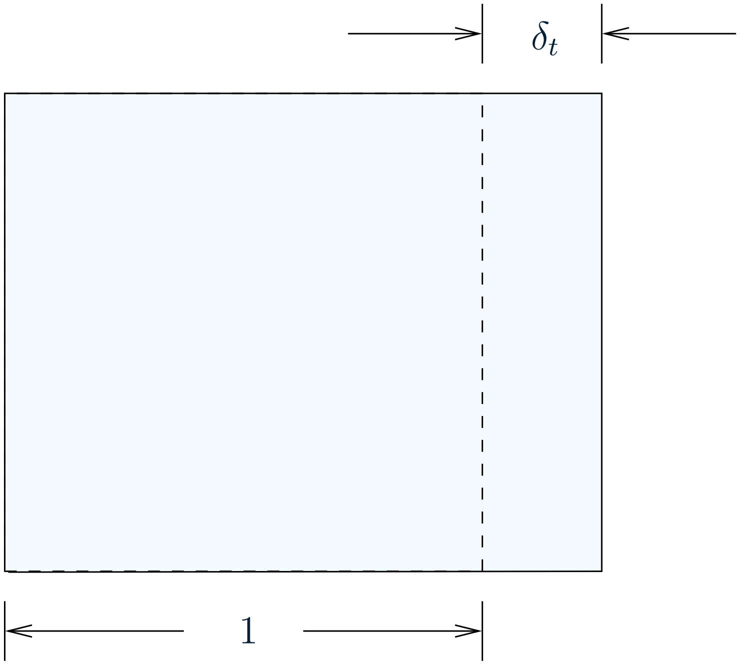


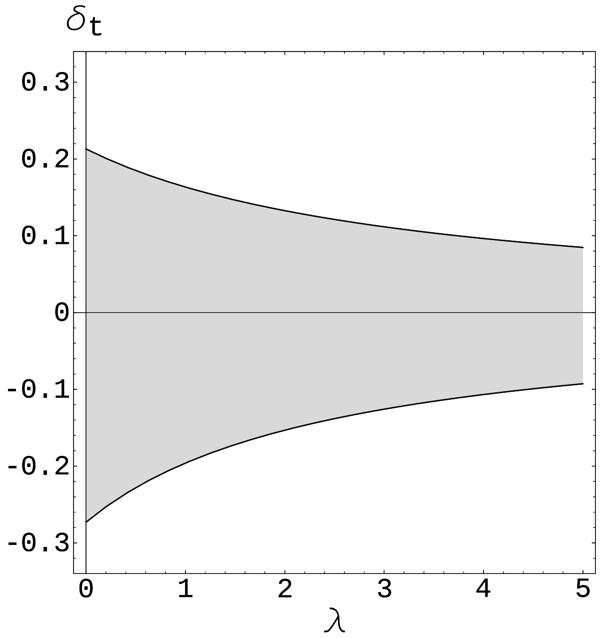


"Small strain" limit of pure shear strain versus λ

Small Strains: Pure Shear and Pure Tensile Strain

Pure tensile strain:





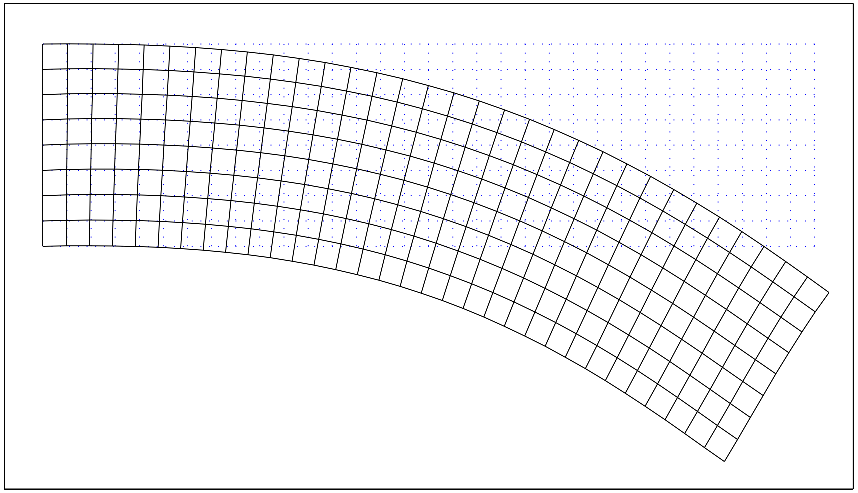
"Small strain" limit of pure tensile strain versus λ

