A Brief Introduction to Mathematical Writing

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1 Introduction

Mathematics is not just about computation. It is about ideas, knowledge, learning, understanding, and perception. We can see this in the linguistic heritage of the word *mathematics*. The Greek verbal base μαθ- (*math-*) derives from the verb μανθάνω (*manthano*), which means "I know". It appears in the nouns ἡ μάθησις (*mathesis*—the act of learning, perceiving, etc. or the desire to learn) and τό μάθημα (*mathema*—the lesson learned, knowledge), and Aristotle and Thucydides used the plural of τό μάθημα (τά μαθήματα—*mathemata*) for knowledge, especially scientific knowledge. In fact Plato used the same work (ὁ μαθηματικός—*mathematikos*) to mean someone disposed to learn as his student Aristotle later used to mean a mathematician.

Communicating your ideas and knowledge through writing and other media is a very important skill to mathematicians. Mathematics often includes concepts and ideas that you cannot easily express using equations and formulae. Mathematicians must write their ideas down to add to the body of mathematical knowledge. They must communicate their thought processes to non-mathematicians, often their employers. The ability to write clearly is as important a skill to mathematicians as solving equations.

Now that you are taking college mathematics classes, you already know far more mathematics than most people. How do you communicate your ideas to others?

Writing mathematics is not the same as showing your work. You do not write papers to demonstrate that you have done your work, but rather to demonstrate how well you understand the ideas and concepts. A list of calculations without any context or explanation demonstrates that you have spent time doing computations, but it omits ideas. It contains no mathematics.

Writing good, clear mathematical explanations will also help you improve your knowledge and understanding of the mathematical ideas and concepts you encounter. The act of writing the explanation will force you to think more carefully about what you are doing. This means clear, carefully-written mathematics will more likely be correct, and the process will help you learn and retain the concepts.

This paper discusses some of the basic ideas involved in writing a mathematical paper. For more information, you can consult reference books such as

Krantz [1997], Higham [1998], and Knuth et al. [1989]. You might also consult Turner [b] for a brief introduction to proofs. If you plan to do much mathematical writing, I suggest learning to use LaTeX, which is the standard tool to typeset papers in mathematics, physics, economics, and other disciplines. In this case, you might find Turner [a] for some basic LaTeX typesetting commands.

2 Mathematical Formulae

Mathematical writing contains one element that is not found in other writing: formulae. You must keep several things in mind when you incorporate mathematical formulae into your writing.

2.1 Combining Words and Formulae

You must incorporate your formulae and equations into your sentences using all the standard rules of English. Symbols may correspond to different parts of speech. They may form a complete sentence:

$$1 + 1 = 2.$$
$$3xy < -2.$$
$$5x \in \mathbb{R}.$$
$$9 - s \neq t.$$

In this example, the symbols "=", "<", "$\in$", and "$\neq$" act as verbs. On the other hand, an expression such as

$$5x^2y - 10y$$

contains no verb and thus is not a complete sentence. Read these expressions aloud to hear the difference.

You must contain formulae and equations in complete sentences with proper punctuation. In particular, every sentence must end with a period. Consult your style manual for the rules regarding other punctuation marks. Pay particular attention to the colon ":". Students often misuse it. It should only follow an independent clause that can stand on its own as a sentence. Consider this example:

> We can use the distance formula
>
> $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
>
> to determine the distance between any two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. For our example, $(x_1, y_1) = (-1, 16)$ and $(x_2, y_2) = (3, 1)$, so plugging these values into the distance formula tell us the distance between the two points:
>
> $$d = \sqrt{(3 - (-1))^2 + (1 - 16)^2} = \sqrt{4^2 + (-15)^2} = \sqrt{241}.$$

You should read your writing aloud exactly as you wrote it, including all the equations and formulae. You will hear the sentence fragments and grammatical errors more easily than you will see them.

