A Musical Scale in Simple Ratios of the Harmonic Series Converted to Cents of Twelve-tone Equal Temperament for Digital Synthesis

PETER LUCAS HULEN
Department of Music
Wabash College
Crawfordsville, Indiana
USA
hulenp@wabash.edu http://persweb.wabash.edu/facstaff/hulenp

Abstract: – Intervals between harmonics in the frequency spectra of pitched sounds occur in a pattern of simple ratios. New musical scales have now been invented with intervals in these ratios, and series of scales sharing common tones allowing for musical modulation. Twelve-tone equal temperament (12Tet) is standard for Eurogenic musical instruments. It divides the octave—the interval between frequencies having a 2:1 ratio—into twelve equal, smaller intervals. Each smaller interval (semitone or half-step) is divided into 100 cents for tuning, resulting in division of the octave into 1,200 fine increments. The simple ratios of the harmonic series, the basis for the new scales, do not conform to the twelve semitones of 12Tet. Digital sound synthesis is very flexible, but writing programs is a tedious distraction from composing music. User interfaces of commercial digital synthesizers and graphic interfaces of virtual digital synthesizer applications tend to correlate fine-tuning with the 1,200 cents of 12Tet. Octave frequencies being a ratio of two, 12Tet dividing octaves into twelve equal intervals, the value of any one thus being the twelfth root of two, a logarithmic equation must be used to calculate values in cents for simple-ratio intervals. Commercial synthesis equipment can be adjusted accordingly, and new music composed with the new scales.

Key-Words: - music, microtonal, computer music, music theory, music acoustics

1 Introduction

This concerns invention and application of new musical scales, exact intervals of which are impractical by acoustic means, but more easily achieved through digital sound synthesis. Intervallic content of these new scales would match that of the harmonic series observable in the frequency spectra of identifiably pitched sounds.

To adjust tuning through the user interfaces of commercial digital synthesis equipment and software, and to graphically represent the new music for performers of acoustic parts using standard music notation, the simple-ratio intervals of the harmonic series (and the new musical scale based on it) must be converted into the 1,200 cent-per-octave scale of standard twelve-tone equal temperament (12Tet) using a logarithmic equation.

2 Challenges

The technical and artistic challenges behind creation of these new scales were fourfold: 1) Create true "overtone" scales, intervals of which are in actual simple ratios of the harmonic series; 2) Create means to realize these scales as a material component of musical composition by mathematically translating simple-ratio frequency intervals of the scales into adjusted musical intervals of standard 12Tet so that: a) commercial synthesis equipment and software can be adjusted according to musically standardized user interfaces; and, b) pitches can be graphically represented for vocal or acoustic performers according to standard music notation; 3) Develop a series of scale transpositions sharing common tones, by which modulation from one to another may be effected; and, 4) Compose, perform and record new music according to the new scales and tonal systems.

2.1 No True Overtone Scale

Certain music of the 20th century is composed applying a Lydian-Mixolydian scale (Fig.1) [1].

Fig.1

When this same scale is applied in jazz improvisation, it is sometimes called an "overtone" scale, or “acoustic” scale [2], even though its standard 12Tet pitches only approximate frequency intervals in the fourth octave of the harmonic series. It is still not a true overtone scale because its intervals do not actually conform to those of the observable harmonic series.

Designing and building nonstandard acoustic instruments that would play a true overtone scale, according to actual intervals of the harmonic series, and then training musicians to play them, would be problematic from a practical standpoint. Writing digital synthesis code for such intervals would be distracting and time consuming for those approaching the electronic music studio as trained creative artists rather than as software engineers. Commercially available synthesis equipment brings its own problems, as detailed below.

2.2 Musical Standards vs. Acoustic Reality

The 12Tet musical pitch standard is widely assumed both for commercial synthesis equipment and software—even that which may be finely adjusted in pitch—and in the standard graphic representation (notation) of music as understood by trained musicians. The actual intervals of the harmonic series conform neither to the intervallic pitch relationships of 12Tet nor to standard musical notation and its uses, inasmuch as they assume and refer to 12Tet.

2.2.1 Synthesis Interface Standards

With some commercial digital synthesizers and synthesis software applications one may produce pitch relations to very fine specifications, but as user interfaces typically translate digital operations into minute divisions of the 12Tet semitone, making correct pitch adjustments so intervals conform to simple ratios of the harmonic series requires a logarithmic formula.