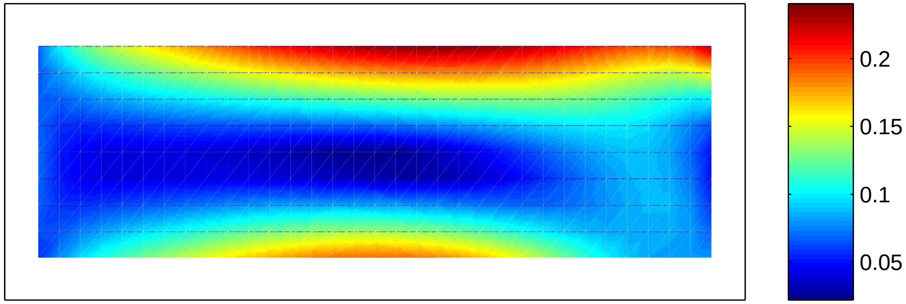
Small strains example

For this configuration our theory holds for $\lambda < 3.25$ ($\nu < 0.38$).

$$\max_{\Omega} \|\Phi^t \Phi - I\|_{\ell^2} = 0.241$$

	λ	$^*\nu$
Steel:	1.22	0.28
Nickel:	1.53	0.30
Aluminum:	2.15	0.34
maximum:	3.25	0.38
Lead:	7.30	0.44
Rubber:	33.3	0.49

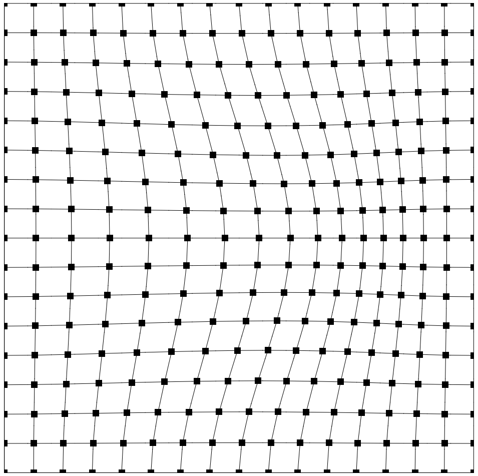




*The Poisson ratio, $\nu = \frac{\lambda}{2(\lambda+1)}$, is another measure of incompressibility.

Numerical Results: Smooth Solutions

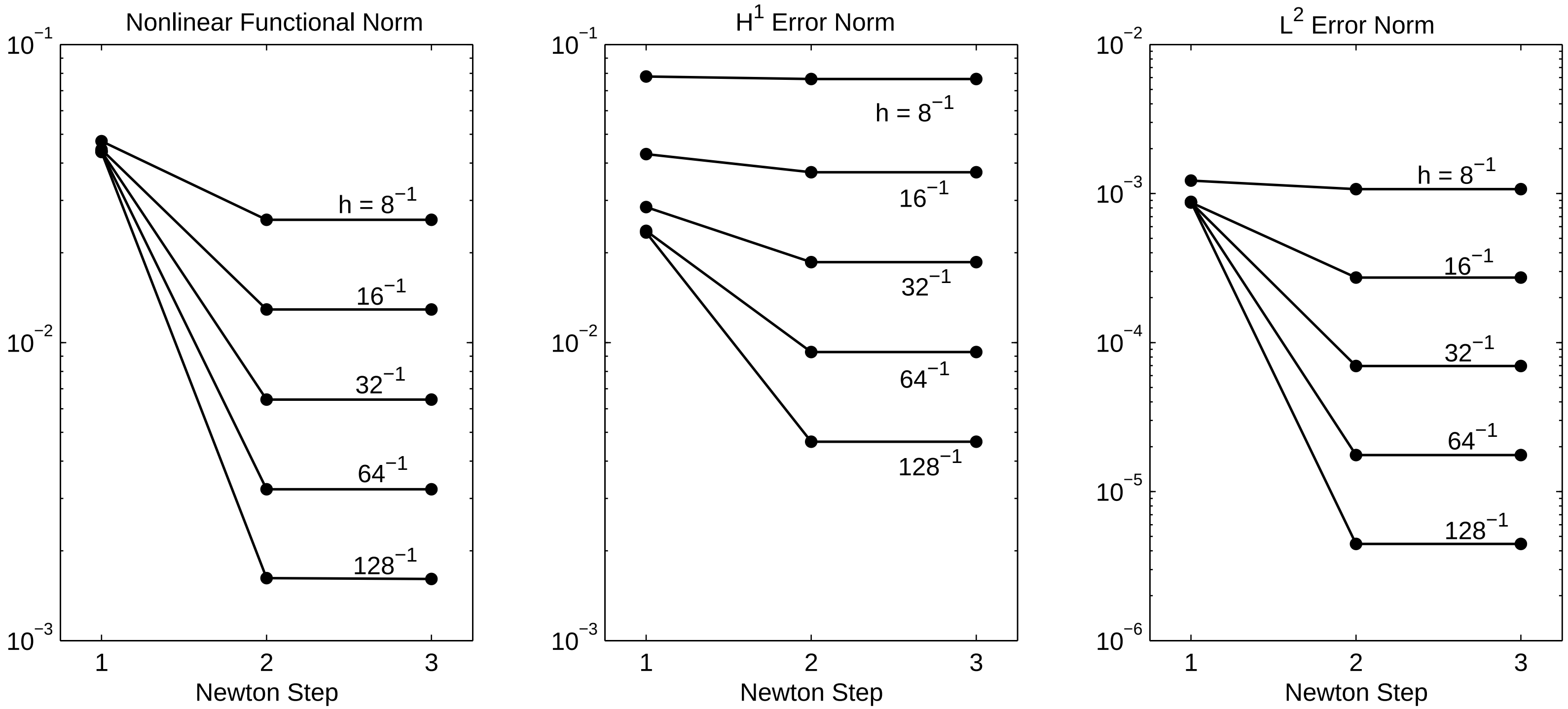
- Displacement boundary conditions on $\Omega = [0,1]^2$
- Known smooth solution $\mathbf{U} \in H^2(\Omega)$
- Bilinear finite elements on uniform meshs, $h^{-1} = 8$, 16, ... , 128
- Computations done using FOSPACK, by J. Ruge
- Lamé constants $\lambda = 2.15$, $\mu = 1.0$ (i.e., $\nu = 0.34$)



