From: John Stuart Mill, *A System of Logic, Ratiocinative and Inductive*, 8th ed. (New York: Harper & Brothers, Publishers, 1882)

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Book II. Of Reasoning.

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Chapter V. Of Demonstration, and Necessary Truths.

… [p. 187]

Chapter VI. The Same Subject Continued.

§ 1. In the examination which formed the subject of the last chapter, into the nature of the evidence of those deductive sciences which are commonly represented to be systems of necessary truth, we have been led to the following conclusions. The results of those sciences are indeed necessary, in the sense of necessarily following from certain first principles, commonly called axioms and definitions; that is, of being certainly true if those axioms and definitions are so; for the word necessity, even in this acceptation of it, means no more than certainty. But their claim to the character of necessity in any sense beyond this, as implying an evidence independent of and superior to observation and experience, must depend on the previous establishment of such a claim in favor of the definitions and axioms themselves. With regard to axioms, we found that, considered as experimental truths, they rest on superabundant and obvious evidence. We inquired, whether, since this is the case, it be imperative to suppose any other evidence of those truths than experimental evidence, any other origin for our belief of them than an experimental origin. We decided, that the burden of proof lies with those who maintain the affirmative, and we examined, at considerable length, such arguments as they have produced. The examination having led to the rejection of those arguments, we have thought ourselves warranted in concluding that axioms are but a class, the most universal class, of inductions from experience; the simplest and easiest cases of generalization from the facts furnished to us by our senses or by our internal consciousness.

While the axioms of demonstrative sciences thus appeared to be experimental truths, the definitions, as they are incorrectly called, in those sciences, were found by us to be generalizations from experience which are not even, accurately speaking, truths; being propositions in which, while we assert of some kind of object, some property or properties which observation shows to belong to it, we at the same time deny that it possesses any other properties, though in truth other properties do in every individual instance accompany, and in almost all instances modify, the property thus exclusively predicated. The denial, therefore, is a mere fiction, or supposition, made for the purpose of excluding the consideration of those modifying circumstances, when their influence is of too trifling amount to be worth considering, or adjourning it, when important to a more convenient moment.

From these considerations it would appear that Deductive or Demonstrative Sciences are all, without exception, Inductive Sciences; that their evidence is that of experience; but that they are also, in virtue of the peculiar character of one indispensable portion of the general formulæ according to which their inductions are made, Hypothetical Sciences. Their conclusions are only true on certain suppositions, which are, or ought to be, approximations to the truth, but are seldom, if ever, exactly true; and to this hypothetical character is to be ascribed the peculiar certainty, which is supposed to be inherent in demonstration. [p. 188]

What we have now asserted, however, cannot be received as universally true of Deductive or Demonstrative Sciences, until verified by being applied to the most remarkable of all those sciences, that of Numbers; the theory of the Calculus; Arithmetic and Algebra. It is harder to believe of the doctrines of this science than of any other, either that they are not truths *a priori*, but experimental truths, or that their peculiar certainty is owing to their being not absolute but only conditional truths. This, therefore, is a case which merits examination apart; and the more so, because on this subject we have a double set of doctrines to contend with; that of the *a priori* philosophers on one side; and on the other, a theory the most opposite to theirs, which was at one time very generally received, and is still far from being altogether exploded, among metaphysicians.

§ 2. This theory attempts to solve the difficulty apparently inherent in the case, by representing the propositions of the science of numbers as merely verbal, and its processes as simple transformations of language, substitutions of one expression for another. The proposition, Two and one is equal to three, according to these writers, is not a truth, is not the assertion of a really existing fact, but a definition of the word three; a statement that mankind have agreed to use the name three as a sign exactly equivalent to two and one; to call by the former name whatever is called by the other more clumsy phrase. According to this doctrine, the longest process in algebra is but a succession of changes in terminology, by which equivalent expressions are substituted one for another; a series of translations of the same fact, from one into another language; though how, after such a series of translations, the fact itself comes out changed (as when we demonstrate a new geometrical theorem by algebra), they have not explained; and it is a difficulty which is fatal to their theory.

