4.2. Arguing from and for alternatives

4.2.0. Overview

Because a disjunction normally says less than its components while a conjunction says more, the two connectives play very different roles in deductive inference.

4.2.1. Proofs by cases
  Since a disjunction says only what is said by both its disjuncts, it entails only things that both of them entail.

4.2.2. Proving disjunctions
  Since a disjunction makes a relatively weak claim, it is easy to state a sound rule to plan for it, but a safe rule that will cover all cases where it holds is more complex.

4.2.3. Further examples
  There are now many choices to be regarding the order in which rules are applied and some differences in the length of derivations can result.

4.2.4. The duality of conjunction and disjunction
  Conjunction and disjunction are, in a certain formal sense, mirror images of one another.

Glen Helman 11 Jul 2012

4.2.1. Proofs by cases

The validity of the argument

> Sam didn't praise the proposal without granting its significance
> Sam didn't condemn the proposal without granting its significance
> Sam either praised or condemned the proposal
> ———
> Sam granted the proposal's significance.

can be accounted for by the validity of the following two arguments:

Sam didn't praise the proposal without granting its significance	Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance	Sam didn't condemn the proposal without granting its significance
Sam praised the proposal	Sam condemned the proposal
Sam granted the proposal's significance	Sam granted the proposal's significance

Each replaces the disjunctive third premise of the original argument by one of its two components. This way of establishing an entailment is sometimes called a *proof by cases*. In this example, the two cases are Sam having praised the proposal and Sam having condemned it. Since the disjunction says all and only what is common to these two claims, what follows from the disjunction in isolation or in addition to other premises is what follows from each of these claims under similar circumstances.

More formally, the idea behind proofs by cases is captured by this principle:

Law for disjunction as a premise. $\Gamma, \varphi \lor \psi \vDash \chi$ if and only if both $\Gamma, \varphi \vDash \chi$ and $\Gamma, \psi \vDash \chi$ (for any set Γ and sentences φ, ψ, and χ).

To see why this law is true note that to separate the members of Γ and $\varphi \lor \psi$ on the one hand from χ on the other, a possible world must make $\varphi \lor \psi$ and all members of Γ true while making χ false. To do this it must make at least one of φ and ψ true, so it must provide a counterexample to at least one of the arguments $\Gamma, \varphi / \chi$ and $\Gamma, \psi / \chi$. So, to say that the original argument is valid is to say that neither of these latter arguments can have its premises and conclusion separated—that is, that both are valid.

This idea appears in derivations by way of a rule we will call *Proof by Cases* (PC); it is shown in Figure 4.2.1-1.

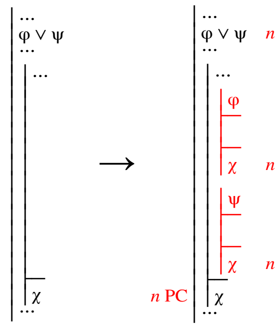
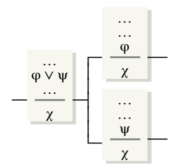


Fig. 4.2.1-1. Developing a derivation by exploiting a disjunction at stage *n*.

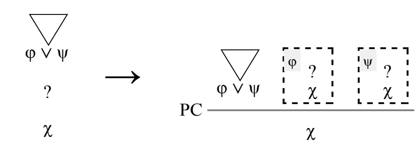
PC divides a gap into two new gaps. Each is a *case argument* that retains the original goal but adds one of the components of the disjunction as a supposition. The function of each supposition is to specify one of the two sorts of case in which the original disjunction is true. A supposition is required because, although our premises tell us that at least one of the disjuncts is true, we do not know which that is and the one that is true will vary among the possible worlds in which the premises are all true.

The safety and soundness (indeed, strictness) of this rule is shown by its effect on proximate arguments, which follows the pattern of law for disjunction as a premise understood as a rule for argument trees:



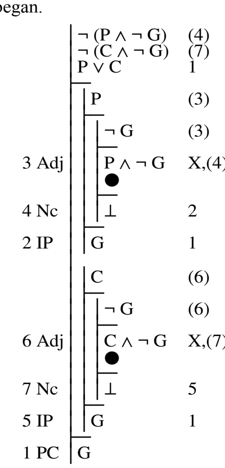
That is, moving left to right, we exploit $\varphi \lor \psi$ and thus drop it from the active resources, and we add suppositions φ and ψ. The goal of the parent gap is carried over to each of its two children. The rule is safe because any counterexample lurking in one of the children is bound to lurk in the parent, too. It must make the resources of the parent true because $\varphi \lor \psi$ is implied by each of φ and ψ, and the requirement to make χ false remains unchanged as we move between the parent and the children. Moverover, the rule is strict because any counterexample lurking in the parent must, in order to make $\varphi \lor \psi$ true, make at least one of φ and ψ true also, so it will lurk in at least one of the children.