- Convergence of Newton’s method
- Finite Element approximation rates
- Multigrid performance
- Nested Iteration efficiency

FE Discretization Accuracy

Nonlinear Functional: $\mathcal{G}(\mathbf{U}; f) = \|\nabla \cdot [(I + \mathbf{U})\Sigma(\mathbf{U})] - f\|^2 + \|\nabla \times \mathbf{U}\|^2$



- Optimal $O(h)$ convergence in H^1 and functional norms
- Optimal $O(h^2)$ convergence in L^2 norm

Computational Cost

Nested Iteration, 3 V(1,1)-pcg cycles per level

h^{-1}	m	$\mathcal{G}(\mathbf{U}^h; f)^{\frac{1}{2}}$	Ratio	$\bar{\rho}$	W_T	time (s)
8	1	2.64e-02		0.29	12.3	1
16	2	1.31e-02	2.01	0.25	16.0	4
32	3	6.61e-03	1.98	0.24	19.0	15
64	4	3.32e-03	1.99	0.24	20.8	60
128	5	1.66e-03	2.00	0.23	21.6	242

m	Newton step
$\mathcal{G}^{\frac{1}{2}}$	Nonlinear functional norm
$\bar{\rho}$	Average V(1,1)-pcg convergence factor
W_T	Total Cumulative Work: in # of Jacobi sweeps relative to current grid

Regularity in Nonsmooth Domains

Loss of smoothness at corners in polygonal domains is well understood (Kondrat'ev, Grisvard, Maz'ya, et. al.)

$$\boxed{\mathbf{u} = \mathbf{u}_0 + \mathbf{S}(r,\theta)}$$

$$\begin{aligned} \mathbf{u}_0 &\in H^2(\Omega) \\ \mathbf{S}(r,\theta) = r^\alpha \boldsymbol{\Theta}(\theta) &\in H^{\alpha+1}(\Omega) \end{aligned}$$

The value of α is determined by

- Boundary Condition Type: displacement, traction, mixed
- Geometry: interior angle at the corner
- Lamé Constants: λ, μ

This difficulty occurs in many second-order problems and is especially problematic for FOSLS discretizations!

Consider the FOSLS two-stage solution of Poisson's equation:

$$\begin{cases} \Delta p = f, & \Omega \\ p = 0, & \partial\Omega \end{cases} \quad \Longrightarrow \quad \begin{cases} \nabla p - \mathbf{u} = 0, & \Omega \\ \nabla \cdot \mathbf{u} = f, & \Omega \\ \nabla \times \mathbf{u} = 0, & \Omega \\ p = 0, & \partial\Omega \\ \boldsymbol{\tau} \cdot \mathbf{u} = 0, & \partial\Omega \end{cases}$$

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$$G_1(\mathbf{u}; f) = \|\nabla \cdot \mathbf{u} - f\|^2 + \|\nabla \times \mathbf{u}\|^2$$

$$G_2(p; \mathbf{u}) = \|\nabla p - \mathbf{u}\|^2$$

$$\mathcal{V} = \{\mathbf{u} \in H^1(\Omega) : \ \boldsymbol{\tau} \cdot \mathbf{u} = 0 \text{ on } \partial\Omega\}$$

