Least-Squares FEM for Problems with Boundary Singularities

Chad Westphal

Wabash College

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Outline

- Regularity of 2nd order problems
- Prototypical div-curl and H(div) least-squares methods
- Weighted-norm minimization
- Theoretical results
- Numerical results
- Conclusions / open questions / current projects

PDEs and Regularity in Nonsmooth Domains

Let Ω be a polygonal domain in $\mathbb{R}^2$.

Consider the Dirichlet problem for Poisson's equation

$$\begin{aligned} \Delta p &= f, \text{ in } \Omega \\ p &= 0, \text{ on } \partial\Omega \end{aligned}$$

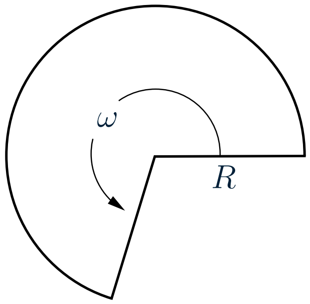
Q: If $f \in L^2(\Omega)$ when can we expect $p \in H^2(\Omega)$?

A: It depends on Ω. If Ω is convex then the problem has full regularity and

$$f \in L^2(\Omega) \quad \Longrightarrow \quad p \in H^2(\Omega).$$

Nonsmooth Domains

$\Omega =$



The solution to Poisson’s equation near the corner may have a component of the form:

$$s(r,\theta) = r^{\alpha} \sin(\alpha\theta), \quad \alpha = \frac{\pi}{\omega}$$

(r,θ) a local polar coordinate system

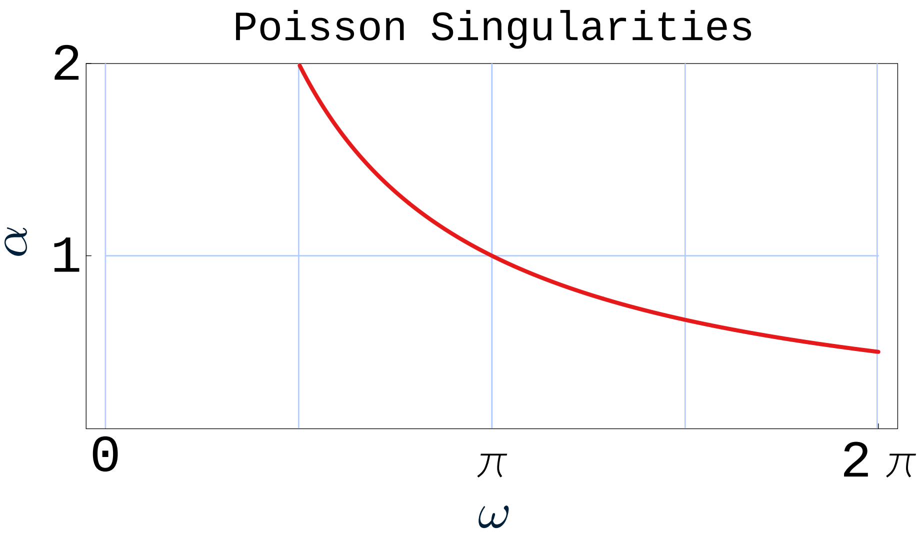
$$p = p_0 + s \qquad p_0 \in H^2(\Omega), \quad s \in H^{1+\alpha}(\Omega)$$

$$\omega > \pi \quad \Longrightarrow \quad \alpha < 1 \quad \Longrightarrow \quad p \notin H^2(\Omega)$$

