Appendices

Appendix A. Reference

A.0. Overview

A.1. Definitions and notation for basic concepts
The full range of deductive properties and relations

A.2. Logical forms
Forms expressed using one or more logical constants together with symbolic and English notation or readings

A.3. Truth tables
Tables that stipulate the meaning of the constants of truth-functional logic

A.4. Derivation rules
A guide to the use of derivation rules with links to the rules themselves

Glen Helman 11 Jul 2012

A.1. Definitions and notation for basic concepts

Most deductive properties or relation concerns a set or some specific number of *assumptions* and a set or some specific number of *alternatives*. When there is only one alternative, it is a *conclusion*. This is shown in the following table, where cells are labeled in boldface by the concept expressed as a noun, with the verbal or adjectival form shown in italics.

assumptions \ *alternatives*	*set* Σ	*two* ψ_1, ψ_2	*one* (*concl.*) ψ	*none*
set Γ	**relative exhaustiveness** Γ *renders* Σ *exhaustive*		**entailment** Γ *entails* ψ	**inconsistency** Γ is *inconsistent*
two φ_1, φ_2				**mutual exclusiveness** φ_1 and φ_2 are *mutually exclusive*
one φ			**implication** φ *implies* ψ	**absurdity** φ is *absurd*
none	**exhaustiveness** Σ is *exhaustive*	**(joint) exhaustiveness** ψ_1 and ψ_2 are (*jointly*) *exhaustive*	**tautologousness** ψ is *tautologous* (or is a *tautology*)	

Not appearing in the table are two relations that each abbreviate conjunctions of two claims drawn from the ones above.

conjunctive relation	*component relations*	
(logical) equivalence φ and ψ are (*logically*) *equivalent*	φ implies ψ	ψ implies φ
contradictoriness φ and ψ are *contradictory*	φ and ψ are mutually exclusive	φ and ψ are jointly exhaustive

There are also two alternative ways of applying the concept of inconsistency:

alternative statements (for assumptions Γ and φ)		
exclusion Γ *excludes* φ	**relative inconsistency** φ is *inconsistent with* Γ	**inconsistency** of the union Γ with φ added is *inconsistent*

Note that in this case all sentences involved count as assumptions.

All concepts appearing in the first table can be defined in the same way, as saying that their assumptions cannot be separated from their alternatives. This idea can be stated more specifically in two ways:

NEGATIVE DEFINITION: there is no possible world in which the assumptions (if any) are all true while the alternatives (if any) are all false.

POSITIVE DEFINITION: in each possible world in which the assumptions (if any) are all true, at least one alternative is true.

When there are no assumptions or no alternatives, the corresponding clause may be dropped from the negative form. The same is true for the clause regarding assumptions in the positive form; and, if there are no alternatives, that definition can be restated as: *in each possible world, the assumptions are not all true* (i.e., *at least one is false*).

The following table gives an explicit definition for each of these concepts and also indicates compact notation for the concept.

concept	*negative definition*	*positive definition*
φ is a *tautology* $\vDash \varphi$	There is no possible world in which φ is false.	φ is true in every possible world.
φ is *absurd* $\varphi \vDash$	There is no possible world in which φ is true.	φ is false in every possible world.
φ *implies* ψ $\varphi \vDash \psi$	There is no possible world in which φ is true and ψ is false.	ψ is true in every possible world in which φ is true.
φ and ψ are *mutually exclusive* $\varphi \vartriangle \psi$	There is no possible world in which φ and ψ are both true.	In each possible world, at least one of φ and ψ is false.
φ and ψ are *(jointly) exhaustive* $\varphi \triangledown \psi$	There is no possible world in which φ and ψ are both false.	In each possible world, at least one of φ and ψ is true.
φ and ψ are *(logically) equivalent* $\varphi \simeq \psi$	There is no possible world in which φ and ψ have different truth values.	In each possible world, φ and ψ have the same truth value as each other.
φ and ψ are *contradictory* $\varphi \bowtie \psi$	There is no possible world in which φ and ψ have the same truth value.	In each possible world, φ and ψ have opposite truth values.
Γ is *inconsistent* $\Gamma \vDash$	There is no possible world in which all members of Γ are true.	In each possible world, at least one member of Γ is false.
Γ is *exhaustive* $\vDash \Gamma$	There is no possible world in which all members of Γ are false.	In each possible world, at least one member of Γ is true.
Γ *entails* φ $\Gamma \vDash \varphi$	There is no possible world in which φ is false while all members of Γ are true.	φ is true in every possible world in which all members of Γ are true.
Γ *excludes* φ $\Gamma, \varphi \vDash$	There is no possible world in which φ is true while all members of Γ are true.	φ is false in every possible world in which all members of Γ are true.
Γ *renders* Σ *exhaustive* $\Gamma \vDash \Sigma$	There is no possible world in which all members of Γ are true while all members of Σ are false.	In each possible world in which all members of Γ are true, at least one member of Σ is true