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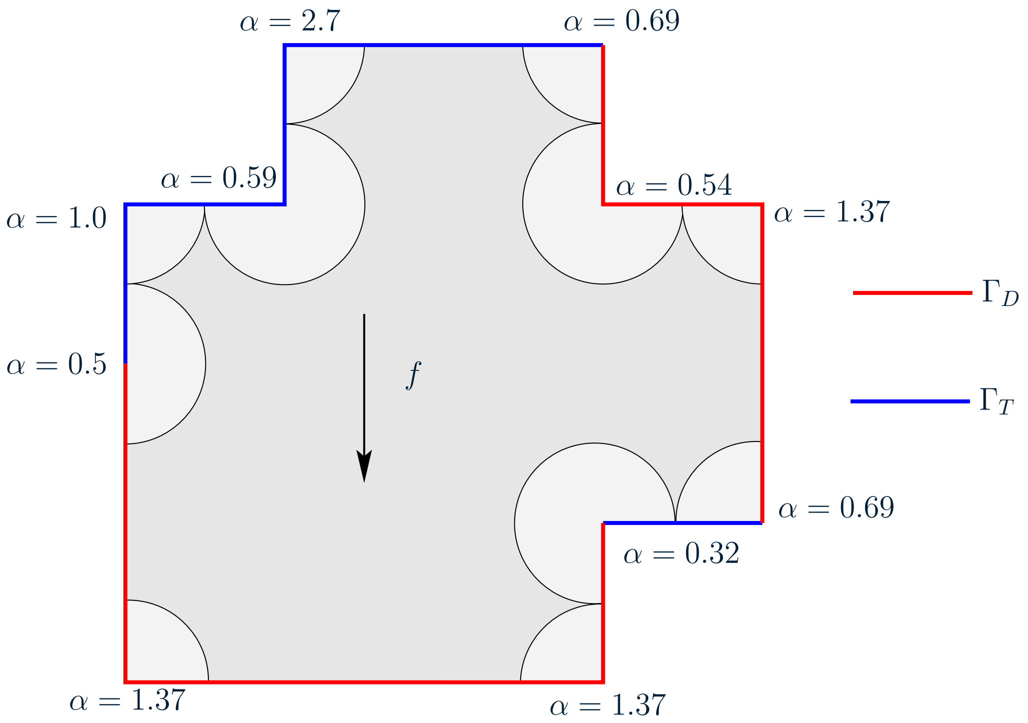
(2) Given $\mathbf{u} \in \mathcal{V}$, choose $p = \underset{q \in \mathcal{W}}{\text{argmin}}\, G_2(q; \mathbf{u})$

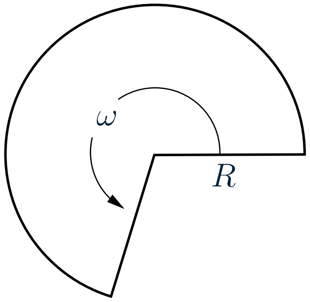
Nonsmooth Domains: Example



- Each corner has a singularity of power α.
- Away from <u>irregular boundary points</u>, the solution is sufficiently smooth.

Nonsmooth Domains

$\Omega =$



The solution to Poisson's equation near the corner may have a component of the form:

$$s(r,\theta) = r^{\alpha} \sin(\alpha\theta), \quad \alpha = \frac{\pi}{\omega}$$

(r,θ) a local polar coordinate system

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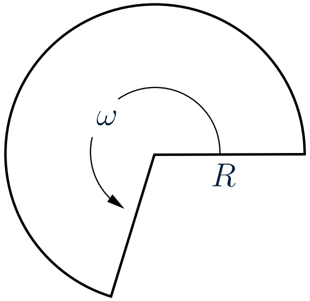
$\Omega =$ 

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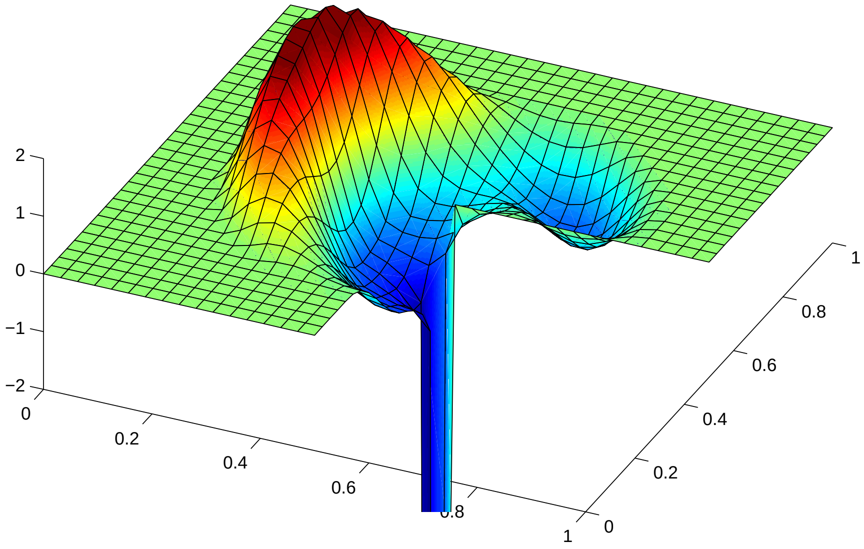
(r,θ) a local polar coordinate system

$$\begin{cases} p = p_0 + s & p_0 \in H^2(\Omega), \quad s \in H^{1+\alpha}(\Omega) \\ \mathbf{u} = \mathbf{u}_0 + \mathbf{S} & \mathbf{u}_0 \in H^1(\Omega), \quad \mathbf{S} \in H^{\alpha}(\Omega) \end{cases}$$