2.2.2 Music Notation Standards

Trained musicians are certainly capable of matching, playing or singing in concert with pitch-adjusted musical tones that do not conform to 12Tet; however, the standard music notation they follow and the ways they follow it assume 12Tet. There is no standard in music notation for indicating the exact measurement in cents of any pitch adjustment in deviation from 12Tet, yet some means of indication must be applied for the sake of trained performers who would interact with pitch-adjusted musical materials according to a notated musical score.

2.3 Need for Means of Modulation

A pitch class includes any and all octave transpositions of a given pitch. The formal dynamics of many kinds of music depend on modulation from one applied set of intra-related pitch classes (or scale) to another, and often, some kind of eventual return to the initial set. In many cases such modulation depends on what pitch classes are shared by a given set and its successor.

As analogous to other types of music, it was found desirable to develop transpositions of the new scales that would produce a network of pitches shared among all transpositions. This would depend on intervals between the primary pitch classes, or *tonics* of each scale transposition. If successful, it would facilitate modulation as a compositional procedure.

2.4 Final Application

Provided simple-ratio intervals of the harmonic series are mathematically translated into 12Tet pitches adjusted by certain numbers of cents, the means of synthesis and graphic notation are adjusted accordingly, and theoretical pitch relationships are developed to provide for musical modulation, it would remain to compose new music applying those materials, and to hear it realized in performance.

3 Realization of Scale and Music

Developing frequency relationships of the harmonic series into musical scales began with regarding the fundamental frequency as *tonic*, against which other pitches of the scales, or *diatonic sets*, were arrayed intervallically. To represent intervals musically, they were translated to values in cents using a logarithmic formula. Once they were represented musically, they were applied in the composition of music, using both digital synthesis resources and music notation intended for reading by performers. The resulting new music was then realized in performance.

3.1 A True Overtone Scale

Taking the frequency of C_1 (32.703 Hz) as an arbitrary fundamental, frequencies of tones that would share the same relationships as harmonics in the harmonic series over that fundamental can be calculated.

So, a true overtone scale based on the actual harmonic series can be expressed initially as a list of frequencies, plus the ratios by which those frequencies are related in series:

Table 1

HARMONIC	FREQUENCY HZ	RATIO
1 (C_1)	32.703	1/1
2 (C_2)	65.406	2/1
3	98.109	3/2
4 (C_3)	130.81	4/3
5	163.52	5/4
6	196.22	6/5
7	228.92	7/6
8 (C_4)	261.62	8/7
9	294.33	9/8
10	327.03	10/9
11	359.73	11/10
12	392.44	12/11
13	425.14	13/12
14	457.84	14/13
15	490.55	15/14
16 (C_5)	523.25	16/15

3.2 Translation into Musical Terms

There is a way to mathematically translate simple-ratio frequency intervals of the harmonic series into musical intervals of the 1,200 cents-per-octave 12Tet standard.

Remembering that the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers:

$$\log xy = \log x + \log y, \qquad (1)$$

it follows that

$$\log x^n = n \log x. \qquad (2)$$

Setting this equation aside for the moment, it is known—given any octave of exactly 2:1—that when the value of an equal semitone in 12Tet is denoted by a, it provides that

$$a^{12} = 2, \qquad (3)$$

thus, a is the twelfth root of two, or

$$a = (2)^{1/12} = 1.05946. \qquad (4)$$

This interval is the tempered semitone. With the cent defined as $^{1}/_{100}$ semitone and ¢ denoting the ratio for it, such that

$$¢^{100} = (2)^{1/12}, \qquad (5)$$

it follows that

$$¢ = (2)^{1/1200}. \qquad (6)$$

As an interval of n cents would be given by the ratio

$$¢^{n} = 2^{n/1200},$$

finding the number of cents n in any interval of frequency ratio R requires that

$$2^{n/1200} = R. \qquad (7)$$

Taking the logarithm of each side gives

$$\log (2^{n/1200}) = \log R,$$

and by applying Eq. (2) it follows that

$$\frac{n}{1200} \log 2 = \log R.$$

The number of the cents in the interval is then given by

$$n = 1200 \, \frac{\log R}{\log 2}$$

so that

$$n = 3986 \log R. \qquad (8)$$

By applying Eq. (8) the number of cents in any frequency ratio n can be calculated. For example, the number of cents in a just fifth, which has a frequency ratio R of $^{3}/_{2}$, per Table 1, can be calculated so:

$$\log R = \log (^{3}/_{2}) = \log 3 - \log 2$$
$$= 0.477 - 0.301 = 0.176,$$

and consequently,

$$n = 3986 \times 0.176$$
$$= 702 \text{ cents}.$$

So, while an equally tempered fifth, at seven semitones, has a value of 700 cents, a just fifth with a frequency ratio of $^{3}/_{2}$ has a value of 702 cents (rounded to the nearest cent) [3].