All these concepts can be expressed in terms of relative exhaustiveness and also in terms of entailment. To express them in terms of relative exhaustiveness, simply list the assumptions (if any) to the left of ⊨ and list the alternatives (if any) to its right. The expression in terms of entailment for the concepts in the first table is shown below.

alternatives

assumptions	Σ	ψ_1, ψ_2	ψ	*none*
Γ	$\Gamma, \Sigma^{\bowtie} \vDash \bot$		$\Gamma \vDash \psi$	$\Gamma \vDash \bot$
φ_1, φ_2				$\varphi_1, \varphi_2 \vDash \bot$
φ			$\varphi \vDash \psi$	$\varphi \vDash \bot$
none	$\Sigma^{\bowtie} \vDash \bot$	${\psi_1}^{\bowtie} \vDash \psi_2$	$\vDash \psi$	

Here $\theta^{\bowtie}$ is any sentence contradictory to θ (such as its negation); and $\Sigma^{\bowtie}$ is any result of replacing each member of Σ by a sentence that is contradictory to it. The joint exhaustiveness of ψ_1 and ψ_2 may also be expressed by ${\psi_2}^{\bowtie} \vDash \psi_1$ and by ${\psi_1}^{\bowtie}, {\psi_2}^{\bowtie} \vDash \bot$. The general rule is that alternatives can be dropped if their contradictories are made assumptions (and vice versa) and that $\bot$ may used as a conclusion if there are no alternatives already.

Glen Helman 09 Sep 2012

A.2. Logical forms

Forms for which there is symbolic notation

	Symbolic notation	English notation *or* English reading	
Negation	¬ φ	not φ	
Conjunction	φ ∧ ψ	both φ and ψ	(φ and ψ)
Disjunction	φ ∨ ψ	either φ or ψ	(φ or ψ)
The conditional	φ → ψ ψ ← φ	if φ then ψ yes ψ if φ	(φ implies ψ) (ψ if φ)
Identity	τ = υ	τ is υ	
Predication	$\theta\tau_1\ldots\tau_n$	θ fits $\tau_1, \ldots, \tau_n$	A series of terms $\tau_1, \ldots, \tau_n$ can be read (series) $\tau_1, \ldots,$ ən τ_n (using the expression ən to distinguish this use of and from its use in conjunction and adding series when necessary to avoid ambiguity)
Compound term	$\gamma\tau_1\ldots\tau_n$	γ of $\tau_1, \ldots, \tau_n$ γ applied to $\tau_1, \ldots, \tau_n$	
Predicate abstract	$[\varphi]_{x_1\ldots x_n}$	what φ says of $x_1\ldots x_n$	
Functor abstract	$[\tau]_{x_1\ldots x_n}$	τ for $x_1\ldots x_n$	
Universal quantification	∀x θx	forall x θx everything, x, is such that θx	
Restricted universal	(∀x: ρx) θx	forall x st ρx θx everything, x, such that ρx is such that θx	
Existential quantification	∃x θx	forsome x θx something, x, is such that θx	
Restricted existential	(∃x: ρx) θx	forsome x st ρx θx something, x, such that ρx is such that θx	
Definite description	Ix ρx	the x st ρx the thing, x, such that ρx	

Some paraphrases of other forms

Truth-functional compounds

neither φ nor ψ	$\neg (\varphi \lor \psi)$ $\neg \varphi \land \neg \psi$
ψ only if φ	$\neg \psi \leftarrow \neg \varphi$
ψ unless φ	$\psi \leftarrow \neg \varphi$

Generalizations

All Cs are such that (… they …)	$(\forall x\colon x \text{ is a } C) \ldots x \ldots$
No Cs are such that (… they …)	$(\forall x\colon x \text{ is a } C) \neg \ldots x \ldots$
Only Cs are such that (… they …)	$(\forall x\colon \neg\, x \text{ is a } C) \neg \ldots x \ldots$

with:		add to the restriction:
with:	among Bs	$x \text{ is a } B$
	except Es	$\neg\, x \text{ is an } E$
	other than τ	$\neg\, x = \tau$