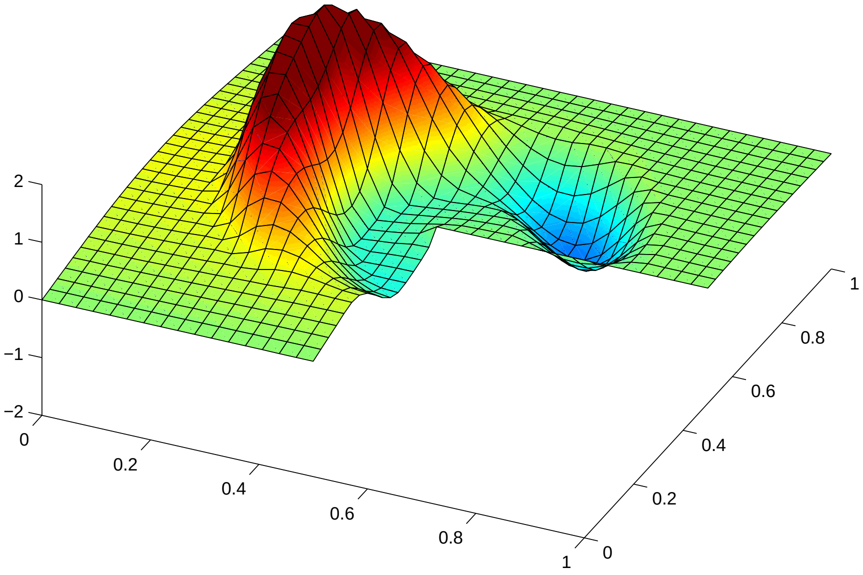
$$\omega > \pi \quad \Longrightarrow \quad \alpha < 1 \quad \Longrightarrow \quad \mathbf{u} \notin H^1(\Omega)$$

- Using standard FE subspaces of H^1 results in stalling convergence and a global loss of accuracy!

The Pollution Effect

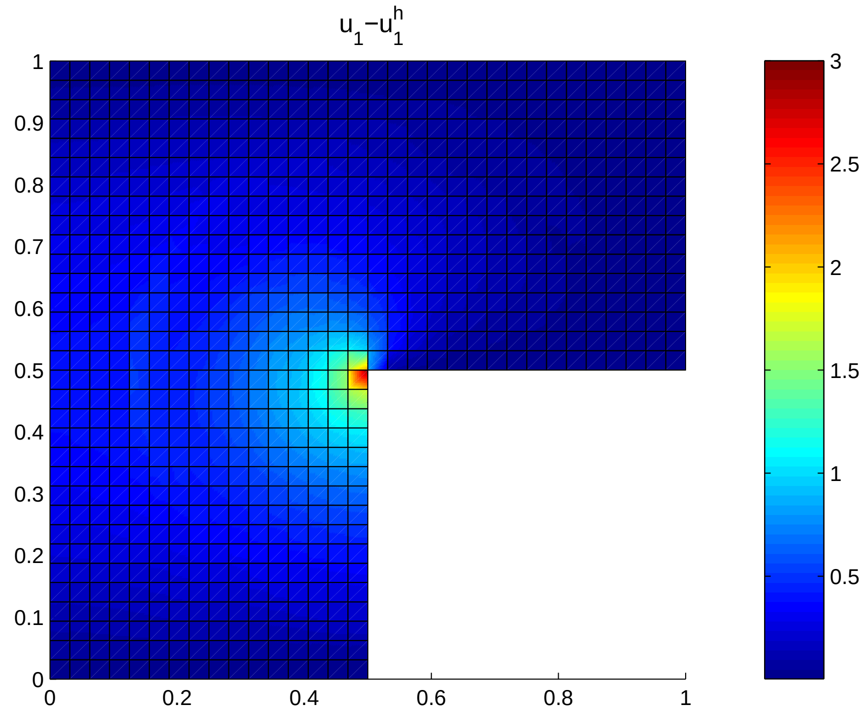


Exact component u_1



u_1^h computed by standard FOSLS

The Pollution Effect



$$u_1 - u_1^h$$

Standard FOSLS

- Even away from the singularity the solution is not accurate!

Existing Methods

- Add singular functions to finite element space
- Inverse-norm methods: H^{-1} least-squares, FOSLL*
- $H(div)$-conforming finite element spaces
- Weighted-norm FOSLS

Weighted-Norm FOSLS

Replace the L^2 norm functionals with locally weighted L^2 norms:

$$G_w(\mathbf{u}; f) = \|w(\nabla \cdot \mathbf{u} - f)\|^2 + \|w \nabla \times \mathbf{u}\|^2, \quad w = r^\beta$$

Let $\mathcal{V}^h \subset \mathcal{V}$ be the FE space. The discrete problem now becomes:
Given $f \in L^2(\Omega)$, choose

$$\mathbf{u}^h = \operatorname*{argmin}_{\mathbf{v}^h \in \mathcal{V}^h} G_w(\mathbf{v}^h; f)$$

Choose the weight parameter β in order to

- make H^1 dense in $H(div) \cap H(curl)$ in the weighted functional norm
- recover the best possible convergence throughout Ω

