Ohm's law for a wire in contact with a thermal reservoir

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Ohm's law is one of the key empirical relations taught in introductory courses on electricity. Students often do a straightforward experiment to measure the current through a resistor as a function of voltage. These experiments typically yield a linear relation between the current and the voltage using carbon film resistors. However, the same measurements on long lengths of conducting wire (such as common wire resistivity kits) typically have a clear nonlinear component. The nonlinear behavior can be modeled using simple principles of heat transfer with a thermal reservoir. Experimental results are shown to agree well with this simple model.

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I. INTRODUCTION

Ohm's law is a key idea in introductory electromagnetism classes, although it is an idea which many students find challenging.[1] I wanted to design a simple experiment to give students experience measuring current versus voltage $(I-V)$ curves for several ohmic resistors (carbon film) as well as a few nonohmic elements. This experiment would give students experience with both types of devices, and a better understanding that, although Ohm's law holds well for a number of special cases,[2] it is not universal. I expected a simple long wire wound into a coil to perform well as an ohmic resistor with a resistance determined by the wire resistivity ρ, the cross-sectional area of the wire A, and the wire length L:

$$R_0 = \rho L/A, \tag{1}$$

as is found in most introductory texts. However, experiments showed the wire coils to behave in a nonlinear fashion with the current dropping below the expected linear dependence based on the initial calculation of the resistance R_0.

The explanation for this behavior is that the temperature of the wire increases as the current flows through it, dissipating power $P=I^2R$, thus causing the resistivity to change. Introductory texts also cover the temperature dependence of the resistivity, and model the change in the resistivity as a linear increase based on the increase in the temperature ΔT times a temperature coefficient of resistivity α. However, the combination of Joule heating leading to a nonlinear $I-V$ curve is not usually discussed.

The following experiment describes the methods used to model and measure this simple nonlinear system. Because the key points in the model are based on ideas covered in introductory physics courses, the theory can be included in introductory labs. The measured results agree well with this model and provide a simple system that demonstrates the power and limitations of Ohm's law when dealing with real measurements.

II. THEORY

Podolsky and Denman have described a macroscopic approach that yields Ohm's law based on symmetry arguments.[3] They also discuss the situation where a material with finite resistance undergoes Joule (or ohmic) heating while maintaining contact with a thermal reservoir. The following discussion extends their theory and presents specific solutions for a conductive wire with a known resistivity.

Consider a homogeneous, isotropic material such as a long conductive wire with resistivity ρ. A steady-state current I is driven through a wire connected to a voltage source with a voltage difference V across the wire. Podolsky and Denman assume that the current caused by the potential drop across the wire can be any function f of the voltage:

$$I = f(V). \tag{2}$$

If the polarity of the voltage source is reversed, the current should also be reversed because the wire is homogeneous and isotropic. The magnitude of the current should remain the same: $|I| = -f(-V)$. This argument implies that the current must be an odd function of the voltage. If the function f is well-behaved near $V=0$, the series expansion of Eq. (2) must be

$$I = c_1 V + c_3 V^3 + \cdots. \tag{3}$$

The lowest order term in the expansion is Ohm's law,

$$I = V/R_0, \tag{4}$$

where the expansion coefficient c_1 is related to the macroscopic resistance R_0 of the wire by $c_1 = 1/R_0$. These arguments hold under steady-state conditions when the wire is held at a uniform constant temperature T_0, pressure, etc. For these conditions deviations from Ohm's law due to high field gradients do not apply.[4]

Now consider a material whose resistance changes linearly as a function of temperature T,

$$R(T) = R_0[1 + \alpha(T - T_0)], \tag{5}$$

where the temperature coefficient of resistivity α is material dependent. This linear relation holds well for the range of temperatures covered in our experiments. Equation (4) becomes, more generally,

$$I = V/R(T). \tag{6}$$

Podolsky and Denman consider what happens if the material is not held at a constant temperature T_0 when the constant voltage V is applied, but if the effects of Joule heating are included. They consider the case where the temperature T is uniform throughout the wire, and the wire can lose energy to its surroundings which are kept at a constant temperature