Singular Functions for 2nd Order PDEs

$$\alpha = \pi/\omega$$

$$p \in H^{1+\alpha}(\Omega)$$

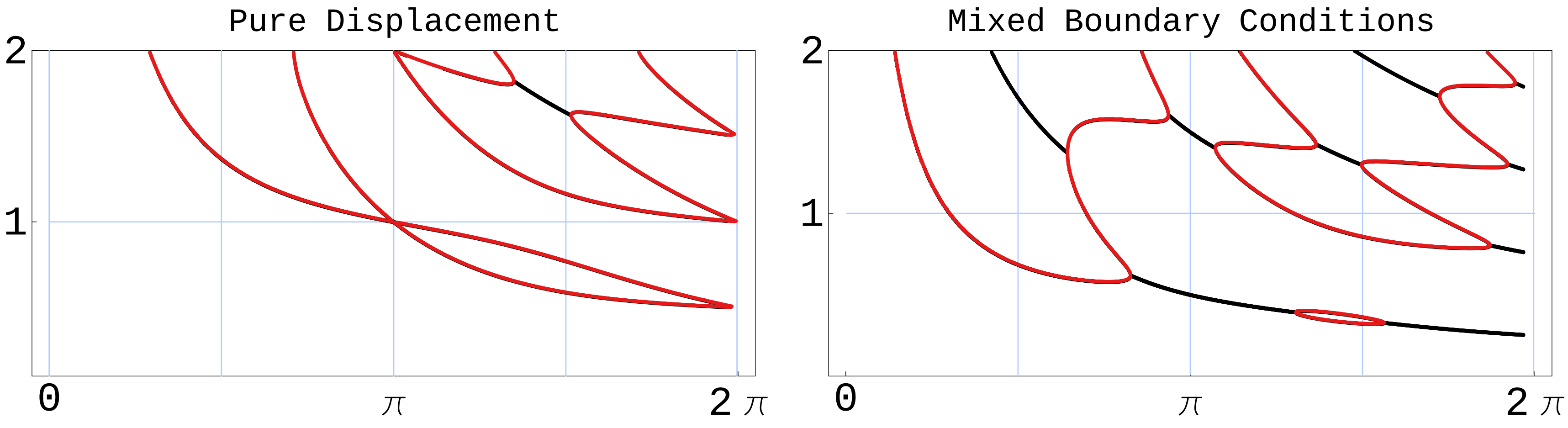


$\alpha < 1 \quad \Longrightarrow$ Loss of regularity for the PDE

This singular behavior is known for many 2nd order PDEs. (Grisvard et. al.)

Other PDEs: 2-d Linear Elasticity

α vs. ω for linear elasticity with Lamé constants $\lambda = 2.15$, $\mu = 1$.



In general, the singular solutions may depend on

- Corner angle, ω
- Boundary condition type
- PDE coefficients
- Nonlinearities

Implications for Numerical Methods

Convergence of standard methods is limited by the smoothness of the solution.

example:

$P_k =$ degree k piecewice polynomial functions on a mesh of size h.

The interpolant onto P_k gives

$$\|p - I^h p\| \le ch^\alpha \|p\|_\alpha \qquad \text{for} \;\; \alpha \le k + 1.$$

$\Longrightarrow$ convergence is limited by both the FE space and α.

Pollution Effect:

Reduced rates are seen globally even though the singularity is local.

Numerical Solutions to PDEs

Let

$$Lu = f$$

be a first-order PDE on a bounded $\Omega \subset \mathbb{R}^d$.

- L first-order differential operator
- $u \in \mathcal{V}$ unknowns
- $\mathcal{V}$ some appropriate function space
- $\mathcal{V}^h \subset \mathcal{V}$ a finite element space
- $f \in L^2(\Omega)$ given data function

Least-Squares Methods

Define the least-squares functional

$$G(u; f) = \|Lu - f\|_X^2$$

Minimizing G over $\mathcal{V}^h$ is equivalent to the weak problem:

$$\text{Find } u^h \in \mathcal{V}^h \quad \text{such that} \quad \langle Lu^h - f, Lv^h \rangle_X = 0 \quad \forall\, v^h \in \mathcal{V}^h.$$

Here, a symmetric variational problem is formed by minimizing the residual of the problem in the norm $\| \cdot \|_X$.

$\mathcal{V}$-ellipticity: $\exists\, c_0, c_1 > 0$ where

$$c_0 \|u\|_{\mathcal{V}} \leq \|Lu\|_X \leq c_1 \|u\|_{\mathcal{V}} \quad \text{for all} \quad u \in \mathcal{V}.$$

$\Longrightarrow$ minimizing $G(u^h, f)$ minimizes the error in the $\mathcal{V}$ norm.

The design of a least-squares finite element method involves...