Define the weighted Sobolev norm and space:

$$\|\mathbf{u}\|_{k,\beta} = \Big(\sum_{|\gamma|\leq k} \|r^{\beta+\gamma-k} D^{\gamma}\mathbf{u}\|^2 \Big)^{\frac{1}{2}},$$

$$H^k_\beta(\Omega) = \{\mathbf{u} : \ \|\mathbf{u}\|_{k,\beta} < \infty\}.$$

for example:

$$\|\mathbf{u}\|^2_{1,\beta} = \|r^{\beta}\nabla\mathbf{u}\|^2 + \|r^{\beta-1}\mathbf{u}\|^2$$

Define the weighted Sobolev norm and space:

$$\|\mathbf{u}\|_{k,\beta} = \Big(\sum_{|\gamma| \le k} \|r^{\beta+\gamma-k} D^{\gamma} \mathbf{u}\|^2 \Big)^{\frac{1}{2}},$$

$$H^k_\beta(\Omega) = \{\mathbf{u} : \ \|\mathbf{u}\|_{k,\beta} < \infty\}.$$

Also define

$$L = \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad \Longrightarrow \quad L\mathbf{u} = \mathbf{f} \quad \text{and}$$

$$G_w(\mathbf{u}^h; \mathbf{f}) = \|L\mathbf{u}^h - \mathbf{f}\|^2_{0,\beta} = \|L(\mathbf{u}^h - \mathbf{u})\|^2_{0,\beta}$$

Theory

Let Ω be a polygonal domain with one reentrant corner of interior angle ω and assume $\mathbf{f} \in L^2(\Omega)$. Let $\mathbf{u} \in H^\alpha(\Omega)$ satisfy $L\mathbf{u} = \mathbf{f}$. If $\mathbf{u}^h \in \mathcal{V}^h$ is chosen to minimize the weighted functional,

$$G_w(\mathbf{u}^h; \mathbf{f}) = \min_{\mathbf{v}^h \in \mathcal{V}^h} \|L\mathbf{v}^h - \mathbf{f}\|_{0,\beta}^2,$$

for $1 - \alpha < \beta < min(1 + \alpha, 2 - \alpha)$, then the approximation error, $\mathbf{u} - \mathbf{u}^h$, satisfies the following bounds:

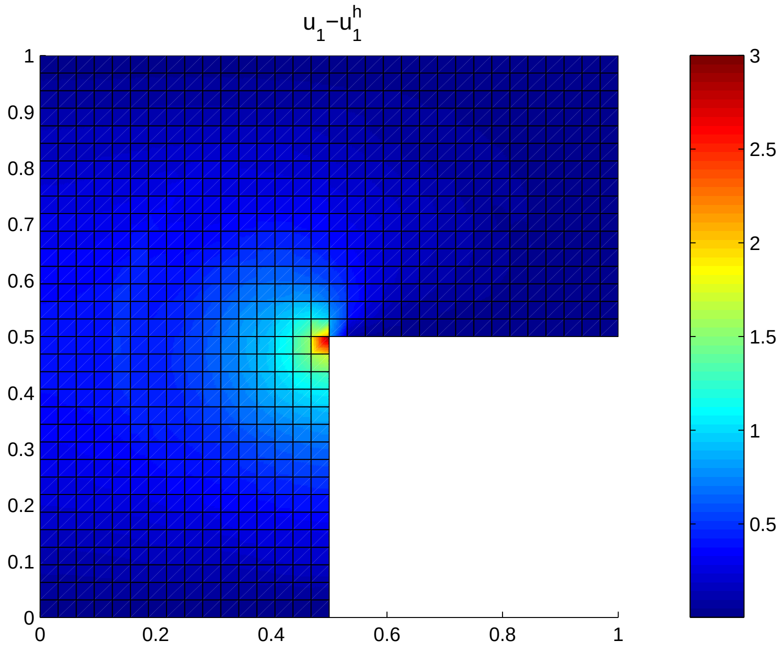
$$(1) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{1,\beta} \leq Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta-1})$$

$$(2) \qquad G_w(\mathbf{u} - \mathbf{u}^h; \mathbf{0})^{\frac{1}{2}} \leq Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta-1})$$

$$(3) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{0,\beta} \leq Ch^{\alpha+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta})$$