It must be acknowledged that there are peculiarities in the processes of arithmetic and algebra which render the theory in question very plausible, and have not unnaturally made those sciences the stronghold of Nominalism. The doctrine that we can discover facts, detect the hidden processes of nature, by an artful manipulation of language, is so contrary to common sense, that a person must have made some advances in philosophy to believe it: men fly to so paradoxical a belief to avoid, as they think, some even greater difficulty, which the vulgar do not see. What has led many to believe that reasoning is a mere verbal process, is, that no other theory seemed reconcilable with the nature of the Science of Numbers. For we do not carry any ideas along with us when we use the symbols of arithmetic or of algebra. In a geometrical demonstration we have a mental diagram, if not one on paper; AB, AC, are present to our imagination as lines, intersecting other lines, forming an angle with one another, and the like; but not so *a* and *b*. These may represent lines or any other magnitudes, but those magnitudes are never thought of; nothing is realized in our imagination but *a* and *b*. The ideas which, on the particular occasion, they happen to represent, are banished from the mind during every intermediate part of the process, between the beginning, when the premises are translated from things into signs, and the end, when the conclusion is translated back from signs into things. Nothing, then, being in the reasoner's mind but the symbols, what can seem more inadmissible than to contend that the reasoning process has to do with any thing more? We seem to have come to one of Bacon's Prerogative Instances; an *experimentum crucis* on the nature of reasoning itself. [p. 189]

Nevertheless, it will appear on consideration, that this apparently so decisive instance is no instance at all; that there is in every step of an arithmetical or algebraical calculation a real induction, a real inference of facts from facts; and that what disguises the induction is simply its comprehensive nature, and the consequent extreme generality of the language. All numbers must be numbers of something: there are no such things as numbers in the abstract. *Ten* must mean ten bodies, or ten sounds, or ten beatings of the pulse. But though numbers must be numbers of something, they may be numbers of any thing. Propositions, therefore, concerning numbers, have the remarkable peculiarity that they are propositions concerning all things whatever; all objects, all existences of every kind, known to our experience. All things possess quantity; consist of parts which can be numbered; and in that character possess all the properties which are called properties of numbers. That half of four is two, must be true whatever the word four represents, whether four hours, four miles, or four pounds weight. We need only conceive a thing divided into four equal parts (and all things may be conceived as so divided), to be able to predicate of it every property of the number four, that is, every arithmetical proposition in which the number four stands on one side of the equation. Algebra extends the generalization still farther: every number represents that particular number of all things without distinction, but every algebraical symbol does more, it represents all numbers without distinction. As soon as we conceive a thing divided into equal parts, without knowing into what number of parts, we may call it *a* or *x*, and apply to it, without danger of error, every algebraical formula in the books. The proposition, $2(a + b) = 2a + 2b$, is a truth co-extensive with all nature. Since then algebraical truths are true of all things whatever, and not, like those of geometry, true of lines only or of angles only, it is no wonder that the symbols should not excite in our minds ideas of any things in particular. When we demonstrate the forty-seventh proposition of Euclid, it is not necessary that the words should raise in us an image of all right-angled triangles, but only of some one right-angled triangle: so in algebra we need not, under the symbol *a*, picture to ourselves all things whatever, but only some one thing; why not, then, the letter itself? The mere written characters, *a*, *b*, *x*, *y*, *z*, serve as well for representatives of Things in general, as any more complex and apparently more concrete conception. That we are conscious of them, however, in their character of things, and not of mere signs, is evident from the fact that our whole process of reasoning is carried on by predicating of them the properties of things. In resolving an algebraic equation, by what rules do we proceed? By applying at each step to *a*, *b*, and *x*, the proposition that equals added to equals make equals; that equals taken from equals leave equals; and other propositions founded on these two. These are not properties of language, or of signs as such, but of magnitudes, which is as much as to say, of all things. The inferences, therefore, which are successively drawn, are inferences concerning things, not symbols; though as any Things whatever will serve the turn, there is no necessity for keeping the idea of the Thing at all distinct, and consequently the process of thought may, in this case, be allowed without danger to do what all processes of thought, when they have been performed often, will do if permitted, namely, to become entirely mechanical. Hence the general language of algebra comes to be used familiarly without exciting ideas, as all other general language is prone to do from mere habit, though in no other case than this can it be done with complete safety. [p. 190] But when we look back to see from whence the probative force of the process is derived, we find that at every single step, unless we suppose ourselves to be thinking and talking of the things, and not the mere symbols, the evidence fails.