As in other cases, the use of numerical annotations in PC reflects the corresponding rule for conclusion trees, which is shown here as a transition from one stage in the construction of the tree to the next.



The conclusion χ is based on three premises (one that we have found among the resources and two more we plan to reach), so in derivations the stage number appears on the right of three lines, the the disjunction that is exploited and the goals of the two new scope lines.

Here is a derivation which uses derivation rule to provide a proof for the example with which we began.



C: Sam condemned the proposal; G: Sam granted the proposal's significance; P: Sam praised the proposal

In the two case arguments, we suppose first that Sam praised the proposal and then that he condemned it and, in each case, we show that he granted the proposal's significance (by showing that he could not have failed to grant it). Since at least one of these two cases must be true whenever the premises are all true, we know that the conclusion must be true also.

The rule for conclusion trees displayed above shows that PC represents a new function for suppositions. Like Lem (or the special case LFR) on the one hand and RAA and IP on the other, we use suppositions in PC to consider the consequences of claims without asserting them. But, while we did this in RAA and IP in order to show the suppositions were false and in Lem and LFR in order to consider the proof of a claim independently of the investigation of its consequences, we do it here to consider separately the consequences of two alternatives without deciding which of the two is true.

The rule for conclusion trees also makes it clear that, apart from the separate consideration of φ and ψ, the form of PC is much like that of Lem. But there is an important difference in the way these rules are employed in proofs. The rule Lem would be used to initiate the search for a proof of its first premise. On the other hand, while the conclusion-tree rule PC might be used in this way, we use PC in derivations instead to derive consequences from a premise φ ∨ ψ that has already been established, and that aspect of the derivation rule is better reflected in the rule for argument trees shown earlier, which displays less analogy with the argument-tree rule for Lem.

Glen Helman 16 Jul 2012

4.2.2. Proving disjunctions

Now let us look at disjunctions as conclusions. An entailment Γ ⊨ φ ∨ ψ will hold if and only if φ ∨ ψ is true in every possible world in which all members of Γ are true. But this is to say that at least one of φ and ψ is true in every such world, and that is a way of saying that Γ renders φ and ψ jointly exhaustive. So we can state the following principle:

Γ ⊨ φ ∨ ψ if and only if Γ ⊨ φ, ψ

Since the right-hand side has two alternatives, this is not a law concerning entailment alone, and we will not take the principle in this form as our account of the role of disjunctions as conclusions. However, we can use the basic law for relative exhaustiveness to restate the right-hand side as claim of entailment. Indeed we have two ways of doing that. If φ and φ′ are contradictory, we can say

Γ ⊨ φ ∨ ψ if and only if Γ, φ′ ⊨ ψ

and if ψ and ψ′ are contradictory, we can say

Γ ⊨ φ ∨ ψ if and only if Γ, ψ′ ⊨ φ

In short, a disjunction is a valid conclusion from premises Γ if and only if adding to our premises a sentence contradictory to one disjunct enables us to validly conclude the other disjunct.

In stating a principle for disjunction we will limit ourselves to cases where a sentence and its negation are the pair of contradictory sentences. But, when the disjuncts are already negative, that leaves us with two choices for each of the pairs φ and φ′ and ψ and ψ′ since each of φ′ and ψ′ might be the result of either adding or dropping a negation. To avoid stating four principles to cover each of these possibilities, we will introduce some notation to capture the general idea of obtaining a contradictory sentence by either adding or dropping a negation. (We will refer to the latter as *de-negation*.) Let the sentence $\neg^{\pm}$ φ be the result of negating φ with an optional added step of deleting a double negation if φ was already negative. Then $\neg^{\pm}$ φ will stand for ¬ φ when φ is not a negation and, when φ is the negation ¬ χ, it will stand for either ¬ ¬ χ or χ. That is, $\neg^{\pm}$ φ is the result of either negating or, perhaps, de-negating φ, which means that $\neg^{\pm}$ φ will either be the negation of φ or have φ as its negation.

This means that $\neg^{\pm}$ φ and φ form a contradictory pair consisting of a sentence and its negation in one order or the other, so we need only two clauses to formulate a principle to account for conclusions that are disjunctions:

Law for disjunction as a conclusion. (i) Γ ⊨ φ ∨ ψ if and only if Γ, $\neg^{\pm}$ φ ⊨ ψ, and (ii) Γ ⊨ φ ∨ ψ if and only if Γ, $\neg^{\pm}$ ψ ⊨ φ (for any set Γ and sentences φ, ψ, and χ).