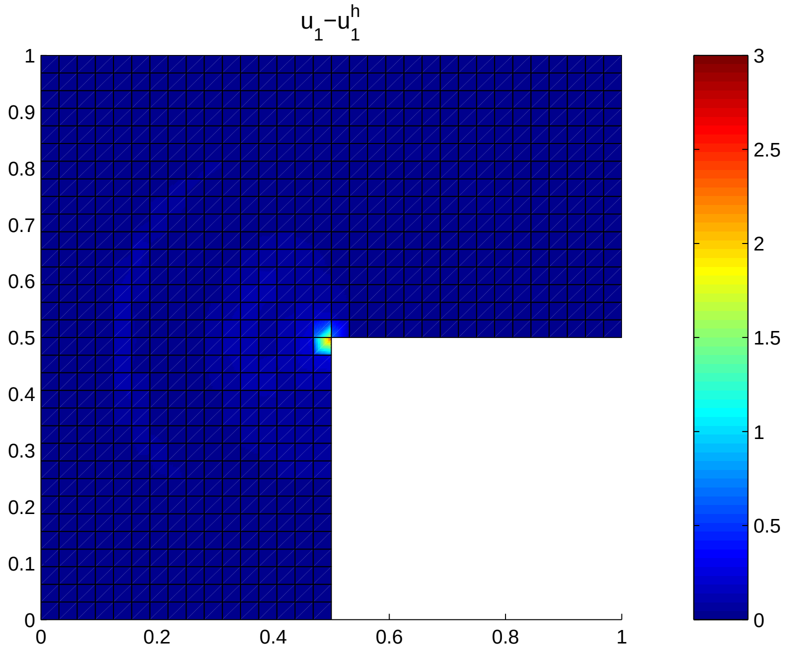
$$(4) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{0} \leq Ch^{\alpha}(\|\mathbf{u}\|_{\alpha} + \|\mathbf{u}\|_{\alpha+\beta,\beta}). \qquad O(h^{\alpha})$$

Reduction of the Pollution Effect



$u_1 - u_1^h$

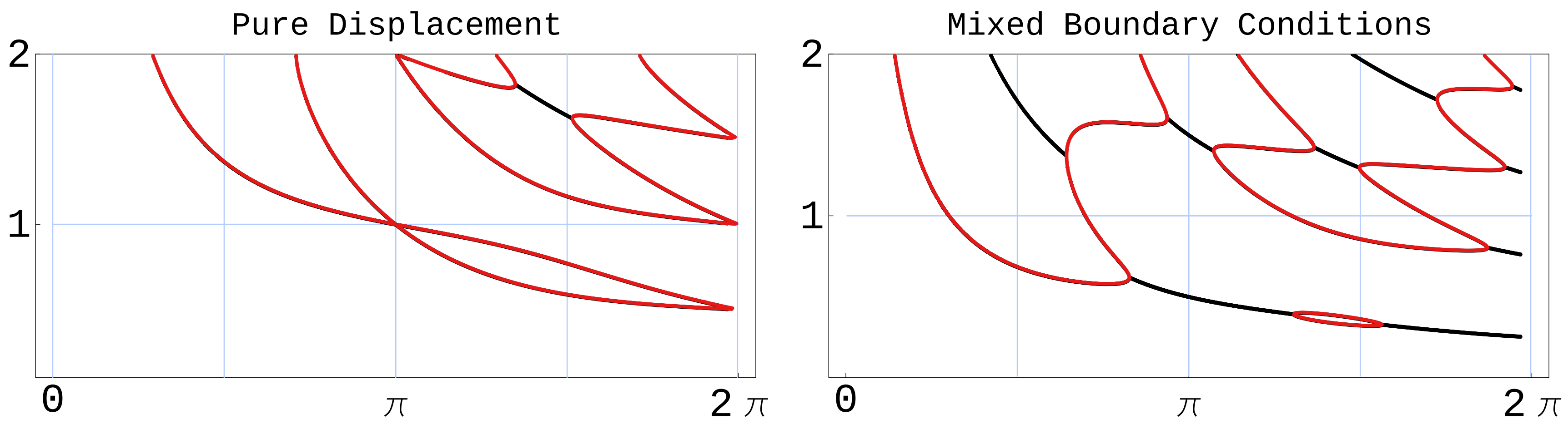
Standard FOSLS, No weighting



$u_1 - u_1^h$

Weighted-norm FOSLS, $\beta = 4/3$

Powers of Singularities: Elasticity



Each plot is the power of the singularity α vs. interior angle ω for linear elasticity with Lamé constants $\lambda = 2.15$, $\mu = 1$.

- Pure Displacement $\omega \geq \pi \implies \mathbf{U} \notin H^1(\Omega)$
- Mixed Boundaries $\omega \gtrsim 0.3\pi \implies \mathbf{U} \notin H^1(\Omega)$