- formulating an appropriate first-order system, $Lu = f$
- finding the right norm, $\| \cdot \|_X$, to minimize over
- proving equivalence of LS functional to a meaningful norm on $\mathcal{V}$
- choosing good FE spaces, $\mathcal{V}^h \subset \mathcal{V}$

2 Specific Examples for Poisson

$$\begin{cases} \Delta p = f, & \Omega \\ p = 0, & \partial\Omega \end{cases}$$

$$(1) \begin{cases} \nabla p - \mathbf{u} = 0, & \Omega \\ \nabla \cdot \mathbf{u} = f, & \Omega \\ \nabla \times \mathbf{u} = 0, & \Omega \\ p = 0, & \partial\Omega \\ \boldsymbol{\tau} \cdot \mathbf{u} = 0, & \partial\Omega \end{cases}$$

div-curl method

$$(2) \begin{cases} \nabla p - \boldsymbol{\sigma} = 0, & \Omega \\ \nabla \cdot \boldsymbol{\sigma} = f, & \Omega \\ p = 0, & \partial\Omega \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma} = 0, & \partial\Omega \end{cases}$$

H(div) method

div-curl Method

$$\begin{cases} \nabla p - \mathbf{u} = 0, & \Omega \\ \nabla \cdot \mathbf{u} = f, & \Omega \\ \nabla \times \mathbf{u} = 0, & \Omega \\ p = 0, & \partial\Omega \\ \boldsymbol{\tau} \cdot \mathbf{u} = 0, & \partial\Omega \end{cases}$$

$$G_1(\mathbf{u}; f) = \|\nabla \cdot \mathbf{u} - f\|^2 + \|\nabla \times \mathbf{u}\|^2$$

$$G_2(p; \mathbf{u}) = \|\nabla p - \mathbf{u}\|^2$$

$$\mathcal{V} = \{\mathbf{v} \in H^1(\Omega) : \ \boldsymbol{\tau} \cdot \mathbf{v} = 0 \text{ on } \partial\Omega\}$$

$$\mathcal{W} = \{q \in H^1(\Omega) : \ q = 0 \text{ on } \partial\Omega\}$$

(1) Given $f \in L^2(\Omega)$, choose $\mathbf{u}$ so that

$$G_1(\mathbf{u}; f) = \min_{\mathbf{v} \in \mathcal{V}} G_1(\mathbf{v}; f)$$

(2) Given $\mathbf{u} \in \mathcal{V}$, choose $\mathbf{u}$ so that

$$G_2(p; \mathbf{u}) = \min_{q \in \mathcal{W}} G_2(q; \mathbf{u})$$

First stage functional, G_1

We would like to solve

$$\begin{cases} \nabla \cdot \mathbf{u} = f, & \Omega \\ \nabla \times \mathbf{u} = 0, & \Omega \end{cases}$$

over a subspace of $H^1(\Omega)$. Naturally,

$$f \in L^2(\Omega) \implies \mathbf{u} \in H(div) \cap H(curl).$$

Full regularity would give

$$f \in L^2(\Omega) \implies p \in H^2(\Omega) \implies \mathbf{u} \in H^1(\Omega).$$

For polygonal domains $H^1(\Omega) = H(div) \cap H(curl) \iff \Omega$ is convex.

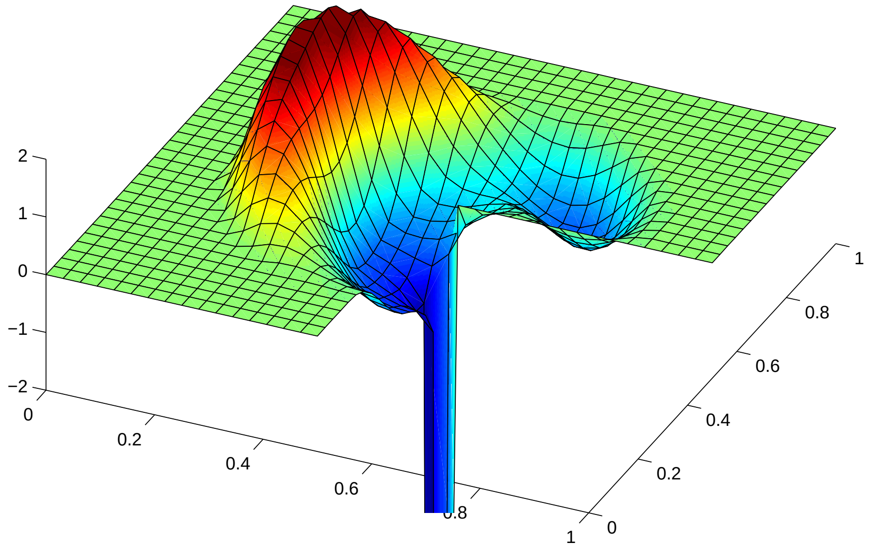
Numerical Approximations and Convergence