There is another circumstance, which, still more than that which we have now mentioned, gives plausibility to the notion that the propositions of arithmetic and algebra are merely verbal. That is, that when considered as propositions respecting Things, they all have the appearance of being identical propositions. The assertion, Two and one is equal to three, considered as an assertion respecting objects, as for instance, "Two pebbles and one pebble are equal to three pebbles," does not affirm equality between two collections of pebbles, but absolute identity. It affirms that if we put one pebble to two pebbles, those very pebbles are three. The objects, therefore, being the very same, and the mere assertion that "objects are themselves" being insignificant, it seems but natural to consider the proposition, Two and one is equal to three, as asserting mere identity of signification between the two names.

This, however, though it looks so plausible, will not bear examination. The expression "two pebbles and one pebble," and the expression "three pebbles," stand indeed for the same aggregation of objects, but they by no means stand for the same physical fact. They are names of the same objects, but of those objects in two different states: though they *de*note the same things, their *con*notation is different. Three pebbles in two separate parcels, and three pebbles in one parcel, do not make the same impression on our senses; and the assertion that the very same pebbles may by an alteration of place and arrangement be made to produce either the one set of sensations or the other, though a very familiar proposition, is not an identical one. It is a truth known to us by early and constant experience: an inductive truth; and such truths are the foundation of the science of Number. The fundamental truths of that science all rest on the evidence of sense; they are proved by showing to our eyes and our fingers that any given number of objects—ten balls, for example—may by separation and re-arrangement exhibit to our senses all the different sets of numbers the sums of which is equal to ten. All the improved methods of teaching arithmetic to children proceed on a knowledge of this fact. All who wish to carry the child's *mind* along with them in learning arithmetic; all who wish to teach numbers, and not mere ciphers—now teach it through the evidence of the senses, in the manner we have described.

We may, if we please, call the proposition, "Three is two and one," a definition of the number three, and assert that arithmetic, as it has been asserted that geometry, is a science founded on definitions. But they are definitions in the geometrical sense, not the logical; asserting not the meaning of a term only, but along with it an observed matter of fact. The proposition, "A circle is a figure bounded by a line which has all its points equally distant from a point within it," is called the definition of a circle; but the proposition from which so many consequences follow, and which is really a first principle in geometry, is, that figures answering to this description exist. And thus we may call "Three is two and one" a definition of three; but the calculations which depend on that proposition do not follow from the definition itself, but from an arithmetical theorem presupposed in it, namely, that collections of objects exist, which while they impress the senses thus, ∵, may be separated into two parts, thus, ∘∘ ∘. This proposition being granted, we term all such parcels Threes, after [p. 191] which the enunciation of the above-mentioned physical fact will serve also for a definition of the word Three.

The Science of Number is thus no exception to the conclusion we previously arrived at, that the processes even of deductive sciences are altogether inductive, and that their first principles are generalizations from experience. It remains to be examined whether this science resembles geometry in the further circumstance, that some of its inductions are not exactly true; and that the peculiar certainty ascribed to it, on account of which its propositions are called Necessary Truths, is fictitious and hypothetical, being true in no other sense than that those propositions legitimately follow from the hypothesis of the truth of premises which are avowedly mere approximations to truth.

§ 3. The inductions of arithmetic are of two sorts: first, those which we have just expounded, such as One and one are two, Two and one are three, etc., which may be called the definitions of the various numbers, in the improper or geometrical sense of the word Definition; and secondly, the two following axioms: The sums of equals are equal, The differences of equals are equal. These two are sufficient; for the corresponding propositions respecting unequals may be proved from these by a *reductio ad absurdum*.

These axioms, and likewise the so-called definitions, are, as has already been said, results of induction; true of all objects whatever, and, as it may seem, exactly true, without the hypothetical assumption of unqualified truth where an approximation to it is all that exists. The conclusions, therefore, it will naturally be inferred, are exactly true, and the science of number is an exception to other demonstrative sciences in this, that the categorical certainty which is predicable of its demonstrations is independent of all hypothesis.

