Derivatives of the identity functor

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Motivation

Goal: to find examples of categories in which the Goodwillie derivatives of the identity functor are the objects of an operad.

Known examples:

- Based spaces (Ching)
- Spectra (Arone-Ching)
- L_n^f-local spaces (Bauer-Sadofsky)
- $\mathcal{DGL}_r$ over $\mathbb{Q}$ (Walter)

Main theorem

Theorem (in progress)

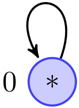
If $\mathcal{M}$ is a pointed, left proper, simplicial, cellular or combinatorial model category in which finite homotopy limits commute with countable directed homotopy colimits, and if there is $n \in \mathbb{Z}_{\geq 0}$ such that $\Sigma^n X \simeq 0$ for all $X \in Ob(\mathcal{M})$, then the derivatives of the identity functor on $\mathcal{M}$ exist and are trivial.

Model structure on $\mathcal{C}at$

We use the canonical model structure on $\mathcal{C}at$, the category of pointed small categories. Define a functor to be

- a weak equivalence if it is an equivalence of categories (equivalently, if it is fully faithful and essentially surjective).
- a cofibration if it is injective on objects.
- a fibration if it has the right lifting property with respect to acyclic cofibrations.

The zero object in $\mathcal{C}at$ is the category with one object and only the identity morphism.



Model structure on $\mathcal{C}at$

- Every object in $\mathcal{C}at$ is cofibrant and fibrant.
- $\mathcal{C}at$ is cofibrantly generated.
- $\mathcal{C}at$ is left proper.

Suspensions

Definition (Quillen)

Given a cofibrant object A in a pointed category $\mathcal{M}$, ΣA is the colimit of the diagram

$$\begin{tikzcd}
A \vee A \arrow[r] \arrow[d] & \mathsf{Cyl}(A) \\
0 &
\end{tikzcd}$$

where the cylinder object $\mathsf{Cyl}(A)$ satisfies:

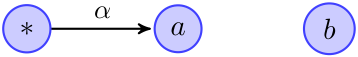
$$\begin{tikzcd}
A \arrow[r] \arrow[rrr, bend right, "="'] & A \vee A \arrow[r, hook] & \mathsf{Cyl}(A) \arrow[r, "\simeq"] & A
\end{tikzcd}$$

Cylinder objects in $\mathcal{C}at$

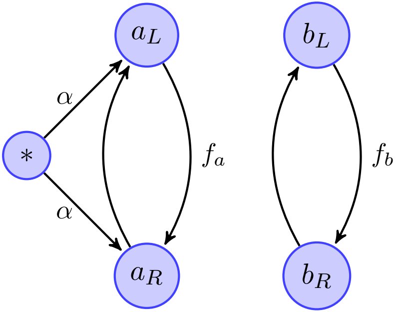
Proposition

Cyl$(\mathcal{C}) = (\mathcal{C} \times I)/(* \times I)$, *where* I *is the category with two objects and a unique isomorphism between them.*

Example: If $\mathcal{C}$ is



then Cyl$(\mathcal{C})$ is



Suspensions in $\mathcal{C}at$

Proposition

Given a category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one object A, and End$(A) = F(n-1)$, where n is the number of components of $\mathcal{C}$.

Proof: In taking the pushout of

$$0 \leftarrow \mathcal{C} \vee \mathcal{C} \rightarrow \mathsf{Cyl}(\mathcal{C}),$$

we identify all the objects of Cyl$(\mathcal{C})$ with the unique object of 0 and all the morphisms of $\mathcal{C} \vee \mathcal{C}$ with the unique morphism of 0. Given $\beta : a \rightarrow b$,

$$f_b \circ \beta = \beta \circ f_a \Rightarrow im(f_b) \circ id_{\mathbf{0}} = id_{\mathbf{0}} \circ f_a \Rightarrow im(f_a) \simeq im(f_b)$$

in the pushout. Thus we end up with one isomorphism for each component that does not contain *.

□

Suspension iterated

Corollary

For any category $\mathcal{C}$, $\Sigma^2\mathcal{C} \simeq 0$.

Proof:

For any category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one component. Thus $\Sigma^2\mathcal{C}$ has one object A and $\text{End}(A) \simeq F(0) \simeq id_A$.