Numerical quantifier phrases

At least 1 C is such that (… it …)	$(\exists x\colon x \text{ is a } C) \ldots x \ldots$
At least 2 Cs are such that (… they …)	$(\exists x\colon x \text{ is a } C)\, (\exists y\colon y \text{ is a } C \land \neg\, y = x)\, (\ldots x \ldots \land \ldots y \ldots)$
Exactly 1 C is such that (… it …)	$(\exists x\colon x \text{ is a } C)\, (\ldots x \ldots \land (\forall y\colon y \text{ is a } C \land \neg\, y = x) \neg \ldots y \ldots)$ *or* $(\exists x\colon x \text{ is a } C)\, (\ldots x \ldots \land (\forall y\colon y \text{ is a } C \land \ldots y \ldots)\, x = y)$

Definite descriptions (on Russell's analysis)

The C is such that (… it …)	$(\exists x\colon x \text{ is a } C \land (\forall y\colon \neg\, y = x) \neg\, y \text{ is a } C) \ldots x \ldots$ *or* $(\exists x\colon x \text{ is a } C \land (\forall y\colon y \text{ is a } C)\, x = y) \ldots x \ldots$

Glen Helman 13 Jul 2012

A.3. Truth tables

Tautology

$\top$
T

Absurdity

$\bot$
F

Negation

φ	$\neg\ \varphi$
T	F
F	T

Conjunction

φ	ψ	$\varphi \land \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

φ	ψ	$\varphi \lor \psi$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional

φ	ψ	$\varphi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Glen Helman 12 Jul 2012

A.4. Derivation rules

Basic system

Rules for developing gaps		
logical form	*as a resource*	*as a goal*
atomic sentence		IP
negation $\neg\varphi$	CR (if φ not atomic & goal is $\bot$)	RAA
conjunction $\varphi \wedge \psi$	Ext	Cnj
disjunction $\varphi \vee \psi$	PC	PE
conditional $\varphi \rightarrow \psi$	RC (if goal is $\bot$)	CP
universal $\forall x\, \theta x$	UI	UG
existential $\exists x\, \theta x$	PCh	NcP

In addition, if the conditions for applying a rule are met except for differences between co-aliases, then the rule can be applied and is notated by adding "="; QED= and Nc= are examples of this.

Rules for closing gaps			
when to close			*rule*
co-aliases	*resources*	*goal*	
	φ	φ	QED
	φ and $\neg\varphi$	$\bot$	Nc
		$\top$	ENV
	$\bot$		EFQ
τ—υ		$\tau = \upsilon$	EC
τ—υ	$\neg\, \tau = \upsilon$	$\bot$	DC
τ_1—$\upsilon_1, \ldots, \tau_n$—$\upsilon_n$	$P\tau_1 \ldots \tau_n$	$P\upsilon_1 \ldots \upsilon_n$	QED=
τ_1—$\upsilon_1, \ldots, \tau_n$—$\upsilon_n$	$P\tau_1 \ldots \tau_n$, $\neg P\upsilon_1 \ldots \upsilon_n$	$\bot$	Nc=

Detachment rules (optional)		
required resources		*rule*
main	*auxiliary*	
$\varphi \rightarrow \psi$	φ	MPP
	$\neg^{\pm}\psi$	MTT
$\varphi \vee \psi$	$\neg^{\pm}\varphi$ or $\neg^{\pm}\psi$	MTP
$\neg(\varphi \wedge \psi)$	φ or ψ	MPT

Additional rules (not guaranteed to be progressive)

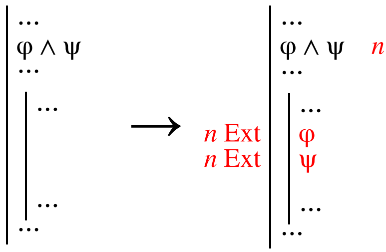
Attachment rules	
added resource	*rule*
$\varphi \wedge \psi$	Adj
$\varphi \rightarrow \psi$	Wk
$\varphi \vee \psi$	Wk
$\neg(\varphi \wedge \psi)$	Wk
$\tau = \upsilon$	CE
$\theta\upsilon_1 \ldots \upsilon_n$	Cng
$\exists x\, \theta x$	EG