On more accurate investigation, however, it will be found that, even in this case, there is one hypothetical element in the ratiocination. In all propositions concerning numbers, a condition is implied, without which none of them would be true; and that condition is an assumption which may be false. The condition is, that 1=1; that all the numbers are numbers of the same or of equal units. Let this be doubtful, and not one of the propositions of arithmetic will hold true. How can we know that one pound and one pound make two pounds, if one of the pounds may be troy, and the other avoirdupois? They may not make two pounds of either, or of any weight. How can we know that a forty-horse power is always equal to itself, unless we assume that all horses are of equal strength? It is certain that 1 is always equal in *number* to 1; and where the mere number of objects, or of the parts of an object, without supposing them to be equivalent in any other respect, is all that is material, the conclusions of arithmetic, so far as they go to that alone, are true without mixture of hypothesis. There are such cases in statistics; as, for instance, an inquiry into the amount of the population of any country. It is indifferent to that inquiry whether they are grown people or children, strong or weak, tall or short; the only thing we want to ascertain is their number. But whenever, from equality or inequality of number, equality or inequality in any other respect is to be inferred, arithmetic carried into such inquiries becomes as hypothetical a science as geometry. All units must be assumed to be equal in that other respect; and this is never accurately true, for one actual pound weight is not exactly equal to another, nor one measured mile's length to another; a nicer balance, or more accurate measuring instruments, would always detect some difference. [p. 192]

What is commonly called mathematical certainty, therefore, which comprises the twofold conception of unconditional truth and perfect accuracy, is not an attribute of all mathematical truths, but of those only which relate to pure Number, as distinguished from Quantity in the more enlarged sense; and only so long as we abstain from supposing that the numbers are a precise index to actual quantities. The certainty usually ascribed to the conclusions of geometry, and even to those of mechanics, is nothing whatever but certainty of inference. We can have full assurance of particular results under particular suppositions, but we can not have the same assurance that these suppositions are accurately true, nor that they include all the data which may exercise an influence over the result in any given instance.

§ 4. It appears, therefore, that the method of all Deductive Sciences is hypothetical. They proceed by tracing the consequences of certain assumptions; leaving for separate consideration whether the assumptions are true or not, and if not exactly true, whether they are a sufficiently near approximation to the truth. The reason is obvious. Since it is only in questions of pure number that the assumptions are exactly true, and even there only so long as no conclusions except purely numerical ones are to be founded on them; it must, in all other cases of deductive investigation, form a part of the inquiry, to determine how much the assumptions want of being exactly true in the case in hand. This is generally a matter of observation, to be repeated in every fresh case; or if it has to be settled by argument instead of observation, may require in every different case different evidence, and present every degree of difficulty, from the lowest to the highest. But the other part of the process—namely, to determine what else may be concluded if we find, and in proportion as we find, the assumptions to be true—may be performed once for all, and the results held ready to be employed as the occasions turn up for use. We thus do all beforehand that can be so done, and leave the least possible work to be performed when cases arise and press for a decision. This inquiry into the inferences which can be drawn from assumptions, is what properly constitutes Demonstrative Science.

It is of course quite as practicable to arrive at new conclusions from facts assumed, as from facts observed; from fictitious, as from real, inductions. Deduction, as we have seen, consists of a series of inferences in this form—*a* is a mark of *b*, *b* of *c*, *c* of *d*, therefore *a* is a mark of *d*, which last may be a truth inaccessible to direct observation. In like manner it is allowable to say, *suppose* that *a* were a mark of *b*, *b* of *c*, and *c* of *d*, *a* would be a mark of *d*, which last conclusion was not thought of by those who laid down the premises. A system of propositions as complicated as geometry might be deduced from assumptions which are false; as was done by Ptolemy, Descartes, and others, in their attempts to explain synthetically the phenomena of the solar system on the supposition that the apparent motions of the heavenly bodies were the real motions, or were produced in some way more or less different from the true one. Sometimes the same thing is knowingly done, for the purpose of showing the falsity of the assumption; which is called a *reductio ad absurdum*. In such cases, the reasoning is as follows: *a* is a mark of *b*, and *b* of *c*; now if c were also a mark of *d, a* would be a mark of *d*; but *d* is known to be a mark of the absence of *a*; consequently *a* would be a mark of its own absence, which is a contradiction; therefore *c* is not a mark of *d*.

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Book III. Of Induction.

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Chapter XXIV. Of The Remaining Laws Of Nature.

… [p. 429]

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§ 5. There is something which seems to require explanation, in the fact that the immense multitude of truths (a multitude still as far from being exhausted as ever) comprised in the mathematical sciences, can be elicited from so small a number of elementary laws. One sees not, at first, how it is that there can be room for such an infinite variety of true propositions, on subjects apparently so limited.