When these are implemented as derivation rules, they give us two ways of planning for a disjunctive goal. The two rules are shown as alternative developments in Figure 4.2.2-1. We will refer to both forms of the rule as *Proof of Exhaustion* (PE) since it is a way of showing that φ and ψ, taken together, exhaust all possibilities left open by the premises.

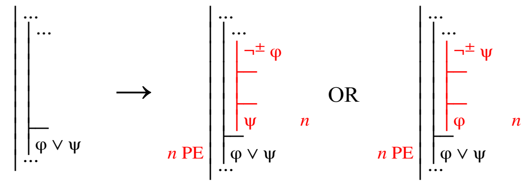


Fig. 4.2.2-1. Alternative ways of developing a derivation by planning for a disjunction at stage *n*.

In each way of developing a gap, we set one of the components of the disjunction as a new goal and add the negation or de-negation of the other component as a supposition. In each way of developing a gap, we set one of the components of the disjunction as a new goal.

Both forms of planning will lead to the same answer in the end, but one or the other may be more efficient in a particular case. There is no simple way of predicting which choice is best but the following rules of thumb may help:

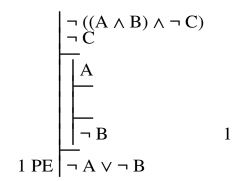
(i) if only one component is a negation, choose it to form the supposition (by dropping its negation);
(ii) if only one component is a non-negative compound choose it as the goal;
(iii) if only one component seems likely to figure in closing the gap and it is not a negation, choose it as the goal.

In many cases none of these suggestions will apply; but, in most such cases, neither one of the two forms of the rule is better than the other.

As an example of this rule, consider the argument below, understanding X was out to be the denial of X was home. The validity of this argument can be established by the English derivation whose first stage is shown at the right.

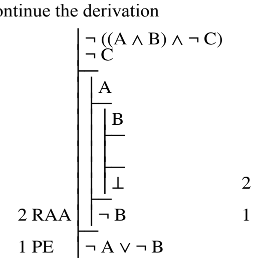
Ann and Bill were not both home without the car being in the driveway
The car was not in the driveway

Either Ann or Bill was out



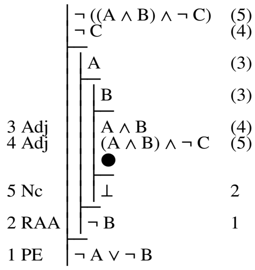
The overall form is that of a argument that we will call "hypothetical" (for reasons discussed below) in which we suppose that Ann was at home (a supposition that is one of the two possibilities for $\neg^{\pm}$ ¬ A) and establish under this supposition that Bill was out. This shows the connection between Ann being out and Bill being out that we claim when we state, outside the scope of the supposition, that at least one was out.

Notice that if we continue the derivation



we plan for the goal ¬ B by supposing B for *reductio*. And this example illustrates the different functions of the two sorts of supposition. We suppose that Ann is home in order to show that ¬ B (i.e., Bill is out) is true in all possible worlds in which ¬ A (i.e., Ann is out) is false. We go on to show that ¬ B is true in these cases by showing that to suppose further that B would rule out all possibilities—i.e., that this supposition would be absurd when added to our premises and the supposition A. From one point of view, both suppositions are merely added assumptions. But we add the first in order to show that to accept the second would be to go too far. That is, we add the second in order to show that we cannot accept it given the first, and we add the first to show that the second is related to it in this way.

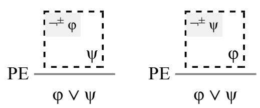
To complete the derivation, we might exploit the first premise by CR, and this is the only way to proceed using basic rules. Doing this would make the conjunction (A ∧ B) ∧ ¬ C our goal; and, since its components are all resources, it is clear that the gap would close. But, seeing this, we might choose instead to derive that conjunction by Adj.



Either way we are completing the *reductio*, in one case under the guidance of the rules and in the other under our own direction.

As noted above, the supposition in PE may be described as *hypothetical*, and this indicates the role it plays, a fourth role on top of those we have seen in Lem and LFR, in RAA and IP, and in PC. In RAA and IP, we make suppositions with the aim of showing that they are false. In Lem and LFR, we make a supposition to consider separately the consequences of a lemma and whether the lemma itself true. In PC, we make a pair of suppositions, having already shown that at least one is true. In PE on the other hand, a supposition is made with no expectation of either truth or falsity. It is made instead simply to establish a connection between it and some other claim. As we argue within the scope of the supposition, we are making a *hypothetical argument*, an argument made "under a hypothesis." The conclusion we draw when we discharge the supposition states a connection between the hypothesis and the conclusion of the hypothetical argument. This statement no longer falls under the supposition, and that can be indicated by saying that it is stated *categorically*.

The two forms of PE are shown below as patterns of argument for conclusion trees.

