5.3. Conditional proofs: bottling inference

5.3.0. Overview

The use of implies for both the conditional and entailment suggests an analogy between the two, and this analogy figures in many of the deductive properties of conditionals.

5.3.1. Conditionalization

The basic grounds for concluding a conditional are the demonstrated ability to move from its antecedent as an assumption to its consequent as a goal.

5.3.2. Detachment

The chief significance of having a conditional as premise is the power to move from its antecedent as a resource to its consequent as a further resource.

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5.3.1. Conditionalization

The truth conditions of the conditional, which count $\varphi \to \psi$ as true except when φ is **T** and ψ is **F**, may have reminded you of the definition of implication, which says that φ implies ψ if and only if there is no possible world in which φ is **T** and ψ is **F**. Even though similar, the two ideas are not the same, and the distinction between material implication on the one hand and logical implication on the other points to the difference between them. Saying that a conditional $\varphi \to \psi$ is true rules out only the actual occurrence of the values **T** for φ and **F** for ψ while saying that φ logically implies, or entails, ψ rules out the occurrence of this pattern in any possible world. The forecast It will rain tomorrow if the front moves through does not commit a meteorologist to the view that It will rain tomorrow is logically implied by The front will move through tomorrow.

This difference can be brought out in another way. In cases where a relation of entailment holds, the corresponding conditional is not only true but tautologous. For example, because It was hot and humid $\vDash$ It was hot, the conditional If it was hot and humid, it was hot tells us nothing; it is a tautology. And we can state this as a general principle: φ entails ψ if and only if $\varphi \to \psi$ is a tautology—in notation, $\varphi \vDash \psi$ if and only if $\vDash \varphi \to \psi$. Either way we are saying that we fail to have φ true and ψ false not merely in the actual world but in all possible worlds.

Since to be a tautology is to be a valid conclusion from no premises at all, the principle just stated provides a partial account of when a conditional is a valid conclusion. To cover cases where there are premises we can use the idea of *implication given*, or *relative to*, a set of additional premises. For example, a weather forecaster might say that the passing of a front "implies" rain, intending to rest this relation between the passing of the front and rain on certain assumptions about the conditions of the atmosphere and laws of meteorology. And when a scientific hypothesis is said to "imply" a certain result for an experimental test, this implication is based on certain assumptions about the behavior of the experimental set up. In such cases we say that a sentence ψ cannot be false when a sentence φ is true, provided that certain further assumptions Γ are true as well. But this is just to say that ψ is entailed by φ taken together with Γ—i.e., that $\Gamma, \varphi \vDash \psi$. So relative implication is really just entailment with one premise singled out for special attention, something that it is quite reasonable to do when, as in the examples above, the set Γ of further premises is large or lacks definite boundaries.

Another way of singling out one assumption from a group of others is to make the conclusion conditional upon it. For example, we might say that, based on certain assumptions about the weather, we can conclude that it will rain if the front passes or that, based on assumptions about the experimental set up, we can conclude that an experiment will yield a certain result if our hypothesis is true. But this way of giving special attention to one of a group of assumptions is equivalent to making a claim of relative implication—that is, a conditional is a valid conclusion from given premises if and only if its antecedent implies its conclusion given those premises. And this gives us our account of conditional conclusions:

LAW FOR THE CONDITIONAL AS A CONCLUSION. $\Gamma \vDash \varphi \rightarrow \psi$ if and only if $\Gamma, \varphi \vDash \psi$ (for any set Γ and any sentences φ and ψ).

To see the truth of this law, note that an entailment $\Gamma \vDash \varphi \rightarrow \psi$ will hold if and only if there is no possible world in which $\varphi \rightarrow \psi$ is false while all members of Γ are true. But the sort of possible world that this rules out is one in which ψ is false while φ and the members of Γ are all true—i.e., one which is a counterexample to the argument Γ, φ / ψ. And to rule out such a possibility is to say that $\Gamma, \varphi \vDash \psi$.

Reading the law above from right to left, we move a premise past the sign $\vDash$, making the conclusion conditional on it. We will use the term *conditionalization* for this operation. Any result of the process is a *conditionalization of* the argument, and we will sometimes say, more specifically, that it is a *conditionalization on* the premise that is moved.