To begin with the science of number. The elementary or ultimate truths of this science are the common axioms concerning equality, namely, "Things which are equal to the same thing are equal to one another," and "Equals added to equals make equal sums" (no other axioms are required),* together with the definitions of the various numbers. Like other so-called definitions, these are composed of two things, the explanation of a name, and the assertion of a fact; of which the latter alone can form a first principle [p. 430] or premise of a science. The fact asserted in the definition of a number is a physical fact. Each of the numbers two, three, four, etc., denotes physical phenomena, and connotes a physical property of those phenomena. Two, for instance, denotes all pairs of things, and twelve all dozens of things, connoting what makes them pairs, or dozens; and that which makes them so is something physical; since it can not be denied that two apples are physically distinguishable from three apples, two horses from one horse, and so forth; that they are a different visible and tangible phenomenon. I am not undertaking to say what the difference is; it is enough that there is a difference of which the senses can take cognizance. And although a hundred and two horses are not so easily distinguished from a hundred and three, as two horses are from three—though in most positions the senses do not perceive any difference—yet they may be so placed that a difference will be perceptible, or else we should never have distinguished them, and given them different names. Weight is confessedly a physical property of things; yet small differences between great weights are as imperceptible to the senses in most situations, as small differences between great numbers; and are only put in evidence by placing the two objects in a peculiar position—namely, in the opposite scales of a delicate balance.

What, then, is that which is connoted by a name of number? Of course, some property belonging to the agglomeration of things which we call by the name; and that property is, the characteristic manner in which the agglomeration is made up of, and may be separated into, parts. I will endeavor to make this more intelligible by a few explanations.

When we call a collection of objects *two*, *three*, or *four*, they are not two, three, or four in the abstract; they are two, three, or four things of some particular kind; pebbles, horses, inches, pounds' weight. What the name of number connotes is, the manner in which single objects of the given kind must be put together, in order to produce that particular aggregate. If the aggregate be of pebbles, and we call it *two*, the name implies that, to compose the aggregate, one pebble must be joined to one pebble. If we call it *three*, one and one and one pebble must be brought together to produce it, or else one pebble must be joined to an aggregate of the kind called *two*, already existing. The aggregate which we call *four*, has a still greater number of characteristic modes of formation. One and one and one and one pebble may be brought together; or two aggregates of the kind called *two* may be united; or one pebble may be added to an aggregate of the kind called *three*. Every succeeding number in the ascending series, may be formed by the junction of smaller numbers in a progressively greater variety of ways. Even limiting the parts to two, the number may be formed, and consequently may be divided, in as many different ways as there are numbers smaller than itself; and, if we admit of threes, fours, etc., in a still greater variety. Other modes of arriving at the same aggregate present themselves, not by the union of smaller, but by the dismemberment of larger aggregates. Thus, *three pebbles* may be formed by taking away one pebble from an aggregate of four; *two pebbles*, by an equal division of a similar aggregate; and so on.

Every arithmetical proposition; every statement of the result of an arithmetical operation; is a statement of

* The axiom, "Equals subtracted from equals leave equal differences," may be demonstrated from the two axioms in the text. If $A = a$ and $B = b$, $A - B = a - b$. For if not, let $A - B = a - b + c$. Then since $B = b$, adding equals to equals, $A = a + c$. But $A = a$. Therefore $a = a + c$, which is impossible.

This proposition having been demonstrated, we may, by means of it, demonstrate the following: "If equals be added to unequals, the sums are unequal." If $A = a$ and B not $= b$, $A + B$ is not $= a + b$. For suppose it to be so. Then, since $A = a$ and $A + B = a + b$, subtracting equals from equals, $B = b$; which is contrary to the hypothesis.