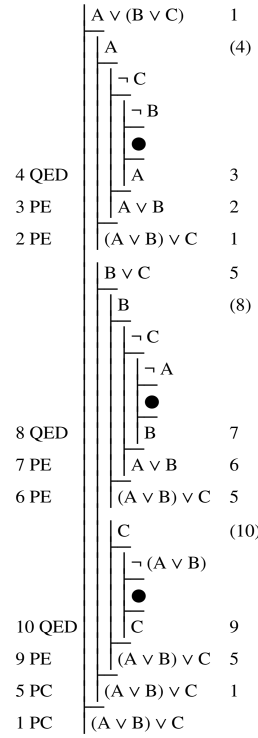


In each case, the disjunction is concluded from one of its components, but the component has been concluded under the supposition that the other is false. This supposition is discharged when drawing the conclusion, and the weakness of the conclusion relative to the premise compensates for its loss. For example, in the second argument the premise φ has been shown to cover a range of possibilities that are limited by the supposition to ones in which ψ is false. And the conclusion is weakened in a way that no longer requires this limitation since, by adding to the premise the qualification `or` ψ, it explicitly covers cases where ψ is true. What is a hypothetical assertion of φ in the premise becomes a categorical assertion of ψ ∨ φ in the conclusion.

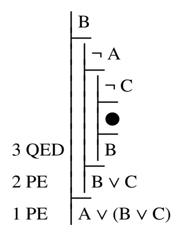
There is some danger of getting tangled in the terminology here, so let's pause and look at it more closely. The terms `hypothetical` and `categorical` derive from an ancient classification of sentences into the "categorical," the "disjunctive," and the "hypothetical." Since disjunctions and "hypothetical sentences" (the conditionals to be studied in the next chapter) are ways of hedging claims, the term `categorical` has acquired the meaning 'unhedged'. Now the disjunctive goal to which we applied this term above certainly hedges each of its components, so it does not state them categorically. But, while sentences in the hypothetical argument are stated only "under a hypothesis"—that is, under the supposition of the hypothetical argument—the disjunction following the argument is no longer hedged in this way. That means it is stated categorically with respect to that supposition (though it may still fall in the scope of earlier ones). In short, when the scope line of a hypothetical argument ends, we move from hedged assertion of some claim (in the English example, the assertion of `Bill was out` under the hypothesis `Ann was not out`) to unhedged assertion of a claim that incorporates a hedge (i.e., `Either Ann or Bill was out` in the example).

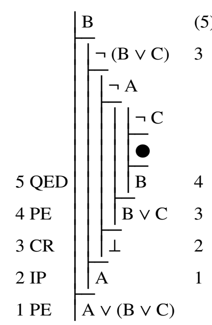
4.2.3. Further examples

Both disjunction rules are illustrated by the derivation at the right, in which one grouping of a three-part disjunction is shown to entail the other. Choices between the two ways of planning for a goal disjunction were made at stages 2, 3, 5, 6, and 7 in accordance with the rules of thumb given above. Each choice helped to shorten the derivation—though only by a few steps. The derivation is contrived to provide several examples of this rule; we might have instead planned for the initial goal at stage 1 before exploiting the premise rather than planning for it later in each of three gaps.



The two derivations below illustrate the scale of the difference you can expect a choice between the two forms of PE to make.





Each chooses a different way of planning for the initial goal at stage 1. Notice that in the second, which makes the less efficient choice, we are led back to the goal B ∨ C in a couple of stages.

4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term dual.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. For example, let us again take X was home and X was out to be contradictories. Then note that to get a sentence contradictory to Ann and Bill were home, we cannot take Ann and Bill were out since both would be false if one of Ann and Bill was home and the other out. To get a contradictory to we need to cover both of those possibilities as well, and Ann or Bill was out will do this. That is, Ann and Bill were home is contradictory to Ann or Bill was out and, similarly, Ann or Bill was home is contradictory to Ann and Bill were out. And this is to say that ¬ Ann and Bill were home ≃ Ann or Bill was out and that ¬ Ann or Bill was home ≃ Ann and Bill were out.