The law for the conditional as a conclusion tells us that an argument Γ / $\varphi \rightarrow \psi$ is valid if and only if the argument Γ, φ / ψ is valid. Moving from the first argument to the second will lead us to consider the latter argument in cases where we do not know the premise φ to be true. In such cases, Γ, φ / ψ will be an argument concerning a hypothetical situation, a hypothetical argument in the sense introduced in 4.2.2. Modifying an example used there, we can see the validity of the argument at the left below by noting the validity of the one at the right.

Ann and Bill were not both home without the car being in the driveway	Ann and Bill were not both home without the car being in the driveway
The car was not in the driveway	The car was not in the driveway
	Ann was at home
If Ann was at home, Bill wasn't	Bill wasn't at home

The first argument is a conditionalization of the second, and the law for the conditional as a conclusion tells that the first is valid if and only if the second is. Someone who offers the first argument is unlikely to know whether or not Ann was at home because there would then be no reason to assert a merely conditional conclusion. Consequently, Ann was at home describes a situation the arguer will regard as hypothetical, and the second argument can be described as a hypothetical argument. This means that we establish conditionals the way we established disjunctions in the last chapter, as compounds that serve to state categorically the upshot of a hypothetical argument.

In derivations, we can plan for a goal that is a conditional by setting out to reach it by a hypothetical argument. The rule embodying this approach, *Conditional Proof* (CP), is shown in Figure 5.3.1-1.

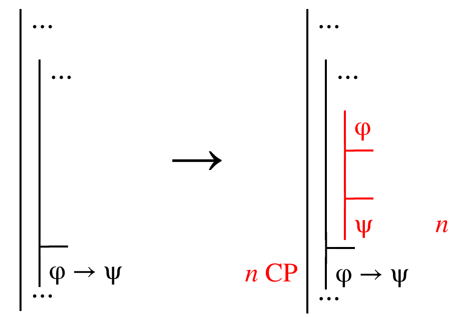


Fig. 5.3.1-1. Developing a derivation by planning for a conditional at stage *n*.

When we apply CP, we add the antecedent of the conditional goal as a supposition and set its consequent as a new goal. We thus plan to carry out, in a vertical direction, the transition indicated by the arrow in the conditional.

As an example, here is a derivation for the argument above.

	$\neg((A \wedge B) \wedge \neg C)$	2
	$\neg C$	(2)
	A	(3)
2 MPT	$\neg(A \wedge B)$	3
3 MPT	$\neg B$	(4)
	●	
4 QED	$\neg B$	1
1 CP	$A \rightarrow \neg B$	

Notice that the proximate argument of the gap after CP is applied is $\neg((A \wedge B) \wedge \neg C), \neg C, A$ / $\neg B$. That is, the ultimate argument of the derivation is a conditionalization on A of the proximate argument that results from CP. In short, when we apply CP, we plan to put ourselves in a position to conditionalize.

Of course, whenever we have premises, we are in a position to conditionalize, and the validity of the argument we have just considered establishes the validity of the result of conditionalization on its second premise: ¬ ((A ∧ B) ∧ ¬ C) / ¬ C → (A → ¬ B). This argument might be put into English as follows:

> Ann and Bill were not both home without the car being in the driveway
>
> ---
>
> Unless the car was in the driveway, Bill wasn't home if Ann was

A derivation for it will incorporate the derivation above, preceded by an initial use of CP.

	¬ ((A ∧ B) ∧ ¬ C)	3
	¬ C	(3)
	A	(4)
3 MPT	¬ (A ∧ B)	4
4 MPT	¬ B	(5)
	●	
5 QED	¬ B	2
2 CP	A → ¬ B	1
1 CP	¬ C → (A → ¬ B)	

After stage 2, we are making two suppositions—that the car is not in the driveway and that Ann is home—and we are thus considering a situation that is doubly hypothetical. And, in general, the most natural way of establishing the validity of a doubly conditional conclusion is by way of such a doubly hypothetical argument.