You should also put important or long formulae on separate lines. This will make your mathematics easier to read and understand. Notice how the equations tend to get lost in the writing in the following example:

> We can use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to determine the distance between any two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$. For our example, $(x_1, y_1) = (-1, 16)$ and $(x_2, y_2) = (3, 1)$, so plugging these values into the distance formula tell us the distance between the two points: $d = \sqrt{(3 - (-1))^2 + (1 - 16)^2} = \sqrt{4^2 + (-15)^2} = \sqrt{241}$.

2.2 Symbols and Words

You must use symbols and words correctly. For example, beginning students often misuse the "=" symbol to mean "the next step is" or "implies." Consider the example

$$3^{2x} - 2(3^x) = -1 = (3^x)^2 - 2(3^x) + 1 = 0 = (3^x - 1)^2 = 0 = 3^x = 1 = x = 0.$$

This example really says $-1 = 0 = 1$! Use the "=" symbol only when you want to say two things are equal.

Use symbols when they aide the reader to understand what you mean, but you should not overuse it. For example, the reader may find the sentence

> Therefore, x plus two is greater than or equal to zero.

hard to read. Instead, you should use

> Therefore, $x + 2 \geq 0$.

On the other hand, do not overuse symbols. Do not use short-cut symbols such as $\Rightarrow$, $\forall$, $\exists$, and $\therefore$. For example, the reader may find

> $x^2 - 1 = 0 \Rightarrow x = \pm 1$

harder to follow than

> $x^2 - 1 = 0$, which means $x = \pm 1$.

Be consistent in your symbols. You should not change the symbol for something in the middle of your paper, nor should you use one symbol to mean two different things. Consider the statement $n = 2^n$. There is no integer n for which this is true. Instead, you probably meant to write $n = 2^k$ for some integer k.

You may find subscripts useful to differentiate symbols. For example, we know we can write any odd integer as $n = 2k + 1$. You can write two odd integers as

$$n_1 = 2k_1 + 1 \text{ and } n_2 = 2k_2 + 1.$$

If you left off the subscripts on the ks, you would imply $n_1 = n_2$.

You should always typeset your symbols and formulae. This does not just mean typing the formulae into the computer using the keyboard, but instead to use the a mathematics mode to properly format the symbols and formulae. In LaTeX you should use math mode, and in Word you should always use the equation editor. For example, you should write x instead of x, and $x^2 - 1 = 0$ instead of `x^2 - 1 = 0`.

You should also structure your sentences to make your symbols and formulae stand out. You should always separate symbols of different formulae with words, and never start a sentence or clause with a symbol. The following sentences demonstrate bad practice:

> For all $x \in I$, $f(x) > 0$.
>
> f is a continuous function.

Instead, start the sentence or clause with a word:

> The function f is continuous.

3 Organization

Organize your paper to aide the reader understand what you are doing. This is particularly important in writing that is as technical and complex as mathematics tends to be.

You should begin your paper with an introduction. Introduce the problem you are solving and give some background describing why it is important. Motivate the readers to read your paper.

After introducing the problem, you should generally state the answer, even before you show how to solve it. This helps hook the readers before they get bogged down in the details, especially if the solution is messy or boring. On the other hand, you can explain the solution before giving the answer if the solution technique is even more interesting than the answer or if you want leave the readers in suspense.

You should introduce your solution with a preview of your solution before you go through all the details. This also helps hook the readers before they get bogged down in the details. It will also allow readers who might not understand as much mathematics as you an insight into what you are doing even if they cannot sift through all the details.

Be sure to organize your paper into paragraphs. Paragraphs let the reader know a new thought has begun. Try to keep paragraphs short. Paragraphs that are too long intimidate the reader and cause them to loose interest.

Use short sentences to help the reader stay focused on the ideas without getting confused in the process. Break complex sentences down into simpler sentences that contain less information. Avoid empty or meaningless words.

Clearly explain your solution in simple terms, and restate your explanation in complimentary ways to give your readers as many ways to think of it as

possible. Beautiful mathematical papers are the easiest to read. You may vary the content of your explanation depending upon your audience. In a course, you may provide a lot of detail to convince someone who knows a lot of mathematics that you too know what you are doing–or get partial credit if you do not. In other cases, you might avoid many of the details if you are trying to show someone who may not know as much mathematics that you got the right answer.