Rule for lemmas	
prerequisite	*rule*
the goal is $\bot$	LFR

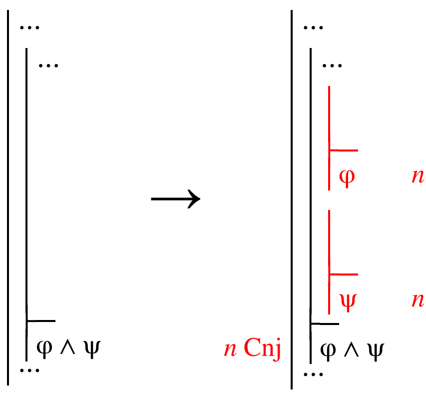
Diagrams

Rules from chapter 2

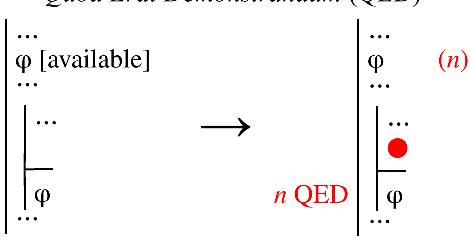
Extraction (Ext)



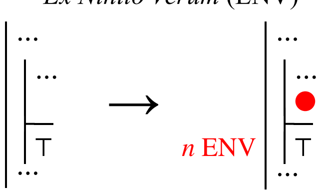
Conjunction (Cnj)



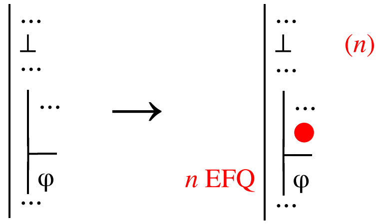
Quod Erat Demonstrandum (QED)



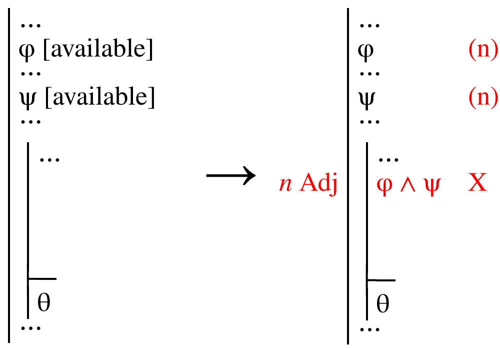
Ex Nihilo Verum (ENV)



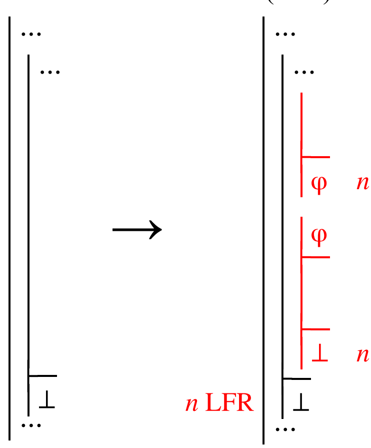
Ex Falso Quodlibet (EFQ)



Adjunction (Adj)

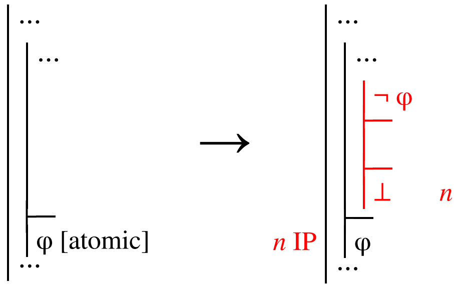


Lemma for *Reductio* (LFR)

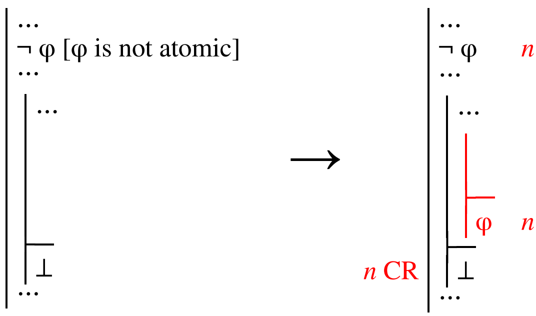


Rules from chapter 3

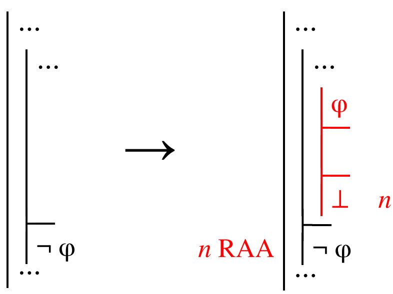
Indirect Proof (IP)



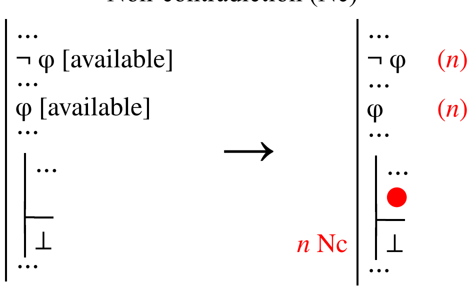
Completing the *Reductio* (CR)