5.3.2. Detachment

The conditional was described by the philosopher Gilbert Ryle (1900-1976) as an *inference ticket*: it confers the right to travel from its antecedent to its consequent in an inference. It is the ability to make this trip that we demonstrate when we use a hypothetical argument to show that a conditional conclusion is valid. It is also true that, when we have a conditional as a resource, we have a ticket we can use to travel from its antecedent to its consequent.

> If it is day, it is light
> It is day
>
> ---
>
> It is light

The pattern of argument employing the latter idea, traditionally known as *modus ponens*, is perhaps the most well-known logical principle. The instance of it at the right was used by the Stoics as their standard example. The hedged character of the conditional means that, like disjunctions and not-both forms, it has no definite implications concerning the truth value of either of its components. *Modus ponens* tells us that if we add to the conditional the information that its antecedent is true, we can detach the consequent and assert it categorically.

In the traditional system of terminology we used for other detachment principles, this pattern of argument deserves the name *modus ponendo ponens*, and the more common form *modus ponens* is an abbreviated form of this. As was the case with disjunction and the not-both form, we have a pair of detachment principles for the conditional. However, due to the asymmetry of the conditional, these two principles take different forms and have different names:

> *Modus ponendo ponens*. φ → ψ, φ ⊨ ψ (for any sentences φ and ψ).
> *Modus tollendo tollens*. φ → ψ, $\neg^{\pm}$ ψ ⊨ $\neg^{\pm}$ φ (for any sentences φ and ψ).

The second is most often known by the abbreviated name *modus tollens*.

Notice that the conditional premise is used in very different ways in these two arguments. Often people who can agree about the truth of a conditional will disagree of the truth values of its components and will be ready to follow the different paths from the conditional that are laid out by these two principles, something that is reflected in the proverb One person's modus ponens is another person's modus tollens. Ann and Bill may agree that it will rain if the front moves through while Ann, who is convinced that the front will move through, concludes that it will rain and Bill, who is convinced that it will not rain, concludes that the front will not move through.

As is the case with other weak compounds, there are weakening principles for the conditional; but again we have two different forms:

> Weakening: ψ ⊨ φ → ψ and $\neg^{\pm}$ φ ⊨ φ → ψ (for any sentences φ and ψ).

Although these weakening principles can be used directly as attachment rules (and we will consider this use in 5.4.2), their most important function is to combine with the detachment principles for the conditional and the law of lemmas to support the detachment rules *Modus Ponendo Ponens* (MPP) and *Modus Tollendo Tollens* (MTT) shown in Figures 5.3.4-1 and 5.3.4-2.

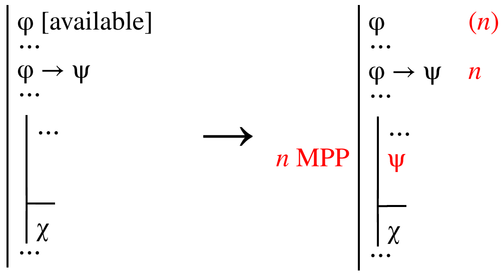


Fig. 5.3.2-1. Developing a derivation at stage *n* by exploiting a conditional whose antecedent is also an active resource.

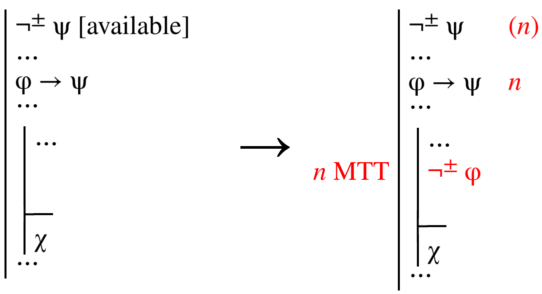
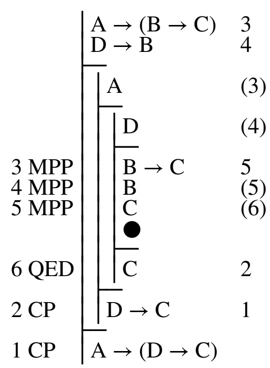


Fig. 5.3.2-2. Developing a derivation at stage *n* by exploiting a conditional when a sentence negating or de-negating its consequent is also an active resource.

