Black Box Linear Algebra

An Introduction to Wiedemann's Approach

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Symbolic Computation

- Sometimes called Computer Algebra
- Symbols or exact arithmetic
- Not floating point numbers (numerical analysis)
- No round off errors
- Algorithms may not be compatible
  - Numerical Analysis
    - Find approximation quickly
    - May never find exact solution
  - Symbolic Computation
    - Find exact solution quickly
    - May never approximate

Probabilistic Algorithms

- Three types:
  - *Monte Carlo:* Always fast, probably correct
  - *Las Vegas:* Probably fast, always correct
    - Certificate
  - *BPP:* Bounded Probabalistic Polynomial Time
    - Probably fast, probably correct
    - Atlantic City?
- Schwartz-Zippel Lemma
  - Probability randomly choose root of polynomial

Solving Linear Systems

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & -2 & 3 \\ 0 & -3 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \\ 6 \\ -4 \end{bmatrix}$$

Gaussian Elimination

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -6 \\ 0 & 0 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \\ 6 \\ 2 \end{bmatrix} \implies x = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

- Row echelon form and back substitution
- Must know how matrix stored
- Matrix changed in calculation
- Sparse matrices may become dense

Black Box Matrix Model

- External view of matrix
- Only matrix-vector products allowed
- Independent of implementation
- Implementations efficient in time or space
- Not unique to symbolic computation
- Can compute Krylov sequence $(A^i v)_{i \in \mathbb{N}}$

Examples

Matrix	storage	time
Arbitrary matrix	n^2	$\mathrm{O}(n^2)$
Sparse matrix (η nonzero entries)	$\mathrm{O}(\eta)$	$\mathrm{O}(\eta)$
Hilbert matrix ($A^{[i,j]} = \frac{1}{i+j-1}$)	$\mathrm{O}(1)$	$\mathrm{O}(n)$
Toeplitz matrix	$\mathrm{O}(n)$	$\mathrm{O}(n \log(n) \operatorname{loglog}(n))$

Linearly Generated Sequences

Let $\mathbb{V}$ be a vector space over field $\mathbb{F}$.

A sequence

$$a = (a_i)_{i \in \mathbb{N}} \in \mathbb{V}^{\mathbb{N}}$$

is *linearly generated* if and only if there exist $m \in \mathbb{N}$ and

$$c_0, \ldots, c_m \in \mathbb{F}, \quad c_m \neq 0$$

such that for all $i \geq 0$

$$\sum_{j=0}^{m} c_j \, a_{i+j} = 0 \text{ or } a_{i+m} = -\frac{1}{c_m} \left(\sum_{j=0}^{m-1} c_j \, a_{i+j} \right)$$

Generating Polynomials

- The polynomial $f(\lambda) = \sum_{j=0}^{m} c_j \, \lambda^j$ generates a
- $$f \bullet a = \left(\sum_{j=0}^{m} c_j \, a_{i+j} \right)_{i \in \mathbb{N}} = (0)_{i \in \mathbb{N}} = 0 \in \mathbb{V}^{\mathbb{N}}$$
- $\mathbb{V}^{\mathbb{N}}$ is an $\mathbb{F}[\lambda]$-module

Minimal Generating Polynomial

- If $f \bullet a = 0$ and $g \in \mathbb{F}[\lambda]$, then

$$(g\, f) \bullet a = g \bullet (f \bullet a) = g \bullet 0 = 0$$

- $\{f \in \mathbb{F}[\lambda] \mid f \bullet a = 0\}$ is an ideal
- $\mathbb{F}[\lambda]$ is a principal ideal domain
- There exists a unique monic generator of minimal degree, the *minimal generating polynomial* of sequence
- Minimal polynomial divides all generating polynomials