Be sure to define your symbols and terms when you first use them. For example, to say the Pythagorean Theorem is "$a^2 + b^2 = c^2$" is not enough. You must tell the readers what a, b, and c are. Instead, you might say:

> Let a right triangle have legs of length a and b and a hypotenuse of length c. Then $a^2 + b^2 = c^2$.

You might modify this organizational structure when you write the solution to a homework problem. You must still introduce the problem, but your goal is now to make your paper self-contained; the readers should be able to determine what you want to say without referring to a textbook or other reference to find the statement of the problem. On the other hand, because the solution to the problem is generally more important to your reader–the professor–than the answer, you might decide to wait to give the answer until after you give the solution.

4 Other Considerations

4.1 Standard Rules of English

All of the standard rules of English apply to your mathematical writing. Your papers should use correct grammar, spelling, and punctuation. Use the correct words: compare and contrast, affect and effect, that and which, etc.

Everyone should own a good style manual such as Diana Hacker's *A Pocket Style Manual* [Hacker, 2003] and *Rules for Writers* [Hacker, 2004]. Use it! Everyone should also read Strunk and White's short style manual [Strunk and White, 2000]. This 85-page book is a classic.

4.2 Formal Writing

Your mathematical papers are a form of formal writing. Avoid contractions. Do not use any irregular abbreviations or shorthand forms that do not conform to standard writing conventions. Do not use jargon.

4.3 Precision in Word Choices

Chose your words carefully. Many words have more precise meanings in mathematics than in everyday speech. The word "if" is a good example. Outside mathematics, you may interpret it to mean a two-way dependence, while in mathematical terms it denotes a strictly one-way dependence. For example,

consider the sentence "If we score, we will win the game." Similarly, we could have phrased the statement as "We will win the game if we score." These sentences say we will win if we score. Logically,

$$\text{we score} \Rightarrow \text{we win.}$$

These sentences do not say we will score if we win the game, nor do they say we will not win the game if we do not score. In other words, we may already be leading the opposing team and have possession of the ball in the final moments of the game. On the other hand, the sentence "We will win the game if and only if we win the game" denotes a mutual dependence. Logically, this sentence says

$$\text{we score} \iff \text{we win.}$$

This is the situation if we are down in the final moments of the game with a last-minute opportunity to score and take the lead, and the opposing team will not have time to score and regain the lead before time expires.

You should also avoid pronouns. Pronouns such as "it" and "they" can lead to confusion. Instead, give names to quantities and use those names. If you must use a pronoun, make sure its antecedent is clear, and make sure the pronoun agrees with its antecedent.

4.4 First Person Plural

Notice how we use the word "we". Mathematicians often use the first person plural in their writing. Contrary to what many may think, mathematicians do not use the "royal we" because they are egotistical–although few people would describe mathematicians as a group as humble. In fact, the truth is quite the opposite: the writer is attempting to *not* be egotistical. The writer wants to stress the participatory nature of mathematical writing. By using "we", mathematicians are trying to draw the readers into the writing and make them feel as though they are taking part in the discovery.

On the other hand, you might use the first person singular "I" when you want to impart to the reader some specific information about what you yourself know. For example, at the end of a paper you might say "At this time I do not know how to prove Conjecture A." Do not force your knowledge, or lack of it, onto the reader. Perhaps the reader *does* know how to prove the conjecture.

4.5 Active Voice

You should favor the active voice. The active voice tends to grab the readers' attention. The active voice indicates action, while the passive voice indicates inactivity. Mathematical writing is boring enough; keep your writing lively.

Often "to be" verbs such as "is" and "are" indicate the passive voice. Whenever you are tempted to use a to be verb, you should carefully consider whether or not you are using the passive voice. If you are, try rewriting the sentence to use the active voice.

Many writers use the first person such as "I" and "we" to avoid the passive voice. Be careful, though. Overusing the first person can turn off readers, too. Remember to allow other objects to be active: results can mean something, data can show something, etc.

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