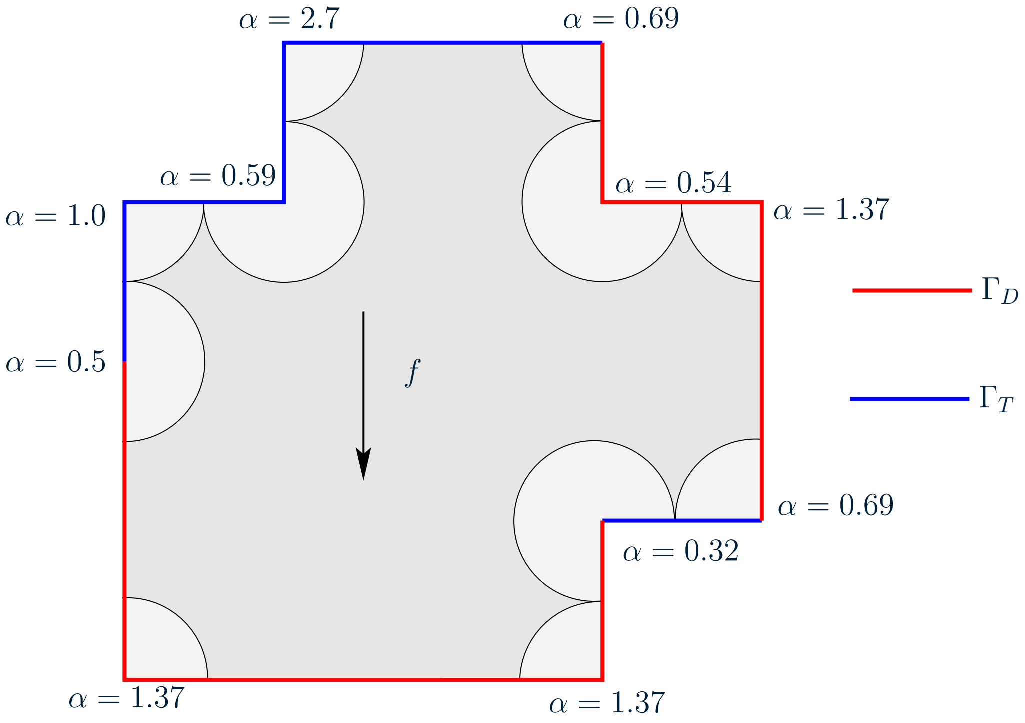
Numerical Results: Elasticity with Singular Solutions

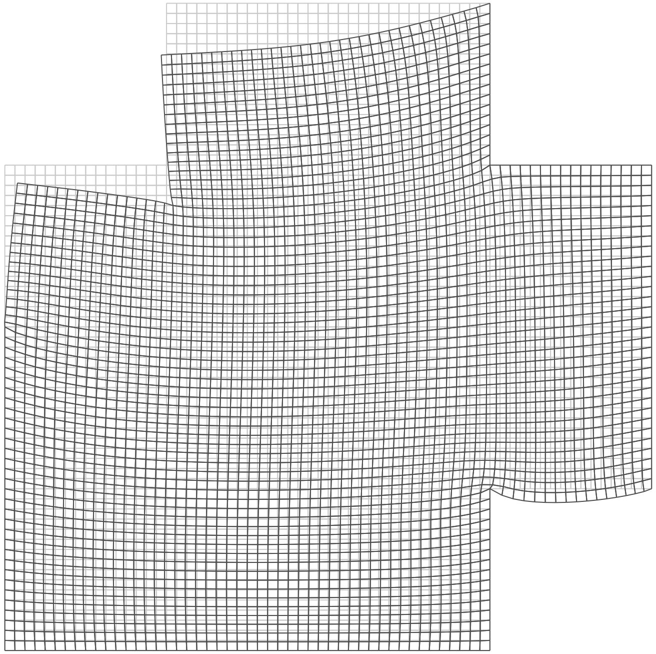
- Mixed boundary conditions on Ω
- Interior body force $\Longrightarrow$ $\mathbf{U} \in L^2(\Omega) \backslash H^1(\Omega)$
- Use weighted functional:

$$G_w(\mathbf{U}; \mathbf{U}_n, \mathcal{F}_n) = \|\nabla \cdot \tilde{A}_n \mathbf{U} - \mathcal{F}_n\|_{0,\beta}^2 + \|\nabla \times \mathbf{U}\|_{0,\beta}^2$$

- Use weight of $\beta = 2 - \alpha$ near each corner

Domain and Boundary Conditions





$$\boxed{\mathbf{U} \sim r^{\alpha-1} \in H^{\alpha}(\Omega)}$$

Numerical Results: Singular Solutions

Nested Iteration, 5 V(1,1)-pcg cycles per level

h^{-1}	m	$\mathcal{G}_w(\mathbf{U}^h; f)^{\frac{1}{2}}$	Ratio	$\bar{\rho}$	W_T	time (s)
8	1	2.88e-01		0.49	20.1	7
16	2	1.94e-01	1.48	0.59	28.3	23
32	3	1.27e-01	1.53	0.76	33.0	90
64	4	8.11e-02	1.56	0.82	37.1	350
128	5	5.14e-02	1.58	0.82	38.9	1376

m	Newton step
$\mathcal{G}_w^{\frac{1}{2}}$	Weighted nonlinear functional norm
$\bar{\rho}$	Average V(1,1)-pcg convergence factor
W_T	Total Cumulative Work: in # of Jacobi sweeps relative to current grid

Concluding Remarks and Open Questions

Summary

- Robust solution strategy for geometrically-nonlinear elasticity problems in both smooth and nonsmooth domains.
- Corner singularities for many problems can be treated with weighted norms, restoring (nearly) optimal discretization convergence rates.

Open Questions

- Elasticity theory for pure traction, mixed boundary conditions
- Elasticity in the incompressible limit, $\lambda \to \infty$
- Improved multigrid solvers for weighted problems