Let $\mathcal{T}^h$ be a regular triangulation of Ω of elements, K, of size $O(h)$, and $P_1(K)$ be the space of linear polynomials on K.

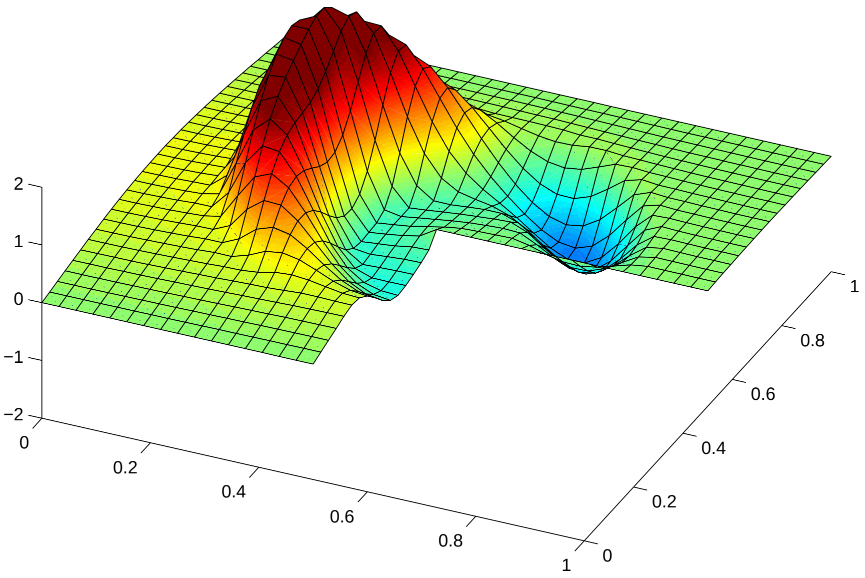
$$\mathbf{u}^h \in \mathcal{V}^h = \underbrace{\{\mathbf{v} \in H^1(\Omega) : \boldsymbol{\tau} \cdot \mathbf{v}|_{\partial\Omega} = 0, \mathbf{v}|_K \in P_1(K) \forall K \in \mathcal{T}^h\}}_{\text{continuous piecewise linears}}$$

- if $\alpha < 1$ then $\mathbf{u} \in H(div) \cap H(curl) \backslash H^1(\Omega)$, and the method will not converge to the right solution.
- if $\alpha \in [1, 2)$ then $\mathbf{u} \in H^1(\Omega) \backslash H^2(\Omega)$, and the method will converge slowly.
- if $\alpha \geq 2$ then $\mathbf{u} \in H^2(\Omega)$, and the method will converge at optimal $O(h)$.