The just fifth from Table 1 can be shown in standard music notation by representing the C_2 as having no deviation from 12Tet—indicated by a zero—and the G_2 has having a deviation of plus two cents. These indications can be added to individual noteheads in the manner of musical articulations (Fig. 2):

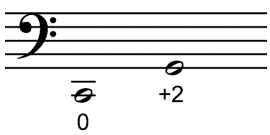

Fig.2

Thus, the entire scale of intervals from Table 1 can be calculated and represented in standard music notation, in terms of each tone's deviation in cents from 12Tet (Fig. 3):

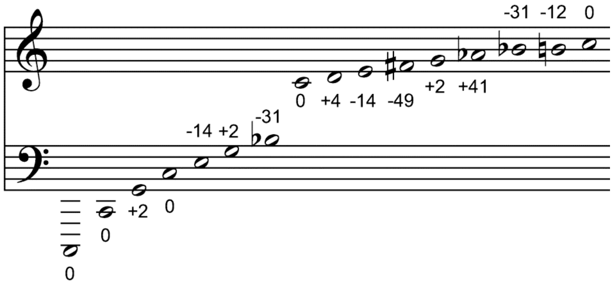

Fig.3

This is a true "overtone" scale, accurately reflecting the intervallic relationships of the harmonic series. Compare the fourth octave of the series as represented in the treble staff of Fig.3 with the Lydian-Mixolydian "overtone" scale in Fig.1.

With cents deviation from 12Tet indicated for each note of the scale, synthesis equipment or software can be adjusted to produce a scale accurate to the intervallic relationships of the harmonic series. Moreover, adjusted pitches can be accurately represented in a musical score for performers expected to match or otherwise interact with them.

It should be noted, in terms of musical style, that the catchall term "microtonal" would be applied to this scale, and to music composed with it. "Microtonal" refers to the broad category of any and all music not conforming to 12Tet. There are a wide variety of approaches that fit into this category, and as wide a variety of interests and reasoning behind them. [4] The electronic generation of music has provided especially fruitful means of experimentation and generation of new pieces that would fall into this stylistic category. [5]

3.3 Developing Means of Modulation

Fig.4 shows the new scale transposed to a fundamental of E_1, at 41.25 Hz. This is the lowest normal pitch produced by conventional stringed instruments, that is, it is the lowest open string on an ordinary contrabass. It is also the lowest E on a piano. Note that the E_1 of 12Tet based on A_4 at 440 Hz is 41.203 Hz. The E_1 given here is unadjusted for equal temperament, as though tuned from A 440 using only the just fifths produced by harmonics on the strings of a contrabass; therefore, it is 41.25 Hz.

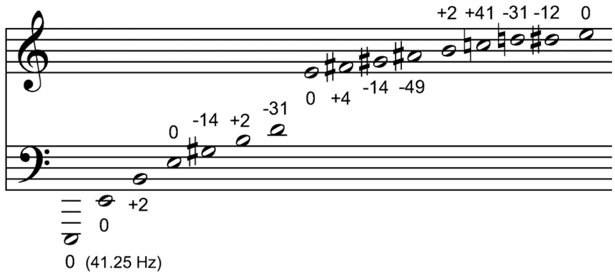

Fig.4

Note that the fourth octave of the diatonic set contains eight tones. In Fig.5 these eight tones have been transposed down three octaves to serve as the tonics for eight transpositions of the diatonic set:

