Suspensions of graphs and categories

Deb Vicinsky

University of Oregon

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Motivation

We are interested in finding examples of categories in which the Goodwillie derivatives of the identity functor have an operad structure (that we understand).

Known results:

- Ching showed that the derivatives of the identity functor on based spaces form an operad in spectra.
- Bauer-Sadofsky did the same for L_n^f-local spaces.
- Walter showed that the derivatives of the rational identity functor in $\mathcal{DGL}_r$ are the spaces of the Lie operad. This implies the result for the identity functor on rational spaces.

Category of directed graphs

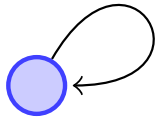
Definition

A directed graph has

- a set of vertices V and
- a set of edges E, where the source and target of each edge is specified.

The category $\mathcal{G}ph$ is the category of directed graphs and morphisms between them.

To make this category pointed, we consider only graphs with a choice of morphism from C_1, the graph with one vertex and one edge.



Model structure on $\mathcal{G}ph$

Theorem (Bisson-Tsemo 2008)

There is a model structure on $\mathcal{G}ph$ in which

- *the weak equivalences are morphisms which are bijections of n-cycles for all $n \geq 1$.*
- *the fibrations are surjectings. A graph morphism $f : X \to Y$ is a surjecting when the set of edges in X with source vertex x surjects onto the set of edges in Y with source vertex $f(x)$ for all vertices $x \in X$.*
- *the cofibrations have the left lifting property with respect to all acyclic fibrations.*

Definition (Quillen)

Given $A \in \mathrm{Ob}(\mathcal{C})$ for a pointed category $\mathcal{C}$, the suspension of A is the colimit of the diagram

$$\begin{tikzcd} A \vee A \arrow[r] \arrow[d] & \mathrm{Cyl}(A) \\ * & \end{tikzcd}$$

where the cylinder object $\mathrm{Cyl}(A)$ satisfies

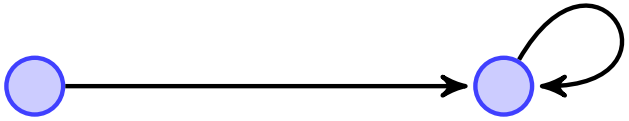
$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend right, "="'] & A \vee A \arrow[r, hook] & \mathrm{Cyl}(A) \arrow[r, "\simeq"] & A \end{tikzcd}$$

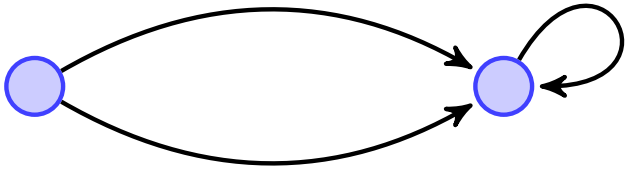
Cylinder object in $\mathcal{G}ph$

Proposition

Cyl$(A) = (A_L \vee A_R)/\sim$, *where* $v_L \sim v_R$ *for all vertices* $v \in A$ *and* $e_L \sim e_R$ *if* e *is part of a cycle.*

Example:

If A is  then Cyl(A) is



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Proof: We need to show

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend right, "="'] & A \vee A \arrow[r, hook] & \mathsf{Cyl}(A) \arrow[r, "\simeq"] & A \end{tikzcd}$$

Let $f : X \to Y$ be an acyclic fibration. Consider a commutative square

$$\begin{tikzcd}
A \vee A \arrow[r, "j"] \arrow[d, "q"] & X \arrow[d, "f"] \\
\mathrm{Cyl}(A) \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

Suppose $e \in \mathrm{Cyl}(A)$ be part of a cycle. Then $g(e)$ is part of a cycle in Y, which lifts to a unique cycle in X since f is a weak equivalence. Define a lift h on e by $h(e) = f^{-1}(g(e))$. Edges that are part of a cycle in $\mathrm{Cyl}(A)$ come from cycles e_L and e_R in $A \vee A$. By the commutativity of the square, since $gq(e_L) = gq(e_R)$, we have $fj(e_L) = fj(e_R)$, and f a weak equivalence implies $j(e_L) = j(e_R)$. Therefore h makes both triangles commute.

$$\begin{tikzcd}
A \vee A \arrow[r, "j"] \arrow[d, "q"] & X \arrow[d, "f"] \\
\mathsf{Cyl}(A) \arrow[r, "g"] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

Now suppose $e \in \mathsf{Cyl}(A)$ is not part of a cycle. Define a lift by $hq(e) = j(e)$. By the commutativity of the diagram, $gq(e) = fj(e)$. Thus $gq(e) = fhq(e)$. Since $q^{-1}(q(e)) = e$, we conclude $g = fh$ for all edges of $\mathsf{Cyl}(A)$ that are not part of a cycle.