Example: Fibonacci Numbers

- $(a_i)_{i\in\mathbb{N}} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$
- Minimal polynomial is $f = \lambda^2 - \lambda - 1$
  - $a_{i+2} = a_{i+1} + a_i$
- $(\lambda + 1)f = \lambda^3 - 2\lambda - 1$ also generates a
  - $a_{i+3} = 2a_{i+1} + a_i$
- $\lambda^k f = \lambda^{k+2} - \lambda^{k+1} - \lambda^k$ also generates a
  - $a_{i+k+2} = a_{i+k+1} + a_{i+k}$
  - Skips first k elements of a

Matrix Power Sequence

- $f \bullet (A^i)_{i \in \mathbb{N}} \iff f(A) = 0$
- $\det(\lambda I - A)$ generates $(A^i)_{i \in \mathbb{N}}$
  - Cayley-Hamilton Theorem
- Let f^A be minimal polynomial of $(A^i)_{i \in \mathbb{N}}$
- $f^A \mid \det(\lambda I - A)$

Krylov Sequence

- $f \bullet (A^i v)_{i \in \mathbb{N}} \iff f(A)\, v = 0$
- $f(A) = 0 \implies f(A)\, v = 0$
- $\{f \mid f \bullet (A^i)_{i \in \mathbb{N}} = 0\} \subset \{f \mid f \bullet (A^i v)_{i \in \mathbb{N}} = 0\}$
- Let $f^{A,v}$ be minimal polynomial of $(A^i v)_{i \in \mathbb{N}}$
- $f^{A,v} \mid f^A \mid \det(\lambda I - A)$

Solving $Ax = b$

- Suppose $g = \sum_{j=0}^{m} c_j \, \lambda^j$, $g(0) = c_0 \neq 0$, and $g \bullet (A^i b)_{i \in \mathbb{N}}$
- If g exists, then $f^{A,b}$ satisfies requirements
- $c_0 \, b + c_1 \, A \, b + \cdots + c_m \, A^m \, b = 0$
- $b = -\frac{1}{c_0} \sum_{j=1}^{m} (A^j \, b) = A \left(-\frac{1}{c_0} \sum_{j=1}^{m} (c_j \, A^{j-1} \, b) \right)$
- $x = -\frac{1}{c_0} \sum_{j=1}^{m} (c_j \, A^{j-1} \, b)$ is a solution

Nonsingular A

- $\det(A) \neq 0$
- $\det(\lambda I - A)|_{\lambda=0} \neq 0$
- $f^{A,b}(0) \neq 0$
- $x = -\frac{1}{c_0}\sum_{j=1}^{m}(c_j\, A^{j-1}\, b)$ is the unique solution

Bilinear Projection Sequence

- $f \bullet (u^{\mathsf{T}} A^i v)_{i \in \mathbb{N}} \iff u^{\mathsf{T}}\, f(A)\, v = 0$
- $f(A)\, v = 0 \implies u^{\mathsf{T}}\, f(A)\, v = 0$
- $\{f \mid f \bullet (A^i v)_{i \in \mathbb{N}} = 0\} \subset \{f \mid f \bullet (u^{\mathsf{T}} A^i v)_{i \in \mathbb{N}} = 0\}$
- Let $f_u^{A,v}$ be minimal polynomial of $(u^{\mathsf{T}} A^i v)_{i \in \mathbb{N}}$
- $f_u^{A,v} \mid f^{A,v} \mid f^A \mid \det(\lambda I - A)$
- If u chosen randomly from finite $S^n \subset \mathbb{F}^n$, then $f_u^{A,v} = f^{A,v}$ with probability at least

$$1 - \frac{\deg(f^{A,v})}{|S|}$$

  - Certificate: $f_u^{A,v} \bullet (A^i v)_{i \in \mathbb{N}} = 0$

Computing Minimal Polynomial

- Must know $\deg(f) < M$
- Extended Euclidean Algorithm
  - $s_j\, f_{-1} + t_j\, f_0 = f_j$
  - Inputs are $f_{-1}(\lambda) = \sum_{i=0}^{2M-1}(a_i\, \lambda^i)$ and $f_0(\lambda) = \lambda^{2M}$
  - Stop when $\deg(f_j) < M < \deg(f_{j-1})$
  - Minimal polynomial is reversal of $s_j(\lambda)$
- Berlekamp-Massey Algorithm
  - From coding theory
  - Interpolates elements of scalar sequence
  - Related to Extended Euclidean Algorithm