The Pollution Effect, $\alpha = 2/3$

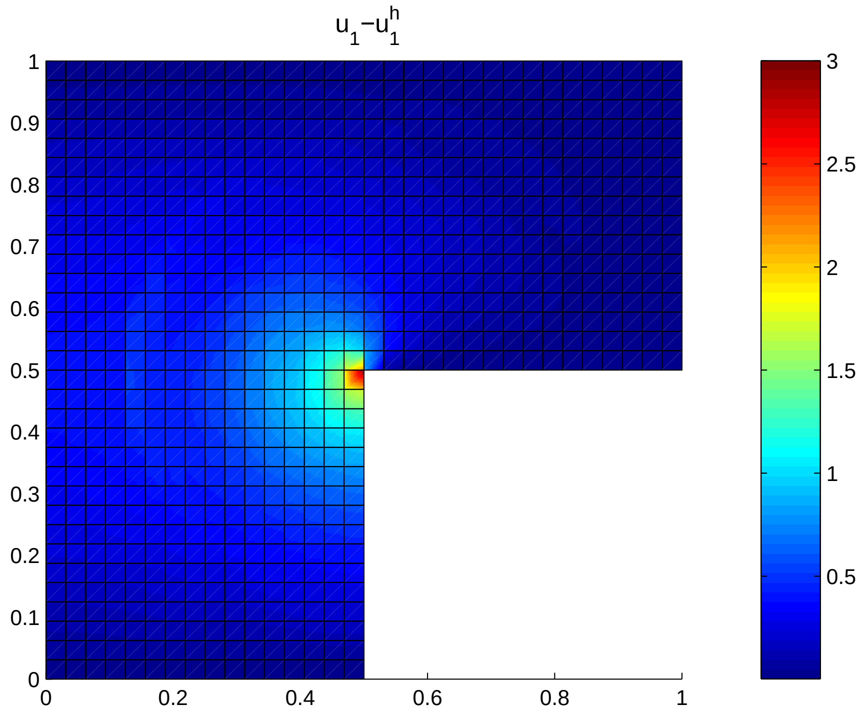


Exact component $\mathbf{u}_1$



$\mathbf{u}_1^h$ computed by standard LS

The Pollution Effect, $\alpha = 2/3$



$\mathbf{u}_1 - \mathbf{u}_1^h$

Standard LS

- Significant global errors persist due to gap between solution space and approximation space.

H(div) Formulation of Poisson

$$\begin{cases} \nabla p - \boldsymbol{\sigma} = 0, & \Omega \\ \nabla \cdot \boldsymbol{\sigma} = f, & \Omega \\ p = 0, & \partial\Omega \\ \boldsymbol{\tau} \cdot \boldsymbol{\sigma} = 0, & \partial\Omega \end{cases} \qquad G(p, \boldsymbol{\sigma}; f) = \|\nabla p - \boldsymbol{\sigma}\|^2 + \|\nabla \cdot \boldsymbol{\sigma} - f\|^2$$

$p \in H^1(\Omega)$ and $\boldsymbol{\sigma} \in H(div)$ naturally

$\Longrightarrow$ use FE spaces that conform to these...

Raviart-Thomas space of order 0: $RT_0(K) = P_0(K)^d + \mathbf{x} P_0(K)$.

$$p^h \in \mathcal{V}^h = \{q \in H^1(\Omega) : q|_{\partial\Omega} = 0, q|_K \in P_1(K) \; \forall \; K \in \mathcal{T}^h\}$$

$$\boldsymbol{\sigma}^h \in \mathcal{W}^h = \{\boldsymbol{\tau} \in H(div) : \boldsymbol{\tau}|_K \in RT_0(K) \; \forall \; K \in \mathcal{T}^h\}$$

This formulation leads to

$$\|p - p^h\|_1 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\| + \|\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}^h)\| \sim O(h^\alpha)$$

- $\alpha \geq 1 \implies$ globally optimal convergence
- $\alpha < 1 \implies$ globally slow convergence

Summary so far...

Least-squares methods work well when full regularity can be guaranteed...

- div-curl methods in H^1 spaces require $\alpha \geq 2$
- H(div) methods in $H^1 \times H(div)$ spaces require $\alpha \geq 1$

Intuition: The pollution effect is caused by the method paying too much attention to the solution where it is difficult to approximate. By shifting more attention away from the singularity we can recover optimal convergence away from the singularity. Near the singularity we can expect no better than the FE space will allow, but our approximation should be as good as the interpolant.

Our Approach: Weighted norm minimization

Again consider solving $Lu = f$ in some nonsmooth Ω, but minimize

$$G(u;f) = \|w(Lu - f)\|^2$$

where w is some weight function of the form $w = r^\beta$ near each singular point.