$\square$

Suspension in $\mathcal{G}ph$

Theorem

For any graph A, $\Sigma A \simeq C_1$.

Proof:

In taking the pushout of the diagram below, we glue all the vertices of $\mathsf{Cyl}(A)$ to the vertex of C_1 and all the edges of $\mathsf{Cyl}(A)$ to the edge of C_1.

$$\begin{tikzcd} A \vee A \arrow[r] \arrow[d] & \mathsf{Cyl}(A) \\ C_1 & \end{tikzcd}$$

$\square$

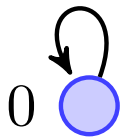
Connection to Goodwillie calculus

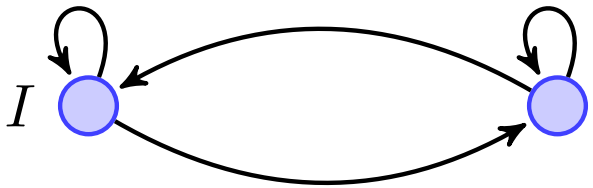
Since the stable category associated to $\mathcal{G}ph$ is trivial, the derivatives of the identity functor on $\mathcal{G}ph$ are trivial.

Model structure on $\mathcal{C}at$

We use the canonical model structure on $\mathcal{C}at$, the category of small categories and functors. Define a functor to be

- a weak equivalence if it is an equivalence of categories (equivalently, if it is fully faithful and essentially surjective).
- a cofibration if it is injective on objects.
- a fibration if it has the right lifting property with respect to acyclic cofibrations (equivalently, if it has the right lifting property with respect to the inclusion $0 \hookrightarrow I$).





Cylinder object in $\mathcal{C}at$

Again, we define the cylinder object $\mathsf{Cyl}(\mathcal{C})$ using the following factorization:

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[rrr, bend right, "="'] & \mathcal{C} \vee \mathcal{C} \arrow[r, hook] & \mathsf{Cyl}(\mathcal{C}) \arrow[r, "\simeq"] & \mathcal{C} \end{tikzcd}$$

Proposition

$\mathsf{Cyl}(\mathcal{C})$ has the same objects as $\mathcal{C}_L \vee \mathcal{C}_R$.

The morphisms of $\mathsf{Cyl}(\mathcal{C})$ include the morphisms of $\mathcal{C}_L$ and $\mathcal{C}_R$ as well as:

For each $A, B \in Ob\, \mathcal{C}$ and for each morphism $\alpha : A \to B$, there are isomorphisms $f_A : A_L \to A_R$ and $f_B : B_L \to B_R$ so that $f_B \circ \alpha = \alpha \circ f_A$ and $f_B^{-1} \circ \alpha = \alpha \circ f_A^{-1}$. If $A = *$, $f_B \circ \alpha = \alpha$.

Cylinder object in $\mathcal{C}at$

Example: If $\mathcal{C}$ is $\begin{tikzcd} * \arrow[r, "\alpha"] & A \end{tikzcd}$ then $\mathrm{Cyl}(\mathcal{C})$ is

$$\begin{tikzcd}
 & A_L \arrow[dd, bend left, "f_A"] \\
* \arrow[ur, "\alpha"] \arrow[dr, "\alpha"'] & \\
 & A_R \arrow[uu, bend left]
\end{tikzcd}$$

Suspension in $\mathcal{C}at$

Theorem

Given a category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one object A, and End$(A) = F(n-1)$, where n is the number of components of $\mathcal{C}$.

Proof:

In taking the pushout of the diagram below, we identify all the objects of Cyl$(\mathcal{C})$ with the unique object of 0 and all the morphisms of $\mathcal{C}_L$ and $\mathcal{C}_R$ with the unique morphism of 0. If A and B are in a component of $\mathcal{C}$ that does not contain the basepoint, then the compatibility condition implies $f_A \sim f_B$ in the pushout.

$$\begin{tikzcd} \mathcal{C} \vee \mathcal{C} \arrow[r] \arrow[d] & \mathrm{Cyl}(\mathcal{C}) \\ 0 & \end{tikzcd}$$

Connection to Goodwillie calculus

Corollary

For any category $\mathcal{C}$, $\Sigma^2\mathcal{C} \simeq 0$.

Proof:

For any category $\mathcal{C}$, $\Sigma\mathcal{C}$ has one component. Thus $\Sigma^2\mathcal{C}$ has one object A and $\mathrm{End}(A) \simeq F(0) \simeq id_A$.

$\square$

This implies that all the derivatives of the identity functor on $\mathcal{C}at$ are trivial.

Future work

I am currently looking at $\mathcal{DGL}_r$ with $\mathbb{Z}/2$ action.

Knowing the derivatives of the identity functor on this category would tell us about equivariant rational spaces.

Thanks for your attention!

References

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