Berlekamp-Massey Algorithm

Input: Scalar sequence $(a_i)_{i \in \mathbb{N}} \in \mathbb{F}^{\mathbb{N}}$ with generator g with
$\deg(g) \leq m$

Output: Minimal polynomial f of sequence

```
f ← 1 {Initial guess}
for r = 0 to 2m − 1 do {f generates a_0, ..., a_{r−1}}
    f = c_0 + c_1 λ + ··· + c_d λ^d
    Δ ← c_0 a_{r−d} + c_1 a_{r−d+1} + ··· + c_d a_r
    if Δ ≠ 0 then {f does not generate a_0, ..., a_r}
        update f to generate a_0, ..., a_r
    end if
end for
```

Example: Fibonacci Numbers

$$(a_i)_{i\in\mathbb{N}} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)$$

r	Δ	recursion relation
0	0	$a_j = 0$
1	1	$a_{j+2} = a_j$
2	1	$a_{j+2} = a_{j+1} + a_j$
3	0	$a_{j+2} = a_{j+1} + a_j$
4	0	$a_{j+2} = a_{j+1} + a_j$
5	0	$a_{j+2} = a_{j+1} + a_j$
6	0	$a_{j+2} = a_{j+1} + a_j$
7	0	$a_{j+2} = a_{j+1} + a_j$

Wiedemann's Algorithm

- $f_u^{A,v} \mid f^{A,v} \mid f^A \mid \det(\lambda I - A)$
- $\deg(f_u^{A,v}) \leq \deg(\det(\lambda I - A)) = n$
- Only need $(u^\top A^i b)_{i=0}^{2n-1}$

Require: nonsingular $A \in \mathbb{F}^{n\times n}$ and $b \in \mathbb{F}^n$

Ensure: $x \in \mathbb{F}^n$ such that $A\,x = b$

1: $u \leftarrow$ random vector in S^n where $S \subset \mathbb{F}$

2: use Berlekamp-Massey to compute

$f^{A,b} = c_0 + c_1\lambda + \cdots + c_m\lambda^m$ {Store only $A^i b$}

3: $x \leftarrow -\frac{1}{c_0}(c_1 b + c_2 A b + \cdots c_m A^{m-1} b)$

Back to original problem

- $$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 3 & 3 & -2 & 3 \\ 0 & -3 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 4 \\ 2 \\ 6 \\ -4 \end{bmatrix}$$
- $n = 4$
- $$u = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

Back to original problem

- $(u^{\mathsf{T}} A^i b)_{i=0}^{2n-1} = (8, -4, 20, -28, 68, -124, 260, -508)$
- $f^{A,b} = f_u^{A,b} = \lambda^2 + \lambda - 2$
- $x = \frac{1}{2}(Ab + b) = \frac{1}{2}\left(\begin{bmatrix} -2 \\ 2 \\ -6 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 6 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$

Singular A

Kaltofen and Saunders (1991):