☐

The category of spectra

Definition (Hovey)

Suppose G is a left Quillen endofunctor of a model category $\mathcal{M}$. A spectrum X is a sequence $\{X_n\}_{n\geq 0}$ of objects of $\mathcal{M}$ together with structure maps $\sigma : GX_n \to X_{n+1}$ for all n.

A map of spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ so that the following diagram commutes for all n:

$$\begin{tikzcd}
GX_n \arrow[r, "\sigma_X"] \arrow[d, "Gf_n"'] & X_{n+1} \arrow[d, "f_{n+1}"] \\
GY_n \arrow[r, "\sigma_Y"] & Y_{n+1}
\end{tikzcd}$$

Denote the category of spectra by $Sp^{\mathbb{N}}(\mathcal{M}, G)$.

The projective model structure

In the **projective model structure**, $f : X \to Y$ is a weak equivalence or fibration in $Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ if and only if $f_n : X_n \to Y_n$ is a weak equivalence or fibration in $\mathcal{C}at$ for all $n \geq 0$.

However, $\Sigma : Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma) \to Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ is not a Quillen equivalence.

Definition

A Quillen adjunction $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is a Quillen equivalence if, for every cofibrant object X in $\mathcal{C}$ and for every fibrant object Y in $\mathcal{D}$, $L(X) \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $X \to R(Y)$ is a weak equivalence in $\mathcal{C}$.

Bousfield localization of model categories

Definition (Hovey)

Let S be a set of maps in a model category $\mathcal{M}$.

1. An $\mathcal{S}$-local object of $\mathcal{M}$ is a fibrant object W such that, for every $f : A \to B$ in $\mathcal{S}$, the induced map
$$map(B, W) \to map(A, W)$$
is a weak equivalence of simplicial sets.
2. An $\mathcal{S}$-local equivalence is a map $g : A \to B$ in $\mathcal{M}$ such that the induced map
$$map(B, W) \to map(A, W)$$
is a weak equivalence of simplicial sets for all $\mathcal{S}$-local objects W.

The stable model structure

The **stable model structure** on $Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ has

- the same cofibrations as in the projective model structure.
- weak equivalences are $\mathcal{S}$-local equivalences, where

$$\mathcal{S} = \{F_{n+1}\Sigma Q\mathcal{C} \xrightarrow{s_n^{Q\mathcal{C}}} F_n Q\mathcal{C}\}.$$

Here

- $F_n(\mathcal{C})$ is the spectrum $\{\ldots, 0, \mathcal{C}, \Sigma\mathcal{C}, \Sigma^2\mathcal{C}, \ldots\}$ that is 0 below level n.
- $\mathcal{C}$ is a domain or codomain of a generating cofibration of $\mathcal{C}at$.
- $s_n^{Q\mathcal{C}}$ is adjoint to the identity on $\Sigma Q\mathcal{C}$.

Existence of Bousfield localization

Remark:
Hovey requires the model category to be left proper and cellular, but only to guarantee that the Bousfield localization exists.

$Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ is **not** cellular (cofibrations are not effective monomorphisms), but it is combinatorial.

Theorem (Smith)

If $\mathcal{M}$ is left proper and $\mathbf{X}$-combinatorial ($\mathbf{X}$ some universe), and S is an $\mathbf{X}$-small set of homotopy classes of morphisms of $\mathcal{M}$, the left Bousfield localization $L_S\mathcal{M}$ of $\mathcal{M}$ along any set representing S exists and satisfies the following conditions.

1. *The cofibrations of $L_S\mathcal{M}$ are exactly those of $\mathcal{M}$.*
2. *The weak equivalences of $L_S\mathcal{M}$ are the S-local equivalences.*

$Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ is trivial

Theorem (Hovey)

$\Sigma : Sp^N(\mathcal{C}at, \Sigma) \to Sp^N(\mathcal{C}at, \Sigma)$ *is a Quillen equivalence with respect to the stable model structure.*

Proposition

Every object in $Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ *is weakly equivalent to the zero object in* $Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$.

Proof:

Σ a Quillen equivalence implies

$$\Sigma^2(X) \simeq 0 \Leftrightarrow X \simeq \Omega^2(0) \simeq 0$$

for all (cofibrant) $X \in \mathrm{Ob}(Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma))$.