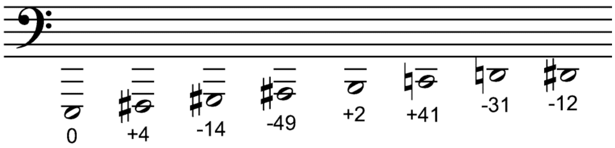

Fig.5

A separate transposition of the basic diatonic set is then created with each of these pitches serving as tonic. Fig.6 shows the diatonic set on F# +4. Note that the relationships between the tones within the F# set (Fig.6) are the same as those in the E set (Fig.4), with values of deviation from 12Tet adjusted by the constant of plus four cents:

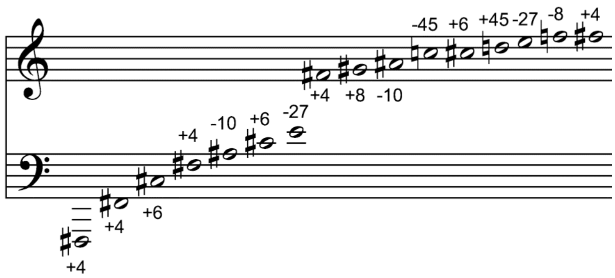

Fig.6

Fig.7 shows the diatonic set on G# -14. All the transposed tones are adjusted to their constant of minus fourteen cents. In addition, arrows pointing from some notes indicate letter names of the tonic(s) of diatonic set(s) containing exactly the same pitches.

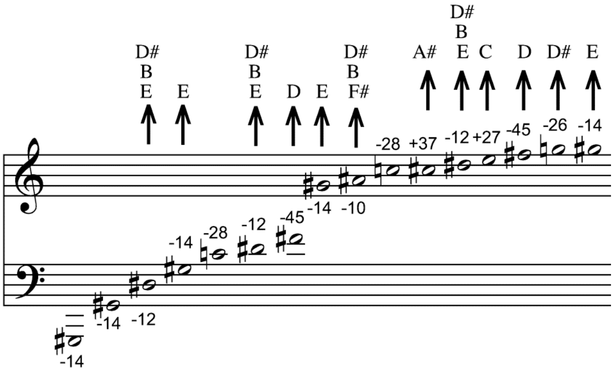

Fig.7

Note the presence of D# -12 and G# -14 from the G# set (Fig.7) in the E diatonic set (Fig.4), and A# -10 from the G# set (Fig.7) in the F# set (Fig.5). Most of the tones in each of the eight diatonic sets are common to at least one other set, and all eight sets have at least one tone in common with every other set, allowing for common-tone transpositions throughout the sets.

3.4 Composition with the New Scale

This scale and system of common-tone relationships suggest procedural possibilities for composition. Polyphony, counterpoint, and harmony have been created applying the scale and interrelated transpositions.

3.4.1 Polyphonic Procedure

Polyphony refers to music with two or more composed parts that sound together but are texturally independent, rather than in unison, simultaneous harmony (homophony), or loosely coordinated but similar parts (heterophony).

One polyphonic procedure with this scale applies tones from the diatonic set in melodic or other events, while the tonic is continuously or intermittently sounded below. When material reaches a tone common to another transposition, the tonic changes to that of the other set, thereby effecting a modulation to the other set. Events continue with tones applied from the new set.

When events are melodic in character, the texture is analogous to 12th century *organum* e.g. of Léonin, wherein the *tenor* comprises a long-breathed chant melody beneath comparatively florid melodic material in the *duplum* [6]. Tonics of diatonic sets in series are analogous to the *tenor*. Melodic material above is analogous to the *duplum*. An example can be taken from the electroacoustic part of the multimedia piece *The Madman's Diary* [7]. Fig.8 shows a musical reduction to simplest terms of mm. 49b-96 of the score (6 mm. before 1C to 1G).

Fig.8

The modulations and *organum*-like texture become clear with the music reduced to its basic structure.

3.4.2 Contrapuntal Procedure

Contrapuntal procedures have been derived from pitch relationships within the diatonic set. Intervals within the scale have been categorized according to an arbitrary but precedented system of "perfect" and "imperfect" consonance, "dissonance," and "*empty*" consonance.

Criteria for categorization are as follows: Octaves/unisons (1200¢/0¢) and just fifths/fourths (702¢/498¢) are discarded as "empty" consonances. Certain thirds and sixths are considered "perfect," *vis à vis* the octaves, fifths and unisons of 16th century practice. These include just major thirds and minor sixths (386¢/814¢), just minor thirds and major sixths (316¢/884¢), as well as a minor third and major sixth 10¢ from equal temperament (290¢/910¢) and a major third and minor sixth thirteen cents from just intonation (373¢/827¢).