The example at the right is typical of the way *modus ponens* functions along with CP. This can be described, very roughly, as a process of cashing in some tickets in order to get a new one with a different itinerary. One of the respects in which this metaphor works only roughly is that the “point of departure” or “destination” are sometimes themselves indicated by conditionals—that is, the “ticket” in question is sometimes more like a voucher for a ticket or some other sort of more abstract right.



5.3.s. Summary

1 The truth conditions of the conditional recall the definition of implication. Indeed, an implication $\varphi \vDash \psi$ will hold if and only if the conditional $\varphi \to \psi$ is a tautology. We can apply similar ideas to conditionals that are conclusions from factual premises by considering a notion of relative implication, implication depending on factual information. This idea appears in our law for the conditional as a conclusion. An entailment $\Gamma \vDash \varphi \to \psi$ holds when $\Gamma, \varphi \vDash \psi$—i.e., when ψ is implied by φ given the further premises Γ. The first of these entailments is a conditionalization of the second, and the second asserts the validity of a hypothetical argument. So an argument with a conditional conclusion is valid if and only if the hypothetical argument it conditionalizes is also valid. The derivation rule implementing this idea is Conditional Proof (CP).

2 The detachment principles for the conditional include the well-known *modus ponendo ponens* (usually called *modus ponens*), which is implemented as a rule Modus Ponendo Ponens (MPP), and a second detachment principle *modus tollendo tollens* (usually called *modus tollens*), which is implemented as a rule Modus Tollendo Tollens (MTT). *Modus ponens* in particular can be understood as the use of a conditional as an inference ticket licensing transitions from its antecedent to its consequent.

5.3.x. Exercise questions

1. Use derivations to establish each of the following. Notice that several are claims of equivalence and require two derivations. All these derivations are designed for the use of detachment rules (especially MPP and MTT), and a number will be quite long if they are not used. Attachment rules from previous chapters will occasionally be useful, and (since we do not yet have a full set of rules for the conditional) they are required in one of the derivations for **k**. Finally, note the leftwards arrow in the second premise of **b**. Although rules like MPP are written using a rightwards arrow they also apply to conditionals written using a leftwards arrow since a conditional ψ ← φ is just an alternative way of writing φ → ψ and plays the same role in derivations.

- **a.** B → C, A → B ⊨ A → C
- **b.** A → B, C ← B, C → D ⊨ A → D
- **c.** A → (B → C) ⊨ (A → B) → (A → C)
- **d.** A → (B → C), A → ¬ C ⊨ B → ¬ A
- **e.** ¬ A ≃ A → ¬ A
- **f.** A → B ≃ ¬ B → ¬ A
- **g.** A → B ≃ ¬ (A ∧ ¬ B)
- **h.** A → (B → C) ≃ (A ∧ B) → C
- **i.** (A → B) ∧ (A → C) ≃ A → (B ∧ C)
- **j.** (A → C) ∧ (B → C) ≃ (A ∨ B) → C
- **k.** (A → B) ∧ (B → C) ≃ (A ∨ B) → (B ∧ C)

2. Give English sentences illustrating **d**, **f**, **g**, and **k** of **1**. (Notice that each of **i-k** tells how to restate, as a single conditional, a particular sort of conjunction of conditionals; the last of the three implies each of the other two.)

The exercise machine is not designed to produce exercises and answers involving only the limited set of rules you have at this point.

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5.3.xa. Exercise answers

1. a.

```
          │B → C        3
          │A → B        2
          ├─
          ││A           (2)
          │├─
2 MPP     ││B           (3)
3 MPP     ││C           (4)
          ││●
          │├─
4 QED     ││C           1
          ├─
1 PC      │A → C
```

b.

```
          │A → B        2
          │C ← B        3
          │C → D        4
          ├─
          ││A           (2)
          │├─
2 MPP     ││B           (3)
3 MPP     ││C           (4)
4 MPP     ││D           (5)
          ││●
          │├─
5 QED     ││D           1
          ├─
1 CP      │A → D
```