Reductio ad Absurdum (RAA)

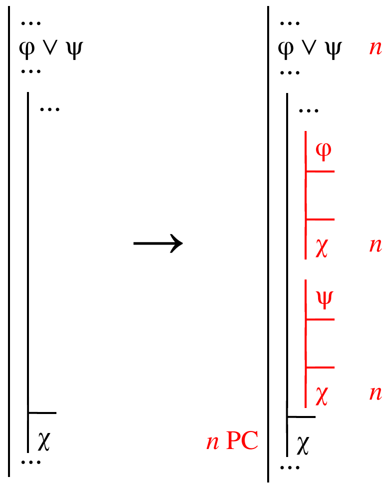


Non-contradiction (Nc)

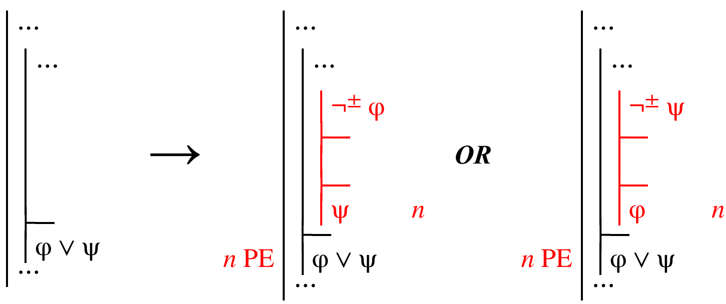


Rules from chapter 4

Proof by Cases (PC)



Proof of Exhaustion (PE)



Modus Tollendo Ponens (MTP)

$\neg^{\pm}\ \varphi$ [available]
…
$\varphi \vee \psi$
…
  …
  ―
  χ
…

$\longrightarrow$

$\neg^{\pm}\ \varphi$ (n)
…
$\varphi \vee \psi$ n
…
  …
  ψ n MTP
  ―
  χ
…

$\neg^{\pm}\ \psi$ [available]
…
$\varphi \vee \psi$
…
  …
  ―
  χ
…

$\longrightarrow$

$\neg^{\pm}\ \psi$ (n)
…
$\varphi \vee \psi$ n
…
  …
  φ n MTP
  ―
  χ
…

Modus Ponendo Tollens (MPT)

φ [available]
…
$\neg\ (\varphi \wedge \psi)$
…
  …
  ―
  θ
…

$\longrightarrow$

φ (n)
…
$\neg\ (\varphi \wedge \psi)$ n
…
  …
  $\neg^{\pm}\ \psi$ n MPT
  ―
  θ
…

ψ [available]
…
$\neg\ (\varphi \wedge \psi)$
…
  …
  ―
  θ
…

$\longrightarrow$

ψ (n)
…
$\neg\ (\varphi \wedge \psi)$ n
…
  …
  $\neg^{\pm}\ \varphi$ n MPT
  ―
  θ
…

Weakening (Wk)

…
 φ [available]
 …
   | …
   |——
   | θ
 …
	⟶	…
		 φ (*n*)
		 …
		   | …
		 n Wk | φ ∨ ψ X
		   |——
		   | θ
		 …

…
 ψ [available]
 …
   | …
   |——
   | θ
 …
	⟶	…
		 ψ (*n*)
		 …
		   | …
		 n Wk | φ ∨ ψ X
		   |——
		   | θ
		 …

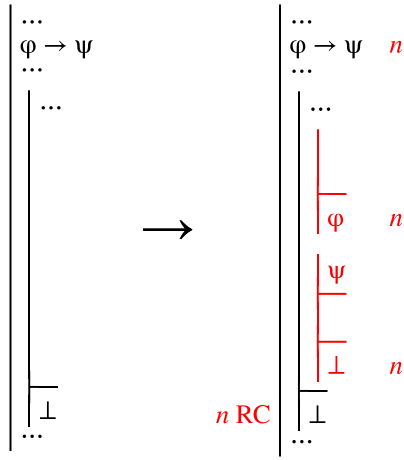
Weakening (Wk)

…
 $\neg^{\pm}$ φ [available]
 …
   | …
   |——
   | θ
 …
	⟶	…
		 $\neg^{\pm}$ φ (*n*)
		 …
		   | …
		 n Wk | ¬ (φ ∧ ψ) X
		   |——
		   | θ
		 …

