

## Overview

### Basic system

Exploitation and planning rules			Rules for closing gaps		
sentence	as a resource	as a goal	when to close		rule
			co-aliases	resources	goal
atomic sentence	none	IP		$\varphi$	$\varphi$ QED
negation $\neg \varphi$	CR (if $\varphi$ not atomic & goal is $\perp$ )	RAA		$\varphi$ and $\neg \varphi$	$\perp$ Nc
conjunction $\varphi \wedge \psi$	Ext	Cnj		any	$\top$ ENV
disjunction $\varphi \vee \psi$	PC	PE		$\perp$	any EFQ
conditional $\varphi \rightarrow \psi$	RC (if goal is $\perp$ )	CP	$\tau \rightarrow \upsilon$	any	$\tau = \upsilon$ EC
universal $\forall x \theta x$	UI	UG	$\tau_1 \rightarrow \upsilon_1, \dots, \tau_n \rightarrow \upsilon_n$	$P\tau_1 \dots \tau_n$	$P\upsilon_1 \dots \upsilon_n$ QED=
existential $\exists x \theta x$	PCh	NcP	$\tau_1 \rightarrow \upsilon_1, \dots, \tau_n \rightarrow \upsilon_n$	$P\tau_1 \dots \tau_n$	$\perp$ Nc=

  

Detachment rules (optional)		
required resources	rule	
main	auxiliary	
$\neg(\varphi \wedge \psi)$	$\varphi$ or $\psi$	MPT
$\varphi \vee \psi$	$\neg^{\pm} \varphi$ or $\neg^{\pm} \psi$	MTP
$\varphi \rightarrow \psi$	$\varphi$	MPP
	$\neg^{\pm} \psi$	MTT

In addition, if the conditions for applying a rule are met except for differences between co-aliases, then the rule can be applied and is notated by adding “=”, QED= and Nc= are examples of this.

### Additional rules

Attachment rules		Rule for lemmas	
resource to be added	rule	prerequisite	rule
$\varphi \wedge \psi$	Adj	the goal is $\perp$	LFR
$\neg(\varphi \wedge \psi)$			
$\varphi \vee \psi$	Wk		
$\varphi \rightarrow \psi$			
$\tau = \upsilon$	CE		
$\theta \upsilon_1 \dots \upsilon_n$	Cng		
$\exists x \theta x$	EG		

## Derivation rules

### Basic system

logical form	Rules for developing gaps	
	as resource	as goal
atomic sentence	no rule	Indirect Proof (IP) $\frac{\dots}{\frac{\dots}{\frac{\dots}{\varphi \text{ [atomic]}} \perp} \neg \varphi} \rightarrow \dots$
negation $\neg \varphi$	Completing the reductio (CR) $\frac{\dots}{\frac{\dots}{\frac{\dots}{\neg \varphi \text{ [}\varphi \text{ is not atomic]}} \perp} \neg \varphi} \rightarrow \dots$	Reductio ad absurdum (RAA) $\frac{\dots}{\frac{\dots}{\frac{\dots}{\varphi} \perp} \neg \varphi} \rightarrow \dots$
	Modus ponendo tollens (MPT) $\frac{\varphi \text{ [available]} \quad \neg(\varphi \wedge \psi)}{\dots} \rightarrow \frac{\dots}{\frac{\dots}{\chi} \neg^{\pm} \psi} \dots$	
conjunction $\varphi \wedge \psi$	Extraction (Ext) $\frac{\dots}{\frac{\dots}{\frac{\dots}{\varphi \wedge \psi} \dots} \varphi} \rightarrow \frac{\dots}{\frac{\dots}{\frac{\dots}{\psi} \dots} \varphi \wedge \psi} \dots$	Conjunction (Cnj) $\frac{\dots}{\frac{\dots}{\frac{\dots}{\varphi} \dots} \psi} \varphi \wedge \psi \rightarrow \dots$