So again, it may be proved that two things, one of which is equal and the other unequal to a third thing, are unequal to one another. If $A = a$ and A not = B, neither is $a = B$. For suppose it to be equal. Then since $A = a$ and $a = B$, and since things equal to the same thing are equal to one another $A = B$; which is contrary to the hypothesis.

one of the modes of formation of a given number. It affirms that a certain aggregate might have been formed by putting together certain other aggregates, or by withdrawing certain portions of some aggregate; and that, by consequence, we might reproduce those aggregates from it, by reversing the process. [p. 431]

Thus, when we say that the cube of 12 is 1728, what we affirm is this: that if, having a sufficient number of pebbles or of any other objects, we put them together into the particular sort of parcels or aggregates called twelves; and put together these twelves again into similar collections; and, finally, make up twelve of these largest parcels; the aggregate thus formed will be such a one as we call 1728; namely, that which (to take the most familiar of its modes of formation) may be made by joining the parcel called a thousand pebbles, the parcel called seven hundred pebbles, the parcel called twenty pebbles, and the parcel called eight pebbles.

The converse proposition that the cube root of 1728 is 12, asserts that this large aggregate may again be decomposed into the twelve twelves of twelves of pebbles which it consists of.

The modes of formation of any number are innumerable; but when we know one mode of formation of each, all the rest may be determined deductively. If we know that *a* is formed from *b* and *c*, *b* from *a* and *e*, *c* from *d* and *f*, and so forth, until we have included all the numbers of any scale we choose to select (taking care that for each number the mode of formation be really a distinct one, not bringing us round again to the former numbers, but introducing a new number), we have a set of propositions from which we may reason to all the other modes of formation of those numbers from one another. Having established a chain of inductive truths connecting together all the numbers of the scale, we can ascertain the formation of any one of those numbers from any other by merely traveling from one to the other along the chain. Suppose that we know only the following modes of formation: $6 = 4 + 2$, $4 = 7 - 3$, $7 = 5 + 2$, $5 = 9 - 4$. We could determine how 6 may be formed from 9. For $6 = 4 + 2 = 7 - 3 + 2 = 5 + 2 - 3 + 2 = 9 - 4 + 2 - 3 + 2$. It may therefore be formed by taking away 4 and 3, and adding 2 and 2. If we know besides that $2 + 2 = 4$, we obtain 6 from 9 in a simpler mode, by merely taking away 3.

It is sufficient, therefore, to select one of the various modes of formation of each number, as a means of ascertaining all the rest. And since things which are uniform, and therefore simple, are most easily received and retained by the understanding, there is an obvious advantage in selecting a mode of formation which shall be alike for all; in fixing the connotation of names of number on one uniform principle. The mode in which our existing numerical nomenclature is contrived possesses this advantage, with the additional one, that it happily conveys to the mind two of the modes of formation of every number. Each number is considered as formed by the addition of a unit to the number next below it in magnitude, and this mode of formation is conveyed by the place which it occupies in the series. And each is also considered as formed by the addition of a number of units less than ten, and a number of aggregates each equal to one of the successive powers of ten; and this mode of its formation is expressed by its spoken name, and by its numerical character.

What renders arithmetic the type of a deductive science, is the fortunate applicability to it of a law so comprehensive as "The sums of equals are equals:" or (to express the same principle in less familiar but more characteristic language), Whatever is made up of parts, is made up of the parts of those parts. This truth, obvious to the senses in all cases which can be fairly referred to their decision, and so general as to be coextensive with nature itself, being true of all sorts of phenomena (for all admit of being numbered), must be considered an inductive truth, or law of nature, of the highest order. And every arithmetical operation is an application of this [p. 432] law, or of other laws capable of being deduced from it. This is our warrant for all calculations. We believe that five and two are equal to seven, on the evidence of this inductive law, combined with the definitions of those numbers. We arrive at that conclusion (as all know who remember how they first learned it) by adding a single unit at a time: $5 + 1 = 6$, therefore $5 + 1 + 1 = 6 + 1 = 7$; and again $2 = 1 + 1$, therefore $5 + 2 = 5 + 1 + 1 = 7$.

§ 6. Innumerable as are the true propositions which can be formed concerning particular numbers, no adequate conception could be gained, from these alone, of the extent of the truths composing the science of number. Such propositions as we have spoken of are the least general of all numerical truths. It is true that even these are co-extensive with all nature; the properties of the number four are true of all objects that are divisible into four equal parts, and all objects are either actually or ideally so divisible. But the propositions which compose the science of algebra are true, not of a particular number, but of all numbers; not of all things under the condition of being divided in a particular way, but of all things under the condition of being divided in any way—of being designated by a number at all.