T_0. In this experiment two thermal reservoirs were used: the air surrounding the wire and a water bath at room temperature. Heat is produced in the wire at a rate

$$P = V^2/R(T), \tag{7}$$

because the current is driven by a constant-voltage source. The temperature T of the wire will become greater than the initial temperature T_0 until equilibrium is reached where the rate of heat input is balanced by the heat losses to the surroundings. The wire is in direct thermal contact with the reservoir so there is a heat current between the wire and the reservoir. Because the heat transfer from the wire to the environment around the wire is primarily due to direct thermal coupling, this model does not apply to the nonohmic behavior of light bulbs whose power losses are primarily due to radiation and are best described by the Stefan-Boltzmann law.[5]

Heat is transferred from the wire to the environment at a rate that depends on the details of how the wire is placed in contact with the reservoir and is parametrized by a thermal coupling coefficient β. The heat current H between the wire and the reservoir is

$$H = \beta(T - T_0). \tag{8}$$

The heat balance equation for the wire is

$$mc\frac{dT}{dt} + \beta(T - T_0) = V^2/R, \tag{9}$$

where m is the mass of the wire and c is its specific heat. When equilibrium is reached, $dT/dt=0$, and because the resistance as a function of temperature is given by Eq. (5), the voltage as a function of temperature is

$$V^2 = R_0[1 + \alpha(T - T_0)]\beta(T - T_0). \tag{10}$$

Equation (10) can be inverted to find the temperature of the wire as a function of the voltage and then inserted into Eq. (6) to give the current as a function of voltage:

$$I = \frac{V}{R_0}\left(\frac{2}{1 + \sqrt{1 + 4(V^2/R_0)(\alpha/\beta)}}\right). \tag{11}$$

The leading term is Ohm's law and Eq. (11) is an odd function of the voltage as expected. If the wire has very good thermal coupling to the environment so that the temperature never rises above T_0, the thermal coupling coefficient β is very large and the term α/β approaches zero. In this limit, the wire obeys Ohm's law, Eq. (4), as expected. The current in Eq. (11) can then be expanded about the voltage

$$I \approx \frac{V}{R_0} - \frac{\alpha}{\beta}\frac{V^3}{R_0^2} + \cdots, \tag{12}$$

which agrees with the expected series expansion from Eq. (3).

III. EXPERIMENTAL SETUP

Copper magnet wire (28 gauge) was used as the resistive element because of its relatively low cost and because the wire is coated with a waterproof enamel insulation that has a high melting point. Bulk copper has a thermal coefficient of resistivity of $\alpha=0.003\,93(°\mathrm{C})^{-1}$ and a resistivity of $\rho_0=1.72\times10^{-8}$ Ωm. The wire diameter is 0.320 mm, and the insulating enamel layer is 0.02 mm thick. Three wires

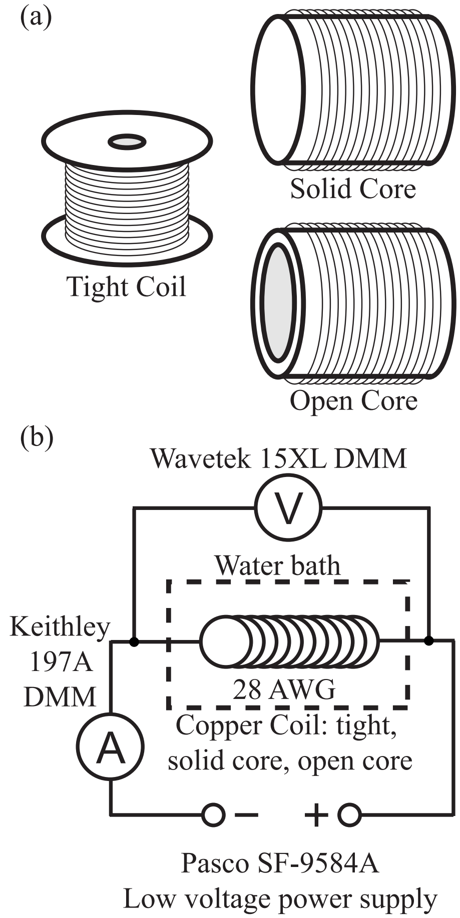

Fig. 1. (a) The same length of copper wire was wound in three configurations. (b) The circuit diagram of the setup used to measure the $I-V$ curves. The data were taken twice—once with the wire coils sitting on a countertop in air and with the coils submerged in a water bath.

with length 19 ± 0.5 m (95% CI, triangular probability density function)[6] were wrapped into different coil configurations, each with an expected resistance of 4.1 ± 0.1 Ω (95% CI) based on Eq. (1). The three coil configurations are illustrated in Fig. 1(a).