Rules for developing gaps		
logical form	as resource	as goal
disjunction $\phi \vee \psi$	<b>Proof by Cases (PC)</b> $\frac{\dots \quad \phi \vee \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \phi \vee \psi \quad n}{\dots}$	<b>Proof of Exhaustion (PE)</b> $\frac{\dots \quad \dots}{\phi \vee \psi} \rightarrow \frac{\dots \quad \neg^{\pm} \phi \quad \dots}{\psi} \quad n$
	<b>Modus Tollendo Ponens (MTP)</b> $\frac{\dots \quad \neg^{\pm} \phi \text{ [available]} \quad \dots \quad \phi \vee \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \neg^{\pm} \phi \quad (n) \quad \dots \quad \phi \vee \psi \quad n}{\dots} \quad n \text{ MTP}$	<b>OR</b> $\frac{\dots \quad \dots}{\phi \vee \psi} \rightarrow \frac{\dots \quad \neg^{\pm} \psi \quad \dots}{\phi} \quad n$
	$\frac{\dots \quad \neg^{\pm} \psi \text{ [available]} \quad \dots \quad \phi \vee \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \neg^{\pm} \psi \quad (n) \quad \dots \quad \phi \vee \psi \quad n}{\dots} \quad n \text{ MTP}$	
conditional $\phi \rightarrow \psi$	<b>Rejecting a Conditional (RC)</b> $\frac{\dots \quad \phi \rightarrow \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \phi \rightarrow \psi \quad n}{\dots} \quad n \text{ RC}$	<b>Conditional Proof (CP)</b> $\frac{\dots \quad \dots}{\phi \rightarrow \psi} \rightarrow \frac{\dots \quad \phi \quad \dots}{\psi} \quad n$
	<b>Modus Ponendo Ponens (MPP)</b> $\frac{\dots \quad \phi \text{ [available]} \quad \dots \quad \phi \rightarrow \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \phi \quad (n) \quad \dots \quad \phi \rightarrow \psi \quad n}{\dots} \quad n \text{ MPP}$	
	<b>Modus Tollendo Tollens (MTT)</b> $\frac{\dots \quad \neg^{\pm} \psi \text{ [available]} \quad \dots \quad \phi \rightarrow \psi \quad \dots}{\dots} \rightarrow \frac{\dots \quad \neg^{\pm} \psi \quad (n) \quad \dots \quad \phi \rightarrow \psi \quad n}{\dots} \quad n \text{ MTT}$	

Rules for developing gaps		
logical form	as resource	as goal
universal $\forall x \theta x$	<b>Universal Instantiation (UI)</b> $\frac{\dots \quad \forall x \theta x \quad \dots}{\dots} \rightarrow \frac{\dots \quad \forall x \theta x \quad \tau:n \quad \dots}{\theta \tau} \quad n \text{ UI}$	<b>Universal Generalization (UG)</b> $\frac{\dots \quad \dots}{\forall x \theta x} \rightarrow \frac{\dots \quad \dots}{\theta a} \quad n$
existential $\exists x \theta x$	<b>Proof by Choice (PCh)</b> $\frac{\dots \quad \exists x \theta x \quad \dots}{\dots} \rightarrow \frac{\dots \quad \exists x \theta x \quad n \quad \dots}{\theta a} \quad n$	<b>Non-constructive Proof (NcP)</b> $\frac{\dots \quad \dots}{\exists x \theta x} \rightarrow \frac{\dots \quad \dots}{\forall x \neg^{\pm} \theta x} \quad n$

The parameter  $a$  used in UG and PCh should be new to the derivation; that is, it should appear only to the right of the scope line it labels

Rules for closing gaps (truth-functional logic)		
when to close	goal	rule
resources	goal	Quod Erat Demonstrandum (QED)
$\varphi$	$\varphi$	$\begin{array}{c} \dots \\ \varphi \text{ [available]} \\ \dots \\ \hline \varphi \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \bullet \\ \dots \\ \varphi \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> QED</p>
$\varphi$ and $\neg \varphi$	$\perp$	Non-contradiction (Nc)
		$\begin{array}{c} \dots \\ \neg \varphi \text{ [available]} \\ \dots \\ \varphi \text{ [available]} \\ \dots \\ \hline \perp \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \varphi \\ \dots \\ \bullet \\ \dots \\ \perp \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> Nc</p>
any	$\top$	Ex Nihilo Verum (ENV)
		$\begin{array}{c} \dots \\ \dots \\ \dots \\ \hline \top \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \bullet \\ \dots \\ \top \\ \dots \end{array}$ <p style="text-align: center;"><math>n</math> ENV</p>
$\perp$	any	Ex Falso Quodlibet (EFQ)
		$\begin{array}{c} \dots \\ \perp \\ \dots \\ \hline \varphi \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \bullet \\ \dots \\ \varphi \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> EFQ</p>