c.

```
          │A → (B → C)            3
          ├─
          ││A → B                 4
          │├─
          │││A                     (3),(4)
          ││├─
3 MPP     │││B → C                 5
4 MPP     │││B                     (5)
5 MPP     │││C                     (6)
          │││●
          ││├─
6 QED     │││C                     2
          │├─
2 CP      ││A → C                 1
          ├─
1 CP      │(A → B) → (A → C)
```

d.

```
          │A → (B → C)   3
          │A → ¬ C       4
          ├─
          ││B            (5)
          │├─
          │││A           (3),(4)
          ││├─
3 MPP     │││B → C       5
4 MPP     │││¬ C         (6)
5 MPP     │││C           (6)
          │││●
          ││├─
6 Nc      │││⊥           2
          │├─
2 RAA     ││¬ A         1
          ├─
1 CP      │B → ¬ A
```

e.

```
          │¬ A          (2)
          ├─
          ││A
          │├─
          ││●
          │├─
2 QED     ││¬ A         1
          ├─
1 CP      │A → ¬ A
```

```
          │A → ¬ A      2
          ├─
          ││A           (2),(3)
          │├─
2 MPP     ││¬ A         (3)
          ││●
          │├─
3 Nc      ││⊥           1
          ├─
1 CP      │¬ A
```

f.

```
          │A → B        2
          ├─
          ││¬ B         (2)
          │├─
2 MTT     ││¬ A         (3)
          ││●
          │├─
3 QED     ││¬ A         1
          ├─
1 CP      │¬ B → ¬ A
```

```
          │¬ B → ¬ A    2
          ├─
          ││A           (2)
          │├─
2 MTT     ││B           (3)
          ││●
          │├─
3 QED     ││B           1
          ├─
1 CP      │A → B
```

g.

```
          | A → B            3
          |--
          | | A ∧ ¬ B        2
          | |--
2 Ext     | | A              (3)
2 Ext     | | ¬ B            (4)
3 MPP     | | B              (4)
          | | ●
          | |--
4 Nc      | | ⊥              1
          |--
1 RAA     | ¬ (A ∧ ¬ B)
```

```
          | ¬ (A ∧ ¬ B)      2
          |--
          | | A              (2)
          | |--
2 MPT     | | B              (3)
          | | ●
          | |--
3 QED     | | B              1
          |--
1 CP      | A → B
```

h.

```
          | A → (B → C)      3
          |--
          | | A ∧ B          2
          | |--
2 Ext     | | A              (3)
2 Ext     | | B              (4)
3 MPP     | | B → C          4
4 MPP     | | C              (5)
          | | ●
          | |--
5 QED     | | C              1
          |--
1 CP      | (A ∧ B) → C
```

```
          | (A ∧ B) → C            4
          |--
          | | A                    (5)
          | |--
          | | | B                  (6)
          | | |--
          | | | | ¬ C              (4)
          | | | |--
4 MTT     | | | | ¬ (A ∧ B)        5
5 MPT     | | | | ¬ B              (6)
          | | | | ●
          | | | |--
6 Nc      | | | | ⊥                3
          | | |--
3 IP      | | | C                  2
          | |--
2 CP      | | B → C                1
          |--
1 CP      | A → (B → C)
```

i.

```
          | (A → B) ∧ (A → C)      1
          |--
1 Ext     | A → B                  3
1 Ext     | A → C                  4
          | | A                    (3),(4)
          | |--
3 MPP     | | B                    (6)
4 MPP     | | C                    (7)
          | | | ●
          | | |--
6 QED     | | | B                  5
          | | | ●
          | | |--
7 QED     | | | C                  5
          | |--
5 Cnj     | | B ∧ C                2
          |--
2 CP      | A → (B ∧ C)
```

```
          | A → (B ∧ C)            3,7
          |--
          | | | A                  (3)
          | | |--
3 MPP     | | | B ∧ C              4
4 Ext     | | | B                  (5)
4 Ext     | | | C
          | | | ●
          | | |--
5 QED     | | | B                  2
          | |--
2 CP      | | A → B                1
          | | | A
          | | |--
7 MPP     | | | B ∧ C              8
8 Ext     | | | B
8 Ext     | | | C                  (9)
          | | | ●
          | | |--
9 QED     | | | C                  6
          | |--
6 QED     | | A → C                1
          |--
1 Cnj     | (A → B) ∧ (A → C)
```