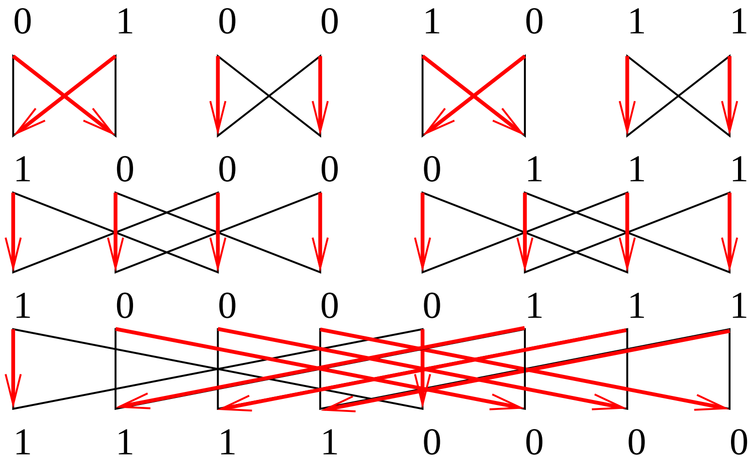
- If
  - $r = \mathrm{rank}(A) < n$ known
  - $r \times r$ leading principal minor nonzero
  - Randomly choose v
  - Solve $r \times r$ nonsingular system $A_r y'_{b,v} = A'(b + A\,v)$
    - A_r is leading $r \times r$ submatrix of A
    - A' is first r rows of A
- Then $y_{b,v} - v = \begin{bmatrix} y'_{b,v} \\ 0 \end{bmatrix} - v$ uniformly samples solution manifold

Precondition Matrix

- Want $r \times r$ leading principal minor of $\tilde{A}$ nonzero
- $\tilde{A} = BA$ or $\tilde{A} = AB$
- Need efficient matrix-vector product
- New linear system $\tilde{A}\tilde{x} = \tilde{b}$

Generic Rank Profile

- *Wiedemann (1986):* $\tilde{A} = A\,P$, P parameterized and can realize any permutation
- *Kaltofen and Saunders (1991):* $\tilde{A} = T_1\,A\,T_2$, T_1 unit upper triangular Toeplitz and T_2 unit lower triangular Toeplitz
- *Chen et al. (2002); Turner (2002):* $\tilde{A} = B_1\,A\,B_2$, B_1, B_2 based on butterfly networks



  - Generic exchange matrix mixes inputs
- *Turner (2002):* Generalize butterfly networks to radix-β switches
  - *Patton and Turner (2004):* Toeplitz (or Hankel) matrix switches

Matrix Rank

Kaltofen and Saunders (1991):

- If
  - $r = \text{rank}(A) < n$ (unknown)
  - Leading principal minors nonzero up to A_r
  - $D = \text{diag}(d_1, \ldots, d_n)$
- Then $r = \deg(f^{AD}) - 1$ with probability at least $1 - \dfrac{n(n-1)}{2\,|S|}$

Matrix Minimal Polynomial

- If v chosen randomly, then $f^{A,v} = f^A$ with probability at least

$$1 - \frac{\deg(f^A)}{|S|} \geq 1 - \frac{n}{|S|}$$

- If u and v chosen randomly, then $f_u^{A,v} = f^A$ with probability at least

$$1 - \frac{\deg(2\, f^A)}{|S|} \geq 1 - \frac{2\, n}{|S|}$$

Rank Preconditioners

- Generic rank profile preconditioner and diagonal matrix
- *Chen et al. (2002):* $\tilde{A} = T_3 \, A \, T_4 \, T_5$

  T_3 and T_4 unit lower triangular Toeplitz, T_5 upper triangular Toeplitz
- *Turner (2002, 2003):* Relax slightly generic rank profile
  - *Patton and Turner (2004):* $\tilde{A} = A \, T \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

    T Toeplitz (or Hankel)
- *Patton and Turner (2004):* $\tilde{A} = A \, T$

Matrix Determinant

- If $f^A = \det(\lambda I - A)$
- Then $\det(A) = (-1)^n f^A(0)$

Determinant Preconditioners

- Generic rank profile preconditioner and diagonal matrix
- *Kaltofen and Pan (1992):* $\tilde{A} = T_1\, A\, T_2$

  T_1 unit upper triangular Toeplitz and T_2 lower triangular Toeplitz
- *Chen et al. (2002):* $\tilde{A} = A \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$
- *Turner (2002, 2003):* $\tilde{A} = A \begin{bmatrix} 1 & a_1 & & \\ & \ddots & \ddots & \\ & & 1 & a_{n-1} \\ & & & 1 \end{bmatrix}$

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