Rules for closing gaps (equations)			
when to close	resources	goal	rule
co-aliases	resources	goal	Equated Co-aliases (EC)
$\tau = \upsilon$	any	$\tau = \upsilon$	$\begin{array}{c} \dots \\ [\tau \text{ and } \upsilon \text{ are co-aliases}] \\ \dots \\ \hline \tau = \upsilon \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \bullet \\ \dots \\ \tau = \upsilon \\ \dots \end{array}$ <p style="text-align: center;"><math>n</math> EC</p>
$\tau = \upsilon$	$\neg \tau = \upsilon$	$\perp$	Distinguished Co-aliases (DC)
			$\begin{array}{c} \dots \\ [\tau \text{ and } \upsilon \text{ are co-aliases}] \\ \dots \\ \neg \tau = \upsilon \\ \dots \\ \hline \perp \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ \neg \tau = \upsilon \\ \dots \\ \bullet \\ \dots \\ \perp \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> DC</p>
$\tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n$	$P\tau_1 \dots \tau_n$	$P\upsilon_1 \dots \upsilon_n$	QED given equations (QED=)
			$\begin{array}{c} \dots \\ [\text{have co-alias relations:} \\ \tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n] \\ \dots \\ P\tau_1 \dots \tau_n \\ \dots \\ \hline P\upsilon_1 \dots \upsilon_n \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ [\text{have co-alias relations:} \\ \tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n] \\ \dots \\ P\tau_1 \dots \tau_n \\ \dots \\ \bullet \\ \dots \\ P\upsilon_1 \dots \upsilon_n \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> QED=</p>
$\tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n$	$P\tau_1 \dots \tau_n$	$\neg P\upsilon_1 \dots \upsilon_n$	Non-contradiction given equations (Nc=)
		$\perp$	$\begin{array}{c} \dots \\ [\text{have co-alias relations:} \\ \tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n] \\ \dots \\ P\tau_1 \dots \tau_n \\ \dots \\ \neg P\upsilon_1 \dots \upsilon_n \\ \dots \\ \hline \perp \\ \dots \end{array} \rightarrow \begin{array}{c} \dots \\ \dots \\ \dots \\ [\text{have co-alias relations:} \\ \tau_1 = \upsilon_1, \dots, \tau_n = \upsilon_n] \\ \dots \\ P\tau_1 \dots \tau_n \\ \dots \\ \bullet \\ \dots \\ \neg P\upsilon_1 \dots \upsilon_n \\ \dots \\ \perp \\ \dots \end{array} \quad (n)$ <p style="text-align: center;"><math>n</math> Nc=</p>

In addition to the following rules for closing gaps, if the conditions for applying any rule are met except for differences between co-aliases, then the rule can be applied and is notated by adding "=" to its label; QED= and Nc= below are examples of this in the case of rules for closing gaps.