The "tight coil" is a cylinder of wire with diameter ≈2 cm and height 1.1 cm wrapped around a 5 mm diameter small plastic rod. The wire in this coil is wrapped in multiple layers and is similar to the type of coil found in commercial resistor kits.[7] In the "solid core" configuration the wire is wrapped tightly around a 2.55 cm diameter solid aluminum cylinder in a single 8.2 cm long layer. The "open core" configuration consists of the wire wrapped around an aluminum tube with an inner diameter of 1.98 cm and an outer diameter of 2.23 cm in a 9.5 cm long single layer. These three configurations were chosen to measure the differences in heat transfer between the wire and the environment and thus the thermal coupling coefficient β from Eq. (8). The open core configuration was designed to give the wire the most thermal contact with the external thermal reservoir, either the ambient room air or the water bath. Because the tight coil has multiple layers, the entire wire is not in direct thermal contact with the environment, and the thermal coupling depends on the type of wire insulation and the tightness of the wire wrapping.

The circuit diagram is shown in Fig. 1(b). The wire coil

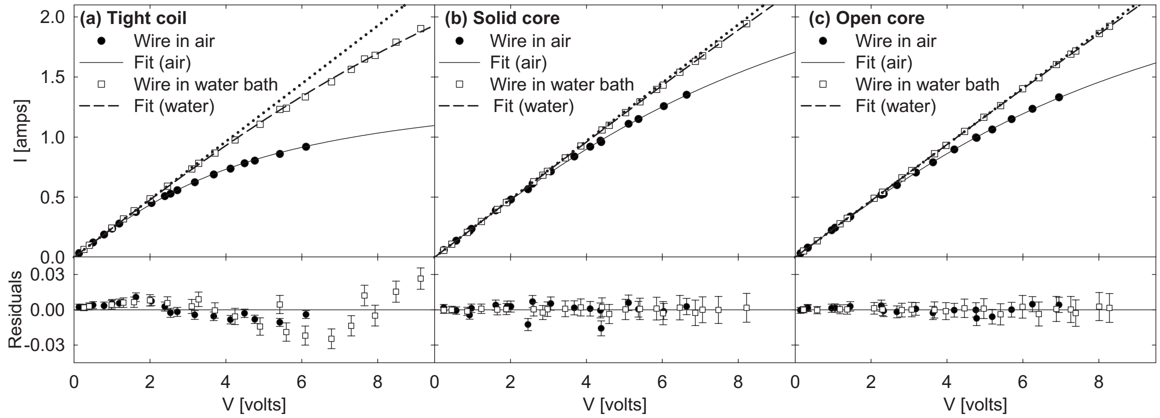

Fig. 2. The current versus voltage curves for (a) the tight coil, (b) the solid core, and (c) the open core wires. Each graph shows the data for the coil taken in air (no water bath) as the solid circles, the data taken with the wire submerged in the water bath (open squares), and the two parameter fits for these data sets following Eq. (13) (solid line and dashed line, respectively). The dotted lines are Ohm's law [Eq. (4)] using the resistance of the wire R_0 found from the fits. The residuals from the fits are shown below each plot. The error bars indicate the effective uncertainty (Ref. 8) in the measurements and are not shown because they are too small to be visible on the data plots.

was connected in series with an ammeter and in parallel with a voltmeter. The first $I-V$ curve was taken with each coil sitting on a lab bench without a liquid bath. All three wire configurations reached a temperature that was very hot to the touch at about 6 W power dissipation. The second set of data was taken with the wire coils submerged in about 8 L of room temperature water that was not stirred during the measurements. The temperature of the water bath was measured periodically with a thermometer and was a constant 23 °C through the duration of the experiment. The insulated wire tails were extended out of the water so that no current could run through the water itself. Because the water kept the wire coils at a much lower temperature, it was possible to collect current data up to the limit of the ammeter (2 A) without running the risk of dangerous high temperatures. As seen in the results shown in Fig. 2, up to 16 W of power were dissipated in the wire submerged in the water bath. If we consider the heat capacity of the water bath and the dissipated power, we see that the temperature of the bath does not change appreciably during the measurement. For each data point the voltage was adjusted and any change in the current was allowed to settle down (about 15 min), indicating that the wire had reached thermal equilibrium.