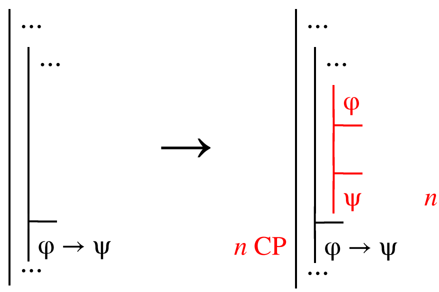
…
 $\neg^{\pm}$ ψ [available]
 …
   | …
   |——
   | θ
 …
	⟶	…
		 $\neg^{\pm}$ ψ (*n*)
		 …
		   | …
		 n Wk | ¬ (φ ∧ ψ) X
		   |——
		   | θ
		 …

Rules from chapter 5

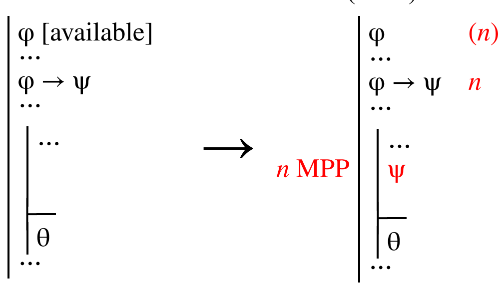
Rejecting a Conditional (RC)



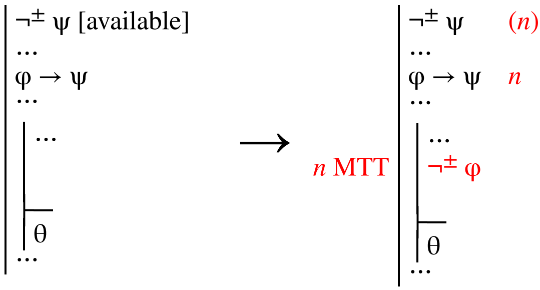
Conditional Proof (CP)



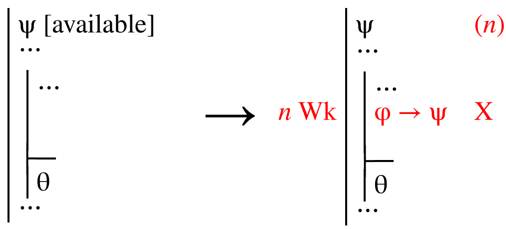
Modus Ponendo Ponens (MPP)



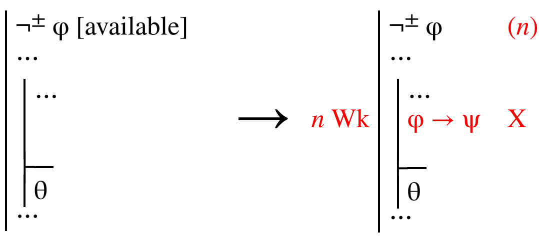
Modus Tollendo Tollens (MTT)



Weakening (Wk)

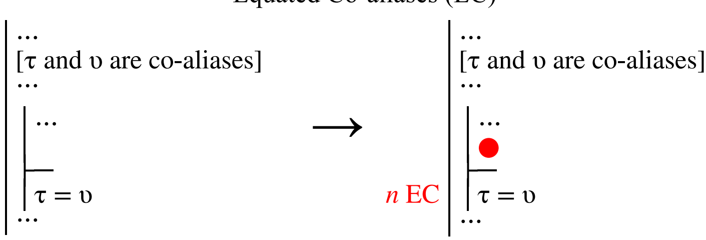


Weakening (Wk)

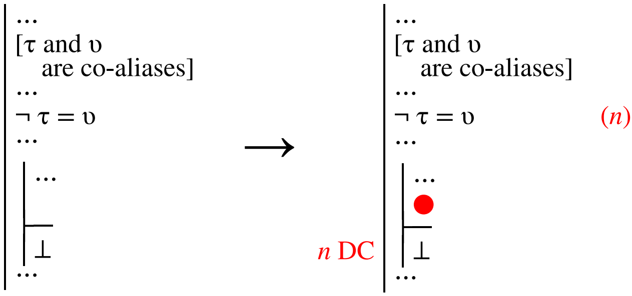


Rules from chapter 6

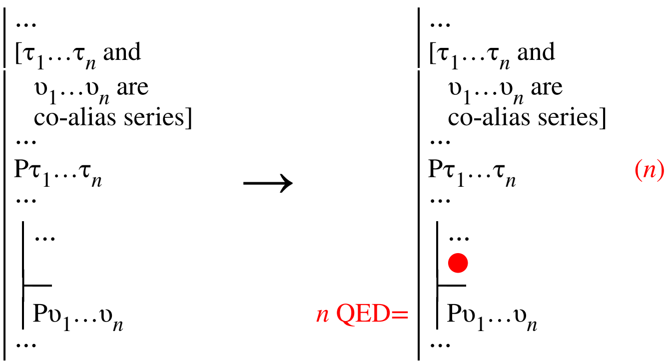
Equated Co-aliases (EC)