"Imperfect" consonances include all other thirds and sixths from 247¢ to 455¢ and 745¢ to 953¢ (technically the extremes of these are large major seconds, small fourths, large fifths and small minor sevenths), as well as seconds of 9:8 (204¢) and larger and their inversions (sevenths), the compound form of the 182¢ second (a ninth) and its inversion (a seventh), and the "just" tritone (first in the harmonic series) of 583¢/617¢.

Dissonances include the simple form of the 182¢ second, all seconds simple and compound smaller than 182¢ with their inversions, and all tritones other then the "just."

These categorizations provide a workable balance of perfect and imperfect consonance and dissonance, divided not only according to ratios and relative dissonance, but also according to more recent culturally accepted ranges of dissonance.

Intervals in these categories have been applied in a way analogous to 16th century counterpoint, according to rhythmic relationships. Fig.9 shows another musical reduction from *The Madman's Diary* [7]. Four-voice counterpoint in the electronic part mm. 358-370 (4B ff.) is represented. The music of the vocal line has been omitted for clarity.

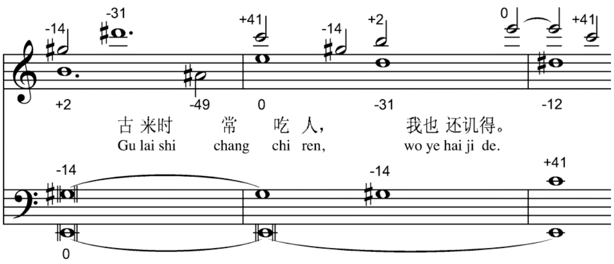

Fig.9

As with Fig.8, contrapuntal texture becomes clear with reduction of the music to its basic structure.

3.4.3 Harmonic Procedure

Chords have been constructed by superimposing the perfect and imperfect consonant intervals. Chords with all perfect consonances are called perfect; those with any combination are termed imperfect. There are fifteen possible triadic combinations, eight possible tetrads and a pentad.

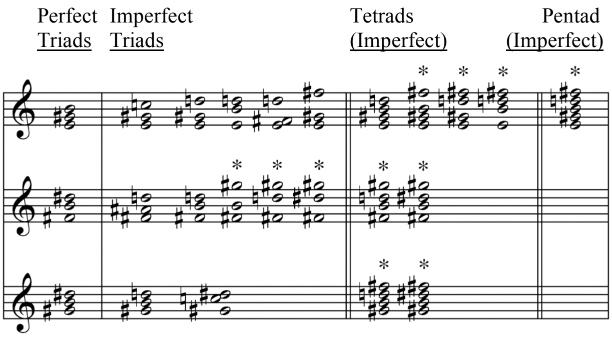

Fig.10

The perfect and imperfect chords for the E diatonic set are shown in Fig.10. Those containing the tonic (E) are in the first line, those remaining that contain F# +4 are in the second line, and those left that contain G# -14 are in the third line. This accounts for all possible combinations. Such systematic inventory is not meant to suggest the primacy of tones by which chords are identified.

This set of relationships is the same at all transpositions. Deviations from 12Tet are omitted here for clarity. Allowance of the 182¢ second as a consonance only if inverted to a seventh or compounded to a 9th—in this case involving the F# and G#—is indicated by the asterisks in Fig.10.

Fig.11

The catalog of chords in Fig.10 does not identify each chord with a particular scale degree, nor are inversions and voice leading addressed. Fig.11 shows another reduction from *The Madman's Diary* [7], this time from mm. 269-340 (11 before 3F to 4A). It is derived from the electronic accompaniment. The chords are triads except as "(4)" indicates a tetrad, are all imperfect except as "P" indicates a perfect triad, and are all to be found on the table in Fig.10.

The progression in Fig.11 is gradual. At each change of bass the chord above is held, and at each change of chord only one member of the chord is changed. This is in keeping with they stylistically minimalist character of the musical texture.

3.5 Extension of the True Overtone Scale

The eight-tone scale, series of scale transpositions through common tones, and contrapuntal and harmonic procedures described above were all developed and applied over the years 1996 to 1999. [8] In 2000 a theoretical and practical extension was added to the system.