Define the weighted Sobolev norm and space:

$$\|\mathbf{u}\|_{k,\beta} = (\sum_{|j|\leq k} \|r^{\beta+j-k} D^j \mathbf{u}\|^2)^{\frac{1}{2}},$$

$$H^k_\beta(\Omega) = \{\mathbf{u} : \ \|\mathbf{u}\|_{k,\beta} < \infty\}.$$

For example:

$$\|\mathbf{u}\|^2_{1,\beta} = \|r^\beta \nabla \mathbf{u}\|^2 + \|r^{\beta-1}\mathbf{u}\|^2$$

These norms induce homogenous weighted spaces, or “Babuška-Kondratiev" spaces, as opposed to nonhomogenous norms which have the same weight for each term. These spaces are natural for our analysis.

Example 1: div-curl Formulation of Poisson

Let $L = \begin{pmatrix} \nabla\cdot \\ \nabla\times \end{pmatrix}$, and $\mathbf{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$.

$$\begin{aligned} G(\mathbf{u};\mathbf{f}) &= \|r^{\beta}(\nabla\cdot\mathbf{u} - f)\|^2 + \|r^{\beta}\nabla\times\mathbf{u}\|^2 \\ &= \|\nabla\cdot\mathbf{u} - f\|_{0,\beta}^2 + \|\nabla\times\mathbf{u}\|_{0,\beta}^2 \\ &= \|L\mathbf{u} - \mathbf{f}\|_{0,\beta}^2 \end{aligned}$$

We (Lee, Manteuffel, Westphal) show the equivalence

$$G(\mathbf{u};\mathbf{0})^{1/2} \sim \|\mathbf{u}\|_{1,\beta} \text{ for } \beta < |1-\alpha|.$$

div-curl Theory

Theorem (Lee, Manteuffel, Westphal) Let Ω be a polygonal domain with one corner admitting a singularity of power α and assume $\mathbf{f} \in L^2(\Omega)$. Let $\mathbf{u} \in H^\alpha(\Omega)$ satisfy $L\mathbf{u} = \mathbf{f}$. If $\mathbf{u}^h \in \mathcal{V}^h$ is chosen to minimize the weighted functional,

$$G_w(\mathbf{u}^h; \mathbf{f}) = \min_{\mathbf{v}^h \in \mathcal{V}^h} \|L\mathbf{v}^h - \mathbf{f}\|_{0,\beta}^2,$$

for $1 - \alpha < \beta < min(1 + \alpha, 2 - \alpha)$, then the approximation error, $\mathbf{u} - \mathbf{u}^h$, satisfies the following bounds:

$$(1) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{1,\beta} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta-1})$$

$$(2) \qquad G_w(\mathbf{u} - \mathbf{u}^h; \mathbf{0})^{\frac{1}{2}} \le Ch^{\alpha+\beta-1}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta-1})$$

$$(3) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{0,\beta} \le Ch^{\alpha+\beta}\|\mathbf{u}\|_{\alpha+\beta,\beta}, \qquad O(h^{\alpha+\beta})$$

$$(4) \qquad \|\mathbf{u} - \mathbf{u}^h\|_{0} \le Ch^{\alpha}(\|\mathbf{u}\|_{\alpha} + \|\mathbf{u}\|_{\alpha+\beta,\beta}). \qquad O(h^{\alpha})$$