Distinguished Co-aliases (DC)

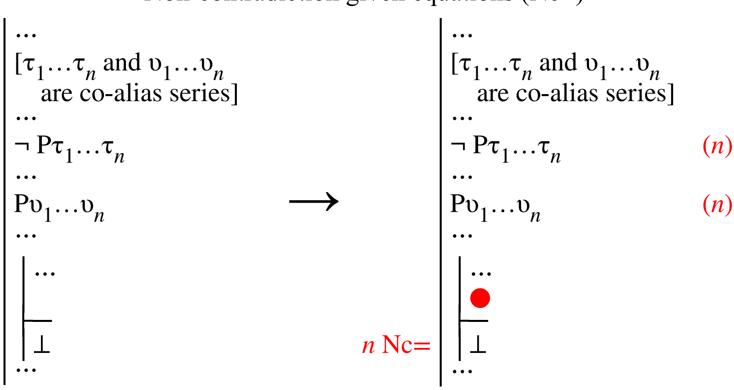


QED given equations (QED=)



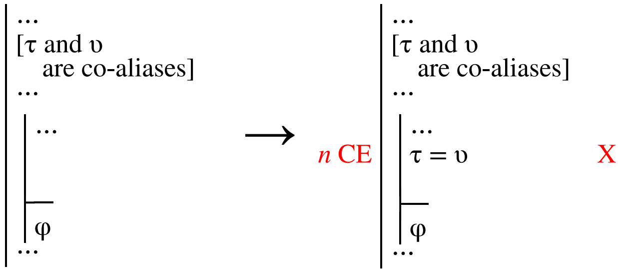
Note: Two series of terms are co-alias series when their corresponding members are co-aliases.

Non-contradiction given equations (Nc=)

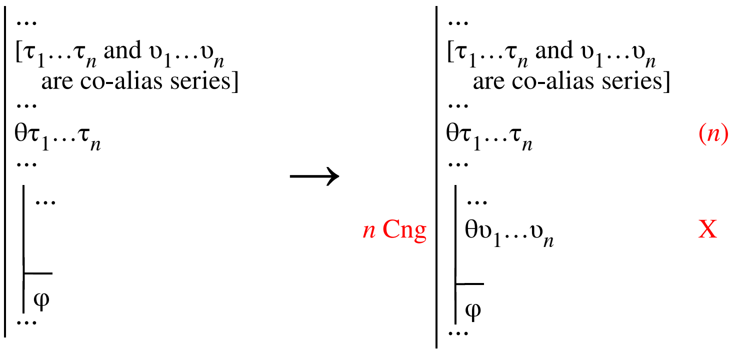


Note: Two series of terms are co-alias series when their corresponding members are co-aliases.

Co-alias Equation (CE)



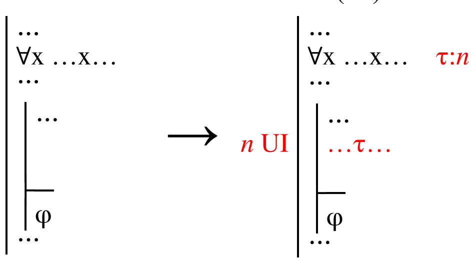
Congruence (Cng)



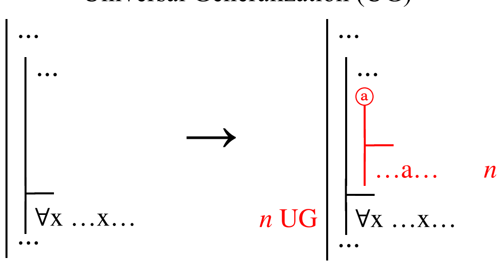
Note: θ can be an abstract, so $\theta\tau_1 \ldots \tau_n$ and $\theta\upsilon_1 \ldots \upsilon_n$ are any formulas that differ only in the occurrence of terms and in which the corresponding terms are co-aliases.