The harmonic series theoretically extends indefinitely. While the fourth octave of the harmonic series contains intervallic relationships that can form the eight-tone scale described above, the fifth octave contains intervals that would form a sixteen-tone scale.

Extending the list of harmonics, frequencies and ratios by which the harmonics are related in series from Table 1 for another octave, results in the following:

Table 2

HARMONIC	FREQUENCY HZ	RATIO
16 (C_5)	523.25	16/15
17	555.95	17/16
18	588.65	18/17
19	621.36	19/18
20	654.06	20/19
21	686.76	21/20
22	719.47	22/21
23	752.17	23/22
24	784.87	24/23
25	817.58	25/24
26	850.28	26/25
27	882.98	27/26
28	915.68	28/27
29	948.39	29/28
30	981.09	30/29
31	1013.8	31/30
32 (C_6)	1046.5	32/31

As these intervals continue in simple ratios, their calculation as tones in 12Tet can be made according to the same formula

$$n = 3986 \log R.$$

Transposed to E, as before, and notated with tunings in cents, a more distinctive sixteen-tone microtonal scale results. Fig.12 shows this scale placed in the mid-treble range.

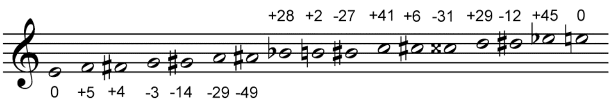

Fig.12

A cycle of scales with tones in common has been constructed, as with the eight-tone scale from the fourth octave of the harmonic series. This allows for modulation in the same manner. Contrapuntal and harmonic procedures could also be developed, though such systems would be considerably more complex than those developed for the eight-tone scale. For this reason, music composed with the sixteen-tone scale has followed the polyphonic procedures outlined above, but without a through set of contrapuntal or harmonic procedures to match.

4 Conclusion

These new scales, the translation of their simple-ratio intervals according to musical standards and graphic representation, and the development of transpositions for modulation provide an innovative system for pitch class relationship in art music. This system comprises an interconnected group of diatonic sets with intervallic relationships found in the harmonic series, and thus it reflects the acoustic properties of all pitched sound. Mathematical translation into 12Tet has allowed this new system realization with digital synthesizers and software.

Procedural possibilities for composition have been successfully applied in musical works intended for both experimentation and public performance. These include the electronic pieces *Organum Duplum* (1997), *Klirrfarbenstrukturen* (1997), *Two Sketches* (1998), and *Picture Tube* (1998); *The Madman's Diary* for tenor, electronic multimedia, strings and percussion (1999); soundtracks for the web-delivered multimedia pieces *Psalm* (2001) [9] and *Spring* (2002); *Accompanied Poetry of Rumi* for reader, electronic music and chorus (2004); and *Scriptures and Religion*, for reader and electronic music—a work in progress (2006).

As with any innovation in the creative arts, continual practice and experimentation with these newly developed materials and procedures will be required to fully realize their artistic implications.

References:

[1] Stefan Kostka, *Materials and Techniques of Twentieth-Century Music*, Prentice-Hall, 1990.

[2] Andrew Jaffe, *Jazz Theory*, Wm. C. Brown Company, 1983.

[3] John Backus, *The Acoustical Foundations of Music* 2d ed., W.W. Norton & Co., 1977.

[4] Brian McLaren, A Brief History of Microtonality in the Twentieth Century, *Xenharmonikon*, No. 17, Spring 1998, 57-110.

[5] Scott R. Wilkinson. *Tuning In: Microtonality in Electronic Music*, Hal Leonard Books, 1988.

[6] Gustave Reese, *Music in the Middle Ages*, W.W. Norton & Co., 1940.

[7] Peter Lucas Hulen, The Madman's Diary: A Monodrama for Tenor, CD-ROM, Strings, and Percussion, Ph.D. diss., Michigan State University, 1999.

[8] Peter Lucas Hulen, Ratiotonic Temperament: A Proposal for the Organization of Tones in Art Music as an Alternative to Dodecaphonic Equal Temperament, *Actes des Journées d'Informatique Musicale 1998*. Marseille, France: Publications du Laboratoire de Mecanique et d'Acoustique, 1998. C4.

[9] Peter Lucas Hulen, *Psalm* (sound recording) on compact disc *Articulations*, Alliance of Michigan State University Composers, 2003.