

Problem of the Fortnight 12

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We must have

and

~~Either~~ $ab - c = 1$, $ac - b = 1$, ~~or~~ $bc - a = 1$. If we were to subtract for example the second equation from the first, we would get $(b+1)(a-c) = 0$ and equations of similar form for the remaining subtractions. Let's first consider the possibility that $b = -1$. Then $ac - (-1) = 1$ and $ac = 0$. This gives us solutions $(0, -1, -1)$, $(-1, -1, 0)$, and by symmetry $(-1, 0, -1)$. Now consider the possibility that $a = c$. Then $b = a^2 - 1 = 1 + \frac{1}{a}$. Setting them equal and solving for zero we get $a^3 - 2a - 1 = (a+1)(a^2 - a - 1) = 0$. We've already talked about when $a = -1$, so let's consider that case when $(a^2 - a - 1) = 0$. Using the quadratic equation we get $a = \frac{1}{2}(1 \pm \sqrt{5}) = c$. By substituting b back in we also get $b = a = c = \frac{1}{2}(1 \pm \sqrt{5})$. This gives us two more solutions: $(\frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}), \frac{1}{2}(1 + \sqrt{5}))$ and $(\frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 - \sqrt{5}), \frac{1}{2}(1 - \sqrt{5}))$.