div-curl Numerical Test Problem

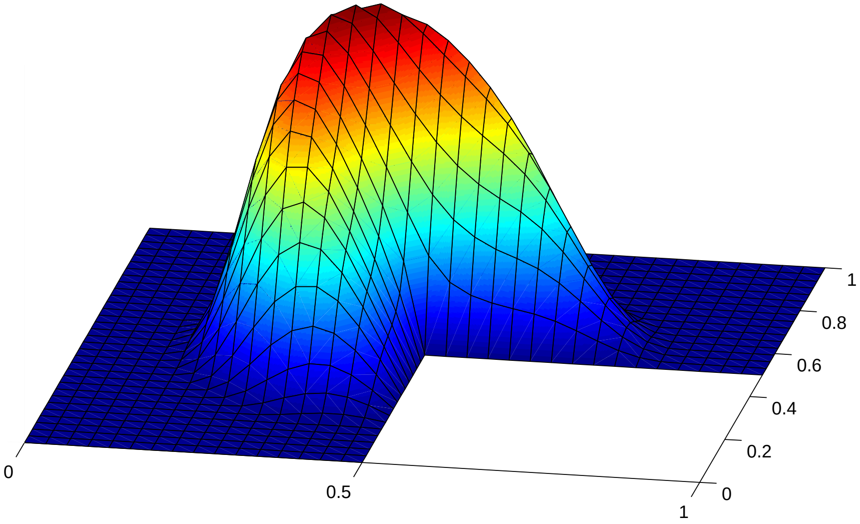
Let

$p(r,\theta) = \delta(r) r^{\frac{2}{3}} \sin(2\theta/3)$,

$\Omega = (0,1)^2 \backslash (0.5,1) \times (0,0.5)$.

Define $f = \Delta p$.

$$f \in L^2(\Omega),\ \ p \notin H^2(\Omega)$$



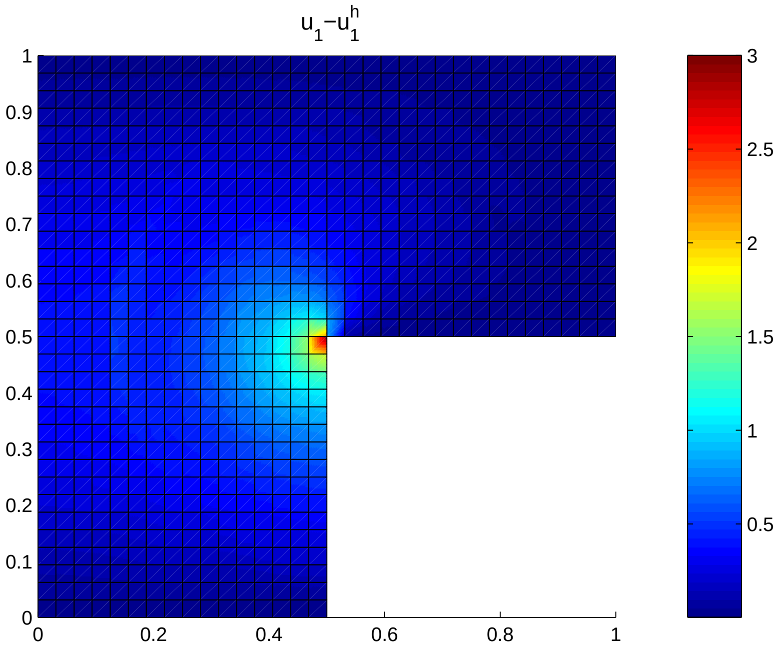
$p(r,\theta)$

We consider minimizing

$$G_w(\mathbf{u}^h;\mathbf{f}) = \|L\mathbf{u}^h - \mathbf{f}\|^2_{0,\beta}$$

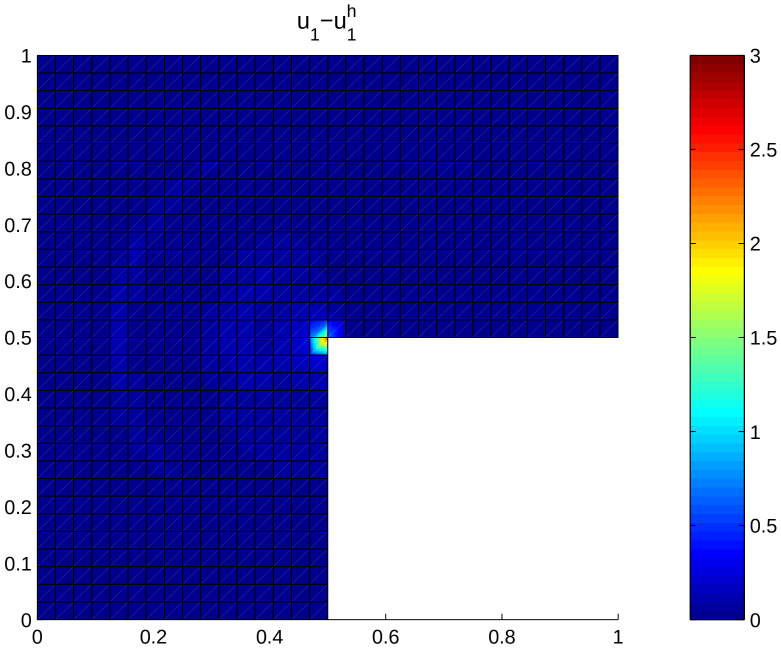
over the space of bilinear finite elements of uniform meshsize h.