Connection to Goodwillie calculus

The derivatives of the identity functor on $\mathcal{C}at$ should be objects in the category of Σ_n-equivariant spectra on $\mathcal{C}at$.

Since $Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ is trivial, Σ_n–$Sp^{\mathbb{N}}(\mathcal{C}at, \Sigma)$ is trivial, so the derivatives of the identity functor on $\mathcal{C}at$ (if they exist) are trivial.

Theorem (in progress)

If $\mathcal{M}$ is a pointed, left proper, simplicial, cellular or combinatorial model category in which finite homotopy limits commute with countable directed homotopy colimits, and if there is $n \in \mathbb{Z}_{\geq 0}$ such that $\Sigma^n X \simeq 0$ for all $X \in$ Ob$(\mathcal{M})$, then the derivatives of the identity functor on $\mathcal{M}$ exist and are trivial.

What does this mean?

We have found an example of a category in which doing Goodwillie calculus is possible but not helpful.

Analogy to regular calculus: consider

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Then $f^{(n)}(0) = 0$ for all n, so $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 \neq f(x)$ for $x \neq 0$.

Similarly, the Taylor tower for the identity on $\mathcal{C}at$ only converges to $Id(X)$ if $X = 0$.

Category of directed graphs

Definition

A directed graph has

- a set of vertices V and
- a set of edges E, where the source and target of each edge is specified.

The category $\mathcal{G}ph$ is the category of directed graphs and morphisms between them.

The zero object in $\mathcal{G}ph$ is the graph with one vertex and one edge.

Model structure on $\mathcal{G}ph$

Theorem (Bisson-Tsemo)

There is a model structure on $\mathcal{G}ph$ in which

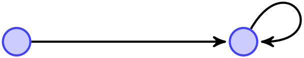
- *the weak equivalences are morphisms which are bijections of n-cycles for all $n \geq 1$.*
- *the fibrations are surjectings. A graph morphism $f : X \to Y$ is a surjecting when the set of edges in X with source vertex x surjects onto the set of edges in Y with source vertex $f(x)$ for all vertices $x \in X$.*
- *the cofibrations have the left lifting property with respect to all acyclic fibrations.*

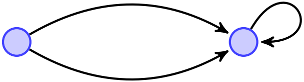
Cylinder objects in $\mathcal{G}ph$

Proposition

Cyl$(A) = (A_L \vee A_R)/\sim$, *where* $v_L \sim v_R$ *for all vertices* $v \in A$ *and* $e_L \sim e_R$ *if* e *is part of a cycle.*

Example:

If A is  then Cyl(A) is



Cylinder objects in $\mathcal{G}ph$

Proposition

Cyl$(A) = (A_L \vee A_R)/\sim$, *where* $v_L \sim v_R$ *for all vertices* $v \in A$ *and* $e_L \sim e_R$ *if* e *is part of a cycle.*

Proof: We need to show

$$\begin{tikzcd}
A \arrow[r] \arrow[rrr, bend right, "="] & A \vee A \arrow[r, hook] & \mathrm{Cyl}(A) \arrow[r, "\simeq"] & A
\end{tikzcd}$$

Suspensions in $\mathcal{G}ph$

Theorem

For any graph A, $\Sigma A \simeq 0$.

Proof:

In taking the pushout of the diagram below, we glue all the vertices of $\mathsf{Cyl}(A)$ to the vertex of 0 and all the edges of $\mathsf{Cyl}(A)$ to the edge of 0.

$$\begin{tikzcd}
A \vee A \arrow[r] \arrow[d] & \mathsf{Cyl}(A) \\
0 &
\end{tikzcd}$$

□

Next steps

- $\mathcal{G}ph$ is left proper and combinatorial. It follows that $Sp^{\mathbb{N}}(\mathcal{G}ph, \Sigma)$ is, also.
- This means we can define the stable model structure on $Sp^{\mathbb{N}}(\mathcal{G}ph, \Sigma)$.
- The derivatives of the identity functor on $\mathcal{G}ph$ will be trivial if they exist.

Future work

I am currently looking at $\mathbb{Z}/2$-Mackey functors into $\mathcal{DGL}_r$.

Knowing the derivatives of the identity functor on this category would tell us about $\mathbb{Z}/2$-equivariant rational spaces.

Thanks for your attention!

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