In cases of contradictoriness captured by the $\neg^{\pm}$ notation, these patterns of equivalence are stated in the following principles:

DE MORGAN'S LAWS. *The denial of a conjunction amounts to a disjunction of denials, and the denial of a disjunction amounts to a conjunction of denials.* That is,

$$\neg(\varphi \wedge \psi) \simeq \neg^{\pm}\varphi \vee \neg^{\pm}\psi$$
$$\neg(\varphi \vee \psi) \simeq \neg^{\pm}\varphi \wedge \neg^{\pm}\psi$$

Although these laws are named after Augustus De Morgan (1806-1871), they were known well before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles of relative exhaustiveness. The table below follows the pattern of the one given for ⊥ and ⊤ in 1.4.8.

	as a premise	as an alternative
Conjunction	$\Gamma, \varphi \wedge \psi \vDash \Sigma$ iff $\Gamma, \varphi, \psi \vDash \Sigma$	$\Gamma \vDash \varphi \wedge \psi, \Sigma$ iff both $\Gamma \vDash \varphi, \Sigma$ and $\Gamma \vDash \psi, \Sigma$
Disjunction	$\Gamma, \varphi \vee \psi \vDash \Sigma$ iff both $\Gamma, \varphi \vDash \Sigma$ and $\Gamma, \psi \vDash \Sigma$	$\Gamma \vDash \varphi \vee \psi, \Sigma$ iff $\Gamma \vDash \varphi, \psi, \Sigma$

(Here iff is used as an abbreviation of if and only if.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise much as disjunction behaves as an alternative and disjunction behaves as premise much as conjunction behaves as an alternative.

Since ⊥ and ⊤ are paired as duals and so are conjunction and disjunction, you might wonder about negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to the two contradictory sentences in order for the results to be contradictory. And the behavior of a negation as a premise is analogous to its behavior as an alternative.

$$\Gamma, \neg\varphi \vDash \Sigma \text{ iff } \Gamma \vDash \varphi, \Sigma$$
$$\Gamma \vDash \neg\varphi, \Sigma \text{ iff } \Gamma, \varphi \vDash \Sigma$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

The term duality points to a certain sort of two-for-one principle. It is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

premise —— alternative
⊥ —— ⊤
negation —— negation
conjunction —— disjunction

So, for example (and to deal only with informal statements of the principles), the principle

> A conjunction as an assumption may be replaced by its components as independent assumptions

(the upper left in the table of principles for conjunction and disjunction above) turns into

> A disjunction as an alternative may be replaced by its components as independent alternatives

(the lower right in that table). And the principle

> A negation as an assumption may be replaced by its immediate component as an alternative

(the first of the principles for negation displayed above) turns into

> A negation as an alternative may be replaced by its immediate component as an assumption

(the second of those principles). We will see more examples of such transformations in the next section but we have already seen some further ones: each of the two forms of De Morgan's laws may be transformed into the other by this association.

Since these transformations treat assumptions and alternatives in a parallel way, not all will apply to entailment, which allows multiple premises but only a single alternative. However, we have also seen that principles for relative exhaustiveness may be transformed still further into principles of entailment by the law for alternatives *via* contradictory assumptions.

Glen Helman 11 Jul 2012

4.2.s. Summary

1 A disjunction $\varphi \vee \psi$ is false only when its disjuncts are both false, and it thus says only what both of them say. The law for disjunction as a premise tell us that we can establish a conclusion using such a premise by showing that it is entailed by each of the disjuncts (given our other premises). This way of exploiting a disjunction is known as an proof by cases and it appears in our system of derivations as the rule Proof by Cases (PC) that leads us to divide a gap into two case arguments, each of which takes over the original goal and adds one of the two disjuncts as a supposition.

2 To show that a disjunction is a valid conclusion, we must show that its disjuncts are rendered jointly exhaustive by the premises. We can do this by showing that one of the disjuncts will follow if we add the contradictory of the other to our premises. We use the notation $\neg^{\pm}\varphi$ to indicate the result of either negating or de-negating φ. The law for disjunction as a conclusion then tells us that we can conclude a disjunction if we can conclude one disjunct provided we take the negation or de-negation of the other disjunct as a premise. The rule implementing this idea is Proof of Exhaustion; it enables us to conclude a disjunction from an argument that may be called hypothetical since it bases a disjunct on an assumption (of the negation or de-negation of the other disjunct) that we may not be prepared to assert categorically. It does not matter for the soundness or safety of PE which disjunct figures as the goal of this hypothetical argument and which is negated or de-negated in its supposition.

3 Derivations, especially those that have a disjuction as a goal as well as a premise can often be developed in different ways. Some of these can be significantly longer than others but the choice between forms of PE will usually have only a limited impact on the length.

4 Conjunction and disjunction are opposite in the sense of being dual. One manifestation of this relation is in De Morgan's laws, which tell how to restate the denial of a conjunction or disjunction as an assertion of the other form of compound. Another manifestation is a pattern in laws of relative exhaustiveness which allows us to interchange conjunctions and disjunctions if at the same time we interchange ⊥ and ⊤ and also premises and alternatives.

Glen Helman 11 Jul 2012

4.2.x. Exercises

1. Use derivations to establish each of the claims of entailment and equivalence shown below. (Remember that claims of equivalence require derivations in both directions.)