Reduction of the Pollution Effect



$\mathbf{u}_1 - \mathbf{u}_1^h$

Standard LS, No weighting



$\mathbf{u}_1 - \mathbf{u}_1^h$

Weighted-norm LS, $\beta = 4/3$

Example 2: H(div) Formulation for Scalar Elliptic Problems

$$\begin{cases} -\nabla \cdot A\nabla p + \mathbf{b} \cdot \nabla p + cp = f, & \Omega \\ p = 0, & \partial\Omega \end{cases}$$

Assume A s.p.d. and denote $Xp := \mathbf{b} \cdot \nabla p + cp$. Define $\boldsymbol{\sigma} = -A\nabla p$ and

$$G(p, \boldsymbol{\sigma}; f) = \|A^{-1/2}(\boldsymbol{\sigma} + A\nabla p)\|_{0,\beta_1}^2 + \|\nabla \cdot \boldsymbol{\sigma} + Xp - f\|_{0,\beta_2}^2$$

for

$$p \in \{q \in H_{\beta_1}^1(\Omega) : q|_{\partial\Omega} = 0\}$$

$$\boldsymbol{\sigma} \in \{\boldsymbol{\tau} \in H_{\beta_1}^0(\Omega) : \nabla \cdot \boldsymbol{\tau} \in H_{\beta_2}^0(\Omega)\}$$

H(div) Approach: Weighted Norms

Define

$$|||(p, \boldsymbol{\sigma})|||_{\beta_1,\beta_2} := \left(\|p\|^2_{1,\beta_1} + \|\boldsymbol{\sigma}\|^2_{0,\beta_1} + \|\nabla \cdot \boldsymbol{\sigma}\|^2_{0,\beta_2}\right)^{1/2}$$

For $\beta_1 \leq \beta_2 \leq \beta_1 + 1$ and $\beta_1 \leq |\alpha|$ we (Cai, Westphal) show that

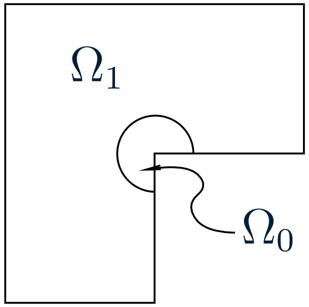
$$G(p, \boldsymbol{\sigma}; 0)^{1/2} \sim |||(p, \boldsymbol{\sigma})|||_{\beta_1,\beta_2}$$

and minimizing over the mixed $H^1 \times H(div)$ spaces leads to:

$$|||(p - p^h, \boldsymbol{\sigma} - \boldsymbol{\sigma}^h)|||_{\beta_1,\beta_2} \leq ch^{\alpha+\beta_1},$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\| \leq ch^{\alpha}.$$

Numerical Results: Dirichlet on L-shaped Domain



Convergence rates for $\alpha = 2/3, \quad \beta_2 = 0.3$

$G(p, \boldsymbol{\sigma}; f)^{1/2}$

β_1	Ω	Ω_0	Ω_1
0.0	0.67	0.65	1.02
0.1	0.76	0.75	1.01
0.2	0.85	0.83	1.00
0.3	0.92	0.90	1.00

$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^h\|$

β_1	Ω	Ω_0	Ω_1
0.0	0.65	0.64	1.27
0.1	0.64	0.64	1.25
0.2	0.63	0.63	1.19
0.3	0.62	0.62	1.14

Conclusions

- Least-squares methods work well, but often have higher regularity requirements than other methods. A weighted-norm least-squares approach is one way to recover from this.
- Directly applies to other applications (fluids, elasticity, nonlinear problems, etc.)
- The method is flexible and trivially implemented into existing LS code.
- It doesn't require specific knowledge of the singular functions, only an estimate of the severity.
- Theory and computation both generalize to 3-d.
- It can be used in conjuction with adaptive refinement.
- Weighted norm idea generalizes to other issues (high convection, e.g.).