j. Stages 3-5 and 7-11 in the derivation at the right could have taken analogous forms; they are varied here to show two approaches, one using attachment rules and the other without them.

```
          | (A → C) ∧ (B → C)      1
          |--
1 Ext     | A → C                  4
1 Ext     | B → C                  6
          | | A ∨ B                3
          | |--
          | | | A                  (4)
          | | |--
4 MPP     | | | C                  (5)
          | | | ●
          | | |--
5 QED     | | | C                  3
          | | | B                  (6)
          | | |--
6 MPP     | | | C                  (7)
          | | | ●
          | | |--
7 QED     | | | C                  3
          | |--
3 PC      | | C                    2
          |--
2 CP      | (A ∨ B) → C
```

```
          | (A ∨ B) → C            4,8
          |--
          | | | A                  (3)
          | | |--
3 Wk      | | | A ∨ B              X,(4)
4 MPP     | | | C                  (5)
          | | | ●
          | | |--
5 QED     | | | C                  2
          | |--
2 CP      | | A → C                1
          | | | B                  (11)
          | | |--
          | | | | ¬ C              (8)
          | | | |--
8 MTT     | | | | ¬ (A ∨ B)        9
          | | | | | | ¬ A
          | | | | | |--
          | | | | | | ●
          | | | | | |--
11 QED    | | | | | | B          10
          | | | | |--
10 PE     | | | | | A ∨ B        9
          | | | |--
9 CR      | | | | ⊥              7
          | | |--
7 IP      | | | C                  6
          | |--
6 CP      | | B → C                1
          |--
1 Cnj     | (A → C) ∧ (B → C)
```

k. Parallel arguments are again completed differently in the two gaps of each derivation—in the first, to show approaches with attachment rules and without them and, in the second, to show two ways of using attachment rules.

```
          | (A → B) ∧ (B → C)      1
          |--
1 Ext     | A → B                  4
1 Ext     | B → C                  5,10
          | | A ∨ B                3
          | |--
          | | | A                  (4)
          | | |--
4 MPP     | | | B                  (5)
5 MPP     | | | C
6 Adj     | | | B ∧ C              X,(7)
          | | | ●
          | | |--
7 QED     | | | B ∧ C              3
          | | | B                  (9),(10)
          | | |--
          | | | | ●
          | | | |--
9 QED     | | | | B                8
10 MPP    | | | | C                (11)
          | | | | ●
          | | | |--
11 QED    | | | | C                8
          | | |--
8 Cnj     | | | B ∧ C              3
          | |--
3 PC      | | B ∧ C                2
          |--
2 CP      | (A ∨ B) → (B ∧ C)
```

```
          | (A ∨ B) → (B ∧ C)      4,10
          |--
          | | | A                  (3)
          | | |--
3 Wk      | | | A ∨ B              X,(4)
4 MPP     | | | B ∧ C              5
5 Ext     | | | B                  (6)
5 Ext     | | | C
          | | | ●
          | | |--
6 QED     | | | B                  2
          | |--
2 CP      | | A → B                1
          | | | B                  (11)
          | | |--
          | | | | ¬ C              (9)
          | | | |--
9 Wk      | | | | ¬ (B ∧ C)        (10)
10 MTT    | | | | ¬ (A ∨ B)        (12)
11 Wk     | | | | A ∨ B            (12)
          | | | | ●
          | | | |--
12 Nc     | | | | ⊥                8
          | | |--
8 IP      | | | C                  7
          | |--
7 CP      | | B → C                1
          |--
1 Cnj     | (A → B) ∧ (B → C)
```

2. **d.** If Ann was there, then Carol was there if Bill was

Carol wasn't there if Ann was

Ann wasn't there if Bill was

f. If Ann was there, Bill was, too

If Bill wasn't there, Ann wasn't either

g. If Ann was there, Bill was there

Ann wasn't there without Bill being there

k. If Ann was there, Bill was there; and if Bill was there, Carol was there

---

If either Ann or Bill was there, then both Bill and Carol were there