**Additional rules** (not guaranteed to be progressive)

Attachment rules		
what is required	added resource	rule
$\varphi$ and $\psi$ are both available	$\varphi \wedge \psi$	<b>Adjunction (Adj)</b> $\frac{\begin{array}{c} \dots \\ \varphi \text{ [available]} \\ \dots \\ \psi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Adj} \frac{\begin{array}{c} \dots \\ \varphi \quad (n) \\ \dots \\ \psi \quad (n) \\ \dots \\ \hline \varphi \wedge \psi \quad X \end{array}}{\dots}$
		<b>Weakening (Wk)</b> $\frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \quad (n) \\ \dots \\ \neg(\varphi \wedge \psi) \quad X \end{array}}{\dots}$
$\neg^{\pm} \varphi$ or $\neg^{\pm} \psi$ is available	$\neg(\varphi \wedge \psi)$	$\frac{\begin{array}{c} \dots \\ \neg^{\pm} \psi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \neg^{\pm} \psi \quad (n) \\ \dots \\ \neg(\varphi \wedge \psi) \quad X \end{array}}{\dots}$
		$\frac{\begin{array}{c} \dots \\ \varphi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \varphi \quad (n) \\ \dots \\ \varphi \vee \psi \quad X \end{array}}{\dots}$
$\varphi$ or $\psi$ is available	$\varphi \vee \psi$	$\frac{\begin{array}{c} \dots \\ \psi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \psi \quad (n) \\ \dots \\ \varphi \vee \psi \quad X \end{array}}{\dots}$
		$\frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \quad (n) \\ \dots \\ \varphi \rightarrow \psi \quad X \end{array}}{\dots}$
$\neg^{\pm} \varphi$ or $\psi$ is available	$\varphi \rightarrow \psi$	$\frac{\begin{array}{c} \dots \\ \psi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \psi \quad (n) \\ \dots \\ \varphi \rightarrow \psi \quad X \end{array}}{\dots}$
		$\frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \text{ [available]} \\ \dots \\ \hline \chi \end{array}}{\dots} \rightarrow n \text{ Wk} \frac{\begin{array}{c} \dots \\ \neg^{\pm} \varphi \quad (n) \\ \dots \\ \varphi \rightarrow \psi \quad X \end{array}}{\dots}$

Attachment rules		
what is required	added resource	rule
$\tau$ and $\upsilon$ are co-aliases	$\tau = \upsilon$	<b>Co-alias Equation (CE)</b> $\frac{\begin{array}{c} \dots \\ [\tau \text{ and } \upsilon \text{ are co-aliases}] \\ \dots \\ \hline \varphi \end{array}}{\dots} \rightarrow n \text{ CE} \frac{\begin{array}{c} \dots \\ [\tau \text{ and } \upsilon \text{ are co-aliases}] \\ \dots \\ \tau = \upsilon \quad X \end{array}}{\dots}$
have co-alias relations $\tau_1 \dashv \upsilon_1, \dots,$ $\tau_n \dashv \upsilon_n$ and $\theta \tau_1 \dots \tau_n$ is available	$\theta \upsilon_1 \dots \upsilon_n$	<b>Congruence (Cng)</b> $\frac{\begin{array}{c} \dots \\ [\text{have co-alias relations:} \\ \tau_1 \dashv \upsilon_1, \dots, \tau_n \dashv \upsilon_n] \\ \dots \\ \hline \theta \tau_1 \dots \tau_n \end{array}}{\dots} \rightarrow n \text{ Cng} \frac{\begin{array}{c} \dots \\ [\text{have co-alias relations:} \\ \tau_1 \dashv \upsilon_1, \dots, \tau_n \dashv \upsilon_n] \\ \dots \\ \theta \tau_1 \dots \tau_n \quad (n) \\ \dots \\ \hline \theta \upsilon_1 \dots \upsilon_n \quad X \end{array}}{\dots}$
$\theta \tau$ is available	$\exists x \theta x$	<b>Existential Generalization (EG)</b> $\frac{\begin{array}{c} \dots \\ \theta \tau \\ \dots \\ \hline \varphi \end{array}}{\dots} \rightarrow n \text{ EG} \frac{\begin{array}{c} \dots \\ \theta \tau \quad (n) \\ \dots \\ \hline \exists x \theta x \quad X \end{array}}{\dots}$

Rule for lemmas	
prerequisite	rule
the goal is $\perp$	<b>Lemma for Reductio (LFR)</b> $\frac{\begin{array}{c} \dots \\ \perp \end{array}}{\dots} \rightarrow \frac{\begin{array}{c} \dots \\ \varphi \quad n \\ \dots \\ \hline \varphi \\ \dots \\ \hline \perp \quad n \\ \dots \\ \hline \perp \end{array}}{n \text{ LFR}}$