- **a.** A ∧ B ⊨ A ∨ B
- **b.** A ∧ B ⊨ B ∨ C
- **c.** A ∨ B, ¬ A ⊨ B
- **d.** A ∨ (A ∧ B) ⊨ A
- **e.** A ∨ B, ¬ (A ∧ C), ¬ (B ∧ C) ⊨ ¬ C
- **f.** A ∧ (B ∨ C) ⊨ (A ∧ B) ∨ C
- **g.** A ∨ B, C ⊨ (A ∧ C) ∨ (B ∧ C)
- **h.** A ∨ B, ¬ A ∨ C ⊨ B ∨ C
- **i.** A ≃ (A ∧ B) ∨ (A ∧ ¬ B)

2. Use derivations to establish each of the claims of equivalence below.

- **a.** A ∨ A ≃ A
- **b.** A ∨ B ≃ B ∨ A
- **c.** A ∨ (B ∨ C) ≃ (A ∨ B) ∨ C
- **d.** A ∨ (B ∧ ¬ B) ≃ A
- **e.** ¬ (A ∨ B) ≃ ¬ A ∧ ¬ B
- **f.** ¬ (A ∧ B) ≃ ¬ A ∨ ¬ B

3. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, confirm a counterexample that lurks in an open gap.

- **a.** A ∨ B, A ⊨ ¬ B
- **b.** A ∨ (B ∧ C) ≃ (A ∨ B) ∧ C
- **c.** ¬ (A ∨ B) ≃ ¬ A ∨ ¬ B

For more exercises, use the exercise machine.

Glen Helman 11 Jul 2012

4.2.xa. Exercise answers

1. a.

```
        | A ∧ B          1
1 Ext   | A
1 Ext   | B              (3)
        | | ¬ A
        | | ●
3 QED   | | B            2
2 PE    | A ∨ B
```

b.

```
        | A ∧ B          1
1 Ext   | A
1 Ext   | B              (3)
        | | ¬ C
        | | ●
3 QED   | | B            2
2 PE    | B ∨ C
```

c.

```
        | A ∨ B          1
        | ¬ A            (3)
        | | A            (3)
        | | | ¬ B
        | | | ●
3 Nc    | | | ⊥          2
2 IP    | | B            1
        | | B            (4)
        | | ●
4 QED   | | B            1
1 PC    | B
```

d.

```
        | A ∨ (A ∧ B)    1
        | | A            (2)
        | | ●
2 QED   | | A            1
        | | A ∧ B        3
3 Ext   | | A            (4)
3 Ext   | | B
        | | ●
4 QED   | | A            1
1 PC    | A
```

e.

```
        | A ∨ B                  2
        | ¬ (A ∧ C)              3
        | ¬ (B ∧ C)              7
        | | C                    (6),(10)
        | | | A                  (5)
        | | | | ●
5 QED   | | | | A                4
        | | | | ●
6 QED   | | | | C                4
4 Cnj   | | | A ∧ C              3
3 CR    | | | ⊥                  2
        | | | B                  (9)
        | | | | ●
9 QED   | | | | B                8
        | | | | ●
10 QED  | | | | C                8
8 Cnj   | | | B ∧ C              7
7 CR    | | | ⊥                  2
2 PC    | | ⊥                    1
1 RAA   | ¬ C
```

f.

```
        | A ∧ (B ∨ C)            1
1 Ext   | A                      (5)
1 Ext   | B ∨ C                  2
        | | B                    (6)
        | | | ¬ C
        | | | | ●
5 QED   | | | | A                4
        | | | | ●
6 QED   | | | | B                4
4 Cnj   | | | A ∧ B              3
3 PE    | | (A ∧ B) ∨ C          2
        | | C                    (8)
        | | | ¬ (A ∧ B)
        | | | ●
8 QED   | | | C                  7
7 PE    | | (A ∧ B) ∨ C          2
2 PC    | (A ∧ B) ∨ C
```

g.

```
        | A ∨ B                        1
        | C                            (5),(9)
        | | A                          (4)
        | | | ¬ (B ∧ C)
        | | | | ●
4 QED   | | | | A                      3
        | | | | ●
5 QED   | | | | C                      3
3 Cnj   | | | A ∧ C                    2
2 PE    | | (A ∧ C) ∨ (B ∧ C)          1
        | | B                          (8)
        | | | ¬ (A ∧ C)
        | | | | ●
8 QED   | | | | B                      7
        | | | | ●
9 QED   | | | | C                      7
7 Cnj   | | | B ∧ C                    6
6 PE    | | (A ∧ C) ∨ (B ∧ C)          1
1 PC    | (A ∧ C) ∨ (B ∧ C)
```

h.

```
        | A ∨ B                  1
        | ¬ A ∨ C                2
        | | A                    (5)
        | | | ¬ A                (5)
        | | | | ¬ B
        | | | | | ¬ C
        | | | | | ●
5 Nc    | | | | | ⊥            4
4 IP    | | | | C              3
3 PE    | | | B ∨ C            2
        | | | C                (7)
        | | | | ¬ B
        | | | | ●
7 QED   | | | | C              6
6 PE    | | | B ∨ C            2
2 PC    | | B ∨ C              1
        | | B                  (9)
        | | | ¬ C
        | | | ●
9 QED   | | | B                8
8 PE    | | B ∨ C              1
1 PC    | B ∨ C
```

i.

```
        │ A                          (3),(7)
        ││ ¬ (A ∧ B)                 5
        │││ ●
3 QED   │││ A                        2
        ││││ B                       (8)
        ││││││ ●
7 QED   ││││││ A                     6
        ││││││ ●
8 QED   ││││││ B                     6
6 Cnj   │││││ A ∧ B                  5
5 CR    ││││ ⊥                       4
4 RAA   │││ ¬ B                      2
2 Cnj   ││ A ∧ ¬ B                   1
1 PE    │ (A ∧ B) ∨ (A ∧ ¬ B)
```

```
        │ (A ∧ B) ∨ (A ∧ ¬ B)   1
        ││ A ∧ B                2
2 Ext   ││ A                    (3)
2 Ext   ││ B
        ││ ●
3 QED   ││ A                    1
        ││ A ∧ ¬ B              4
4 Ext   ││ A                    (5)
4 Ext   ││ ¬ B
        ││ ●
5 QED   ││ A                    1
1 PC    │ A
```

2. a.

```
        │ A ∨ A   1
        ││ A      (2)
        ││ ●
2 QED   ││ A      1
        ││ A      (3)
        ││ ●
3 QED   ││ A      1
1 PC    │ A
```

```
        │ A       (2)
        ││ ¬ A
        ││ ●
2 QED   ││ A      1
1 PE    │ A ∨ A
```

b.

```
        │ A ∨ B     1
        ││ A        (3)
        │││ ¬ B
        │││ ●
3 QED   │││ A       2
2 PE    ││ B ∨ A    1
        ││ B
        │││ ¬ A     (5)
        │││ ●
5 QED   │││ B       4
4 PE    ││ B ∨ A    1
1 PC    │ B ∨ A
```

```
        │ B ∨ A     2
        ││ ¬ A      (5)
        │││ B       (3)
        │││ ●
3 QED   │││ B       2
        │││ A       (5)
        ││││ ¬ B
        ││││ ●
5 Nc    ││││ ⊥      4
4 IP    │││ B       2
2 PC    ││ B        1
1 PE    │ A ∨ B
```

c.

```
        │ (A ∨ B) ∨ C   3
        ││ ¬ A         (6)
        │││ ¬ B        (8)
        ││││ A ∨ B     4
        │││││ A        (6)
        ││││││ ¬ C
        ││││││ ●
6 Nc    ││││││ ⊥       5
5 IP    │││││ C        4
        │││││ B        (8)
        ││││││ ¬ C
        ││││││ ●
8 Nc    ││││││ ⊥       7
7 IP    │││││ C        4
4 PC    ││││ C         3
        ││││ C         (9)
        ││││ ●
9 QED   ││││ C         3
3 PC    │││ C          2
2 PE    ││ B ∨ C       1
1 PE    │ A ∨ (B ∨ C)
```

This is the second of the two derivations needed; the first appears in 4.2.3 . In that one, disjunctive resources are exploited before disjunctive goals are planned for while the derivation at the left here illustrates the opposite approach.

d.

```
        │ A ∨ (B ∧ ¬ B)   1
        ││ A              (2)
        ││ ●
2 QED   ││ A              1
        ││ B ∧ ¬ B        3
3 Ext   ││ B              (5)
3 Ext   ││ ¬ B            (5)
        │││ ¬ A
        │││ ●
5 Nc    │││ ⊥             4
4 IP    ││ A              1
1 PC    │ A
```

```
        │ A               (2)
        ││ ¬ (B ∧ ¬ B)
        ││ ●
2 QED   ││ A              1
1 PE    │ A ∨ (B ∧ ¬ B)
```

e.

```
        │ ¬ (A ∨ B)     3,7
        │││ A           (5)
        │││││ ¬ B
        │││││ ●
5 QED   │││││ A         4
4 PE    ││││ A ∨ B      3
3 CR    │││ ⊥           2
2 RAA   ││ ¬ A          1
        │││ B           (9)
        │││││ ¬ A
        │││││ ●
9 QED   │││││ B         8
8 PE    ││││ A ∨ B      7
7 CR    │││ ⊥           6
6 RAA   ││ ¬ B          1
1 Cnj   │ ¬ A ∧ ¬ B
```

```
        │ ¬ A ∧ ¬ B    1
1 Ext   │ ¬ A          (4)
1 Ext   │ ¬ B          (5)
        ││ A ∨ B       3
        │││ A          (4)
        │││ ●
4 Nc    │││ ⊥          3
        │││ B          (5)
        │││ ●
        │││ ⊥          3
3 PC    ││ ⊥           2
2 RAA   │ ¬ (A ∨ B)
```

f.

```
        │ ¬ (A ∧ B)     3
        ││ A            (5)
        │││ B           (6)
        │││││ ●
5 QED   │││││ A         4
        │││││ ●
6 QED   │││││ B         4
4 Cnj   ││││ A ∧ B      3
3 CR    │││ ⊥           2
2 RAA   ││ ¬ B          1
1 PE    │ ¬ A ∨ ¬ B
```

```
        │ ¬ A ∨ ¬ B    3
        ││ A ∧ B       2
2 Ext   ││ A           (4)
2 Ext   ││ B           (5)
        │││ ¬ A        (4)
        │││ ●
4 Nc    │││ ⊥          3
        │││ ¬ B        (5)
        │││ ●
5 Nc    │││ ⊥          3
3 PC    ││ ⊥           1
1 RAA   │ ¬ (A ∧ B)
```

3. a.

```
        │ A ∨ B   2
        │ A
        ││ B
        │││ A
        │││ ○     A, B ⊭ ⊥
        │││ ⊥     2
        │││ B
        │││ ○     A, B ⊭ ⊥
        │││ ⊥     2
2 PC    ││ ⊥      1
1 RAA   │ ¬ B
```

A	B	A	∨	B	,	A	/	¬	B
T	T		Ⓣ			Ⓣ		Ⓕ	

b.

```
        │ A ∨ (B ∧ C)   3,8
        │││ ¬ A         (5)
        ││││ A          (5)
        │││││ ¬ B
        │││││ ●
5 Nc    │││││ ⊥        4
4 IP    ││││ B          3
        ││││ B ∧ C
6 Ext   ││││ B          7
6 Ext   ││││ C
        ││││ ●
7 QED   ││││ B          3
3 PC    │││ B           2
2 PE    ││ A ∨ B        1
        │││ A
        ││││ ¬ C
        ││││ ○          A, ¬ C ⊭ ⊥
        ││││ ⊥          9
9 IP    │││ C           8
        │││ B ∧ C       10
10 Ext  │││ B
10 Ext  │││ C           11
        │││ ●
11 QED  │││ C           8
8 PC    ││ C            1
1 Cnj   │ (A ∨ B) ∧ C
```

Since entailment fails in one direction, equivalence must fail, so a second derivation for entailment in the other direction need not be pursued; but that entailment does hold, as is shown below.

```
        │ (A ∨ B) ∧ C   1
1 Ext   │ A ∨ B         2
1 Ext   │ C             (8)
        ││ A            (4)
        │││ ¬ (B ∧ C)
        │││ ●
4 QED   │││ A           3
3 PE    ││ A ∨ (B ∧ C)  2
        ││ B            (7)
        │││ ¬ A
        ││││ ●
7 QED   ││││ B          6
        ││││ ●
8 QED   ││││ C          6
6 Cnj   │││ B ∧ C       5
5 PE    ││ A ∨ (B ∧ C)  2
2 PC    │ A ∨ (B ∧ C)
```

Each of the following lurks in the one open gap:

A	B	C	A	∨	(B	∧	C)	/	(A	∨	B)	∧	C
T	T	F		Ⓣ		F				T		Ⓕ	
T	F	F		Ⓣ		F				T		Ⓕ	

c.

	¬ (A ∨ B)	3
	A	(5)
	B	
	¬ B	
	●	
5 QED	A	4
4 PE	A ∨ B	3
3 CR	⊥	2
2 RAA	¬ B	1
1 PE	¬ A ∨ ¬ B	

The following counterexamples lurk in the first and second open gap, respectively:

A	B	¬	A	∨	¬	B	/	¬	(A	∨	B)
F	T	T		Ⓣ	F			Ⓕ		T	
T	F	F		Ⓣ	T			Ⓕ		T	

	¬ A ∨ ¬ B	2
	A ∨ B	3,5
	¬ A	(4)
	A	(4)
	●	
4 Nc	⊥	3
	B	
	○	¬ A, B ⊭ ⊥
	⊥	3
3 PC	⊥	2
	¬ B	(6)
	A	
	○	A, ¬ B ⊭ ⊥
	⊥	5
	B	(6)
	●	
6 Nc	⊥	5
5 PC	⊥	2
2 PC	⊥	1
1 RAA	¬ (A ∨ B)	

Glen Helman 16 Jul 2012