Since it is impossible for different numbers to have any of their modes of formation completely in common, it is a kind of paradox to say, that all propositions which can be made concerning numbers relate to their modes of formation from other numbers, and yet that there are propositions which are true of all numbers. But this very paradox leads to the real principle of generalization concerning the properties of numbers. Two different numbers can not be formed in the same manner from the same numbers; but they may be formed in the same manner from different numbers; as nine is formed from three by multiplying it into itself, and sixteen is formed from four by the same process. Thus there arises a classification of modes of formation, or in the language commonly used by mathematicians, a classification of Functions. Any number, considered as formed from any other number, is called a function of it; and there are as many kinds of functions as there are modes of formation. The simple functions are by no means numerous, most functions being formed by the combination of several of the operations which form simple functions, or by successive repetitions of some one of those operations. The simple functions of any number x are all reducible to the following forms: $x + a$, $x - a$, ax, x/a, x^a, $x^{1/a}$, log.x (to the base a), and the same expressions varied by putting x for a and a for x, wherever that substitution would alter the value: to which, perhaps, ought to be added sin x, and arc (sin$=x$). All other functions of x are formed by putting some one or more of the simple functions in the place of x or a, and subjecting them to the same elementary operations.

In order to carry on general reasonings on the subject of Functions, we require a nomenclature enabling us to express any two numbers by names which, without specifying what particular numbers they are, shall show what function each is of the other; or, in other words, shall put in evidence their mode of formation from one another. The system of general language called algebraical notation does this. The expressions a and $a^2 + 3a$ denote, the one any number, the other the number formed from it in a particular manner. The expressions a, b, n, and $(a + b)^n$, denote any three numbers, and a fourth which is formed from them in a certain mode.

The following may be stated as the general problem of the algebraical calculus: F being a certain function of a given number, to find what function [p. 433] F will be of any function of that number. For example, a binomial $a + b$ is a function of its two parts a and b, and the parts are, in their turn, functions of $a + b$: now $(a + b)^n$ is a certain function of the binomial; what function will this be of a and b, the two parts? The answer to this question is the binomial theorem. The formula $(a + b)^n = a^n + n/1\ a^{n-1}\ b + (n.n-1)/1.2\ a^{n-2}\ b^2$ +, etc., shows in what manner the number which is formed by multiplying $a + b$ into itself n times, might be formed without that process, directly from a, b, and n. And of this nature are all the theorems of the science of number. They assert the identity of the result of different modes of formation. They affirm that some mode of formation from x, and some mode of formation from a certain function of x, produce the same number.

Besides these general theorems or formulæ, what remains in the algebraical calculus is the resolution of equations. But the resolution of an equation is also a theorem. If the equation be $x^2 + ax = b$, the resolution of this equation, viz., $x = -1/2\ a \pm \sqrt{(1/4\ a^2 + b)}$, is a general proposition, which may be regarded as an answer to the question, If b is a certain function of x and a (namely $x^2 + ax$), what function is x of b and a? The resolution of equations is, therefore, a mere variety of the general problem as above stated. The problem is—Given a function, what function is it of some other function? And in the resolution of an equation, the question is, to find what function of one of its own functions the number itself is.

Such, as above described, is the aim and end of the calculus. As for its processes, every one knows that they are simply deductive. In demonstrating an algebraical theorem, or in resolving an equation, we travel from the *datum* to the *quæsitum* by pure ratiocination; in which the only premises introduced, besides the original hypotheses, are the fundamental axioms already mentioned—that things equal to the same thing are equal to one another, and that the sums of equal things are equal. At each step in the demonstration or in the calculation, we apply one or other of these truths, or truths deducible from them, as, that the differences, products, etc., of equal numbers are equal.

It would be inconsistent with the scale of this work, and not necessary to its design, to carry the analysis of the truths and processes of algebra any further; which is also the less needful, as the task has been, to a very great extent, performed by other writers. Peacock's Algebra, and Dr. Whewell's *Doctrine of Limits*, are full of instruction on the subject. The profound treatises of a truly philosophical mathematician, Professor De Morgan, should be studied by every one who desires to comprehend the evidence of mathematical truths, and the meaning of the obscurer processes of the calculus, and the speculations of M. Comte, in his *Cours de Philosophie Positive*, on the philosophy of the higher branches of mathematics, are among the many valuable gifts for which philosophy is indebted to that eminent thinker.

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