Rules from chapter 7

Universal Instantiation (UI)

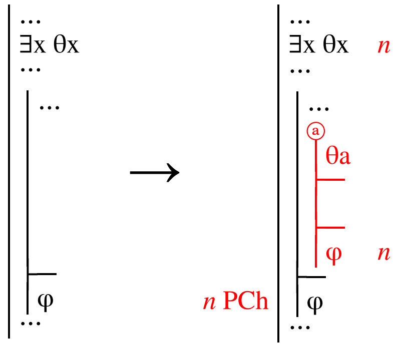


Universal Generalization (UG)

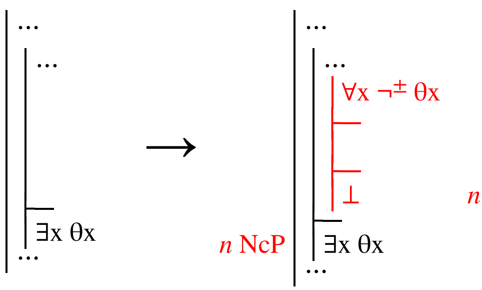


Rules from chapter 8

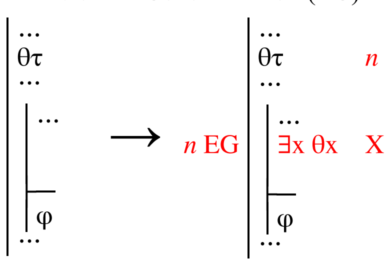
Proof by Choice (PCh)



Non-constructive Proof (NcP)



Existential Generalization (EG)



Glen Helman 13 Jul 2012

Appendix B. Laws for relative exhaustiveness

Atomic sentences

The first of the following laws is stated only for unanalyzed sentences because laws of the same form for equations and other predications are special cases of the second and third laws:

$\Gamma, A \vDash A, \Sigma$
$\Gamma \vDash \tau = \upsilon, \Sigma$ (where τ and υ are co-aliases given the equations in Γ)
$\Gamma, P\tau_1 \ldots \tau_n \vDash P\upsilon_1 \ldots \upsilon_n, \Sigma$ (where τ_i and υ_i, for *i* from 1 to *n*, are co-aliases given the equations in Γ)

Non-atomic sentences

For each logical constant which forms non-atomic sentences, there are two laws, one for cases where it appears among the assumptions and one for cases where it appears among the alternatives.

Constant	*As an assumption*	*As an alternative*
$\top$	$\Gamma, \top \vDash \Sigma$ if and only if $\Gamma \vDash \Sigma$	$\Gamma \vDash \top, \Sigma$
$\bot$	$\Gamma, \bot \vDash \Sigma$	$\Gamma \vDash \bot, \Sigma$ if and only if $\Gamma \vDash \Sigma$
$\neg$	$\Gamma, \neg\varphi \vDash \Sigma$ if and only if $\Gamma \vDash \varphi, \Sigma$	$\Gamma \vDash \neg\varphi, \Sigma$ if and only if $\Gamma, \varphi \vDash \Sigma$
$\wedge$	$\Gamma, \varphi \wedge \psi \vDash \Sigma$ if and only if $\Gamma, \varphi, \psi \vDash \Sigma$	$\Gamma \vDash \varphi \wedge \psi, \Sigma$ if and only if both $\Gamma \vDash \varphi, \Sigma$ and $\Gamma \vDash \psi, \Sigma$
$\vee$	$\Gamma, \varphi \vee \psi \vDash \Sigma$ if and only if both $\Gamma, \varphi \vDash \Sigma$ and $\Gamma, \psi \vDash \Sigma$	$\Gamma \vDash \varphi \vee \psi, \Sigma$ if and only if $\Gamma \vDash \varphi, \psi, \Sigma$
$\rightarrow$	$\Gamma, \varphi \rightarrow \psi \vDash \Sigma$ if and only if both $\Gamma \vDash \varphi, \Sigma$ and $\Gamma, \psi \vDash \Sigma$	$\Gamma \vDash \varphi \rightarrow \psi, \Sigma$ if and only if $\Gamma, \varphi \vDash \psi, \Sigma$
$\forall$	$\Gamma, \forall x\, \theta x \vDash \Sigma$ if and only if $\Gamma, \forall x\, \theta x, \theta\tau \vDash \Sigma$	$\Gamma \vDash \forall x\, \theta x, \Sigma$ if and only if $\Gamma \vDash \theta\alpha, \Sigma$
$\exists$	$\Gamma, \exists x\, \theta x \vDash \Sigma$ if and only if $\Gamma, \theta\alpha \vDash \Sigma$	$\Gamma \vDash \exists x\, \theta x, \Sigma$ if and only if $\Gamma \vDash \theta\tau, \exists x\, \theta x, \Sigma$

where τ is any term and α is independent in the sense that it does not appear in θ, Γ, or Σ