IV. RESULTS

The results for all three wires are shown in Fig. 2. The $I-V$ data were fit to a two-parameter function based on Eq. (11),

$$I_{\text{fit}} = \frac{V}{a}\left(\frac{2}{1+\sqrt{1+4bV^2/a}}\right). \tag{13}$$

The fit parameters a and b are weighted by the uncertainty of the current and voltage measurements following the method proposed in Ref. 8. The current and voltage measurement uncertainties are specified by the manufacturers of the meters in terms of a relative percentage of each measurement. The fit parameter a is equivalent to the resistance R_0 of the wire at temperature T_0 from Eq. (11). The parameter b is the ratio of the thermal resistivity coefficient to the thermal coupling to the environment $b=\alpha/\beta$. The residuals (the difference between the data and the fit at each point) are shown below each plot. The residuals for the tight coil data [Fig. 2(a)] do not appear completely random due to the remaining systematic effect of incomplete thermalization of the coil due mainly to the multiple wire layers in this configuration. The tight coil took significantly longer to come to thermal equilibrium with the environment and several of these points indicate the systematic shift below the fit (when the coil was warming up) and above the fit (when the coil was cooling down). Future experiments might investigate this hysteresis.

The parameter b was used to calculate the thermal coupling coefficient β using the known temperature coefficient of resistivity for copper. The results for the resistance of the wire R_0, the thermal coupling coefficient β, and the reduced χ^2 of the fits are listed in Table I.

The fits for the solid and open cores in the water bath have unusually small reduced χ^2 values. Both systems have very

Table I. Results from the model fits. The results agree for the initial resistance R_0 for each wire between the air and water data.

	(95% CI, t-distribution)					
	Tight coil		Solid core		Open core	
Result	Air	Water	Air	Water	Air	Water
R_0 (Ω)	4.15 ± 0.04	4.14 ± 0.03	4.13 ± 0.03	4.15 ± 0.03	4.28 ± 0.04	4.24 ± 0.04
β (W/°C)	0.0356 ± 0.0014	0.384 ± 0.033	0.183 ± 0.015	3.0 ± 2.0	0.162 ± 0.013	3.9 ± 3.3
Reduced χ^2	3.2	3.1	1.3	0.05	0.4	0.04

large thermal couplings to the water reservoir and thus have small corrections to Ohm's law. These reduced χ^2 values indicate a lower confidence in the accuracy of the parameter b, which is reflected in the larger uncertainties in the thermal coupling coefficients.

The increase in temperature of the wire coils can be found by calculating the wire resistance at each temperature using Eq. (6) and substituting the result into Eq. (5). The maximum change in the temperature $T-T_0$ for these experiments occurred in the tight coil in air and was approximately 150 °C. The temperature of the other two coils in air also increased by approximately 50 °C, indicating poor thermal coupling to the environment for all three configurations. However, the results from the coils in the water bath give a maximum increase of about 40 °C for the tight coil and an increase of only a few degrees for the other two coils. These results are consistent with the expected improvement in thermal coupling in the water bath.

V. CONCLUSIONS

Because Ohm's law holds only for a limited range of materials and situations, it is important to give students experience with situations where the current is not linearly proportional to the voltage. This experiment is an example where the nonlinear current can be explained using simple concepts from introductory physics courses. The model applies to wires in almost any realistic situation. Multiple measurements of different wire coils in a single lab section would lead to a discussion of the ideal case in which the wire obeys Ohm's law. Although the current was measured with an expensive 5.5 digit multimeter, the magnitude of the nonlinear effect for wire coils in air could easily be measured with an inexpensive digital multimeter. An improvement that might yield a more linear $I-V$ curve would be to use either a stirred water bath or flowing water as a thermal bath to ensure a constant temperature and efficient cooling.

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[7] The tight coil is the type of wire coil typically found in wire resistor kits such as the Resistance Coils kit from Science First (part no. 10–143).

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