8.5. Proofs by choice and proofs of existence

8.5.0. Overview

Although formal proofs for disjunction involve some new ideas, these are mainly recombinations of ideas used for disjunction and universals.

8.5.1. Proof by choice

A conclusion can be derived from an existential by choosing a new name for the example whose existence it claims.

8.5.2. Constructive and non-constructive proof

A claim of exemplification can be established either by constructing an example or by reducing to absurdity the assumption that there is no such example.

8.5.3. Derivations for existentials

Our selection of derivation rules for existentials is analogous to that for disjunction, with two basic rules supplemented by an often useful attachment rule.

8.5.4. First-order logic

This completes our account of entailment for first-order logic, which has come to replace the theory of syllogisms as the generally accepted core of deductive logic.

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8.5.1. Proof by choice

As has been the case elsewhere in this chapter, our discussion of principles of entailment for existentials can build on our discussion of universals in the last chapter. The differences between the principles governing universal and existential quantifiers will, in most cases, be analogous to differences between the principles for conjunction and disjunction. The laws of entailment for the universal quantifiers were modifications of laws for conjunction, and the rules for the existential quantifiers will nearly all be based in a similar way on rules for disjunction. Our planning rule for existential sentences is the one exception to this, and even it is analogous to a rule that could have been used for disjunction.

These analogies with the universal quantifier on the one hand and with disjunction on the other derive from the truth conditions for the unrestricted existential, which follow the conditions for disjunction in precisely the way the conditions for the universal follow those for conjunction. A sentence ∃x θx is true in a structure if and only if it has at least one true instance in a language expanded by the range **R** of that structure. In other words, an existential claim behaves like a disjunction of its instances when these instances are taken from a language that has a name for each reference value in the structure. However, as was the case with the universal, the set of instances can vary from one structure to another, so general laws of entailment cannot employ any definite information about what the instances of an existential sentence are.

We will begin our discussion of principles of entailment with the role of an unrestricted existential as a premise. First, recall the corresponding principle for disjunction. A disjunctive premise may be used to draw a conclusion by way of a proof by cases. In such a proof, we suppose in turn that each of the disjuncts is true and argue for the conclusion in each case. A comparable way of arguing from an existential would be to establish many case arguments, each one considering an instance of the existential as one case. Since we cannot associate the existential with any definite set of instances, there is no way to delimit the range of case arguments we would need to consider, so we must use adapt a device from our treatment of the universal: we need to set out the indefinitely many arguments by offering a general pattern. That is, to use an existential premise to draw a conclusion, we draw the conclusion from one instance of the existential in a way that sets a pattern for all other instances.

This sort of argument may be called a *proof by choice*, a name which reflects another way of looking at the principle behind it. Consider the two arguments below.

Anyone who worked late got overtime	Anyone who worked late got overtime
If anything broke down, Tom worked late	If anything broke down, Tom worked late
Something broke down	*E* broke down
Tom got overtime	Tom got overtime

The validity of the argument on the left can be traced to the validity of the one on the right. In the latter, we use the premise *E* broke down in place of the existential Something broke down, so we argue for the conclusion from an instance of the existential. When we replace an existential by an instance of it, we are choosing *E* as a name for an example that the existential claims to exist, so this is an argument to proceeds by way of the choice of a name.

Of course, we cannot assume that the "something" claimed to exist by an existential premise is some thing that we have other information about. That is, choosing a name really means choosing a *new* name. For an unrestricted existential tells us nothing about the example it claims to exist except for the property it is said to exemplify. So the name we choose must be one that could apply to anything that has this property. And that returns us to the first way of looking at proofs by choice: they must argue from one instance of an existential in a way that sets a pattern for all such instances.

Recalling the test we used for the generality of arguments in the case of the universal quantifiers, we can expect our analysis of the role of an existential as a premise to make reference to a term that is *independent* in an appropriate sense. We will want a term α that has no connection to the premises and conclusion of the argument—including the existential $\exists x\ \theta x$—apart from the assumption $\theta\alpha$. So suppose the term α is unanalyzed term and does not appear in the set Γ, the sentence φ, or the existential $\exists x\ \theta x$, and consider the two arguments

$$\Gamma, \exists x\ \theta x \ /\ \varphi$$
$$\Gamma, \theta\alpha \ /\ \varphi.$$

We can argue that each is valid if and only if the other is if we can show that each is divided by a structure if and only if the other is. If a structure S divides the premises and conclusion of the first, it will assign θ a non-empty extension, and we can form a structure S′ that divides the second argument by assigning a value in this extension to the term α. For this assignment will not change the extension of θ or the truth values of φ and the members Γ since α does not appear in these expressions, so $\theta\alpha$ will be true and the conclusion and the other premises will keep the same truth values. On the other hand, any structure dividing the second argument will give θ a non-empty extension (because the value of the term α will be in it) so this structure will make ∃x θx true and also divide the first argument. Thus we will have a structure dividing one argument if and only if we have a structure dividing the other, and each argument is valid if and only if the other is.

This gives us our principle describing the role of the unrestricted existential as a premise.

Law for the unrestricted existential as a premise. Γ, ∃x θx ⊨ φ if and only if Γ, θα ⊨ φ (for any set Γ, predicate θ, and sentence φ and any unanalyzed term α that does not appear in Γ, θ, or φ)

The corresponding principle for the restricted existential combines these ideas with the properties of conjunction:

Γ, (∃x: ρx) θx ⊨ φ if and only if Γ, ρα, θα ⊨ φ
(for any set Γ, predicates ρ and θ, and sentence φ and any unanalyzed term α that does not appear in Γ, ρ, θ, or φ)

That is, having a restricted existential as an assumption comes to the same thing as assuming that an independent term refers to something that is in the domain and has the attribute.

8.5.2. Constructive and non-constructive proof

Let us now turn to the task of establishing an existential conclusion. As the title suggests, we will consider two ways of doing this. In the first and most general of these, we establish a claim ∃x θx that a property θ is exemplified by reducing to absurdity the claim ∀x ¬± θx that nothing has this property. This way of drawing an existential conclusion is called a *non-constructive proof* because it enables us to establish a claim of exemplification without ever describing a particular example. The use of the term construction here can be traced historically to geometry, where claims of exemplification are typically established by a geometric construction of the figure that is claimed to exist; but the term construction has come to be applied in mathematics to other techniques that specify particular examples.

We will adopt the idea of non-constructive proof as our basic principle for the existential as a conclusion.

LAW FOR THE UNRESTRICTED EXISTENTIAL AS A CONCLUSION. Γ ⊨ ∃x θx if and only if Γ, ∀x ¬± θx ⊨ ⊥ (for any set Γ and predicate θ)

This principle does not explain the role of the existential as a conclusion directly, but instead makes a connection with the role of the universal as a premise. We are led to do things in this way by entailment's focus on a single sentence as the conclusion. Were we to consider conditional exhaustiveness rather than entailment, a law for ∃ that makes no reference to ∀ would be easier to state because the consideration of multiple alternatives makes it possible to formulate a principle dual to the principle for the universal as a premise; see appendix B for the form this principle takes.

The second rule for existential conclusions takes a more direct approach. A *constructive proof* of a claim of exemplification establishes the claim by first producing an example of the sort that is claimed to exist. The move from an example to a claim of exemplification appears formally as a step from an instance of an existential to the existential itself. The principle of entailment governing this step is commonly known as *existential generalization*:

θτ ⊨ ∃x θx (for any term τ)

The conclusion of this entailment is not a generalization in the sense in which we have been using the term. But it may be said of someone who is making heavy use of words like something and someone that he is "speaking in generalities" and is not being specific. The principle of existential generalization is a license to move from a specific claim to a generality of an existential sort. We cannot rely on this principle alone, but it does provide a useful supplement in the way the principle of weakening supplements the law for disjunction as a conclusion. And, like weakening, we will count existential generalization as an attachment principle. (What is attached? In form, we could say it is the existential quantifier; in what is said, it is the other instances of the conclusion, the other ways in which it could be true.)

Although non-constructive proofs of exemplification have been common in modern mathematics, some have questioned their value. The doubts about them have not usually been doubts about their validity (though Brouwer, who was mentioned in 3.1.3, could be said to have doubted that—in spite of the fact that early in his career he produced some non-constructive proofs for which he is still famous). The feature of non-constructive proofs that lies behind these doubts is a different sort of weakness that is granted even by those who accept such proofs happily: because they do not produce an example, non-constructive proofs may provide little insight into the reasons why a claim of exemplification is true.

The deepest concerns about non-constructive proof are focused on arguments about abstract and, especially, infinite structures, and even Brouwer thought that non-constructive proofs were valid for reasoning about ordinary claims concerning the world of sense experience. Still, the indirection and uninformativeness of non-constructive arguments can be felt with ordinary reasoning and is often unnecessary, so it is worthwhile considering the alternative.

Before considering the implementation of these principles of entailment in derivations, let us look a little more closely at the reasons why our general account of the existential as the conclusion has been made parasitic on our account of the universal as a premise. First, recall our account of the role of disjunction as conclusion. In one of its forms it is this: $\Gamma \vDash \varphi \vee \psi$ if and only if Γ, $\neg^{\pm} \varphi \vDash \psi$. We could have avoided the asymmetric treatment of the two components in this principle if we had resorted to an even heavier use of negation; applying the idea behind IP to the right side of the law, we get this: $\Gamma \vDash \varphi \vee \psi$ if and only if Γ, $\neg^{\pm} \varphi$, $\neg^{\pm} \psi \vDash \bot$. That is, a disjunction is a valid conclusion if and only if we can reduce to absurdity the supposition that its components are both false. A strict analogue for the existential of this rule for disjunction would say that we can conclude an existential $\exists x\ \theta x$ from premises Γ if and only if we can reduce to absurdity the result of adding denials of all the instances of $\exists x\ \theta x$ to Γ. But, since there is no definite set of instances, we cannot take this approach literally. So, instead of adding the denials of all instances of the existential, we add the corresponding generalization $\forall x \neg^{\pm} \theta x$.

Any approach to existential conclusions that is more analogous to the principle for disjunctions as a conclusion would force us to consider repeated partial planning for existential goals as we consider we consider repeated partial exploitation of universal premises. And that would involve a far greater modification of the system of derivations than the rules we will now go on to consider.

8.5.3. Derivations for existentials

To implement the laws we have just been considering, we will again use ideas introduced in connection with universals. In particular, a proof by choice will be marked by a veil of ignorance flagged by an independent term, and it will have a supposition that sets out the example chosen. However, the complications that appeared with the rules for exploiting universals may be left with those rules, since we manage planning for an existential conclusion simply by passing the buck on to universals.

The two basic rules for the unrestricted existential are *Proof by Choice* (PCh) and *Non-constructive Proof* (NcP):

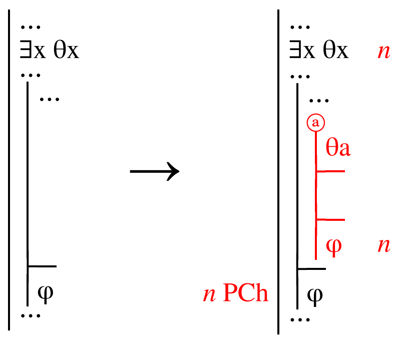


Fig. 8.5.3-1. Developing a derivation at stage *n* by exploiting an unrestricted existential; the independent term a is new to the derivation.

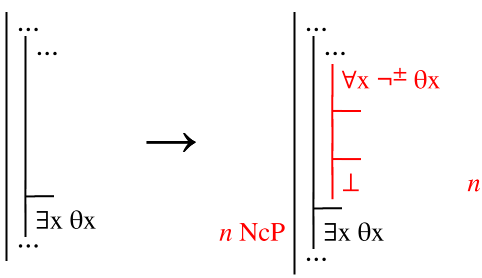


Fig. 8.5.3-2. Developing a derivation at stage *n* by planning for an unrestricted existential.

Notice that the existential is rendered inactive in the first rule. Also remember that the independent term that is used in this rule should be new to the derivation; that will insure that the supposition that is introduced represents the only information about this independent term that may be used in closing the gap.

The second rule will often be a very indirect way of reaching an existential goal, and the attachment rule, *Existential Generalization* (EG), which implements the idea of constructive proof, can simplify derivations considerably:

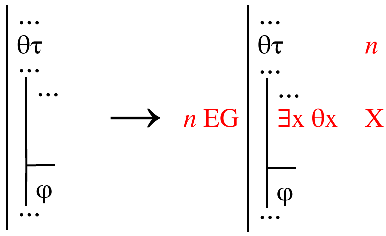


Fig. 8.5.3-3. Developing a derivation at stage *n* by adding an unrestricted existential that has an instance among the active resources.

Although this is an attachment rule and therefore not part of the basic system, you should be as ready to use it as the two above.

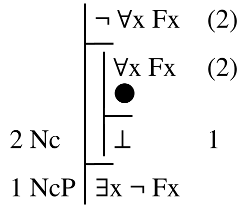
Here are two derivations that illustrate these rules. Each shows that a claim of uniformly general exemplification implies the corresponding claim of general exemplification without a claim of uniformity.

	∃x ∀y Rxy	1
	ⓐ ∀y Ray	b:3
3 UI	ⓑ Rab	(6)
	∀x ¬ Rxb	a:5
5 UI	¬ Rab	(6)
	●	
6 Nc	⊥	4
4 NcP	∃x Rxb	2
2 UG	∀y ∃x Rxy	1
1 PCh	∀y ∃x Rxy	

	∃x ∀y Rxy	1
	ⓐ ∀y Ray	b:3
3 UI	ⓑ Rab	(4)
4 EG	∃x Rxb	X,(5)
	●	
5 QED	∃x Rxb	2
2 UG	∀y ∃x Rxy	1
1 PCh	∀y ∃x Rxy	

The derivation on the left uses a non-constructive proof of the existential that is set as the goal in stage 2 while the one on the right uses EG to give a constructive proof of this existential. Both derivations begin by exploiting the existential premise, but derivations for the same entailment could have been developed by planning for the initial conclusion first; and, when NcP is used, it would be possible to postpone the exploitation of the initial premise until after NcP is applied. (It would be a good exercise at this point to write down these other derivations for this argument.) The savings in length and complexity that were achieved by using EG in this case are typical.

Since EG can be used only when the resources entail an existential, it often cannot be used in derivations that fail, and NcP is required even in some derivations for valid existential conclusions. A derivation showing the obversion principle ¬ ∀x Fx ⊨ ∃x ¬ Fx is simple example of this.



EG could not have been applied here because the premise does not entail any sentence ¬ Fτ from which we could generalize.

Arguments for the soundness and completeness of this system carry over from 7.7 without any new wrinkles. We solved all the key problems there, and a number are not even repeated here. However, we cannot avoid the consequences of the failure of decisiveness. If we wish to find finite counterexamples whenever they exist, we need to use a modified rule for exploiting existential resources in the way the rule for planning for a universal goal was modified in 7.8.1. Without such a rule, we will not reach dead-end open gap in any derivation whose resources contain a weak, though unrestricted, claim of general exemplification (e.g., a sentence of the form ∀x ∃y Rxy). The modified rule needed to search for finite counterexamples is *Supplemented Proof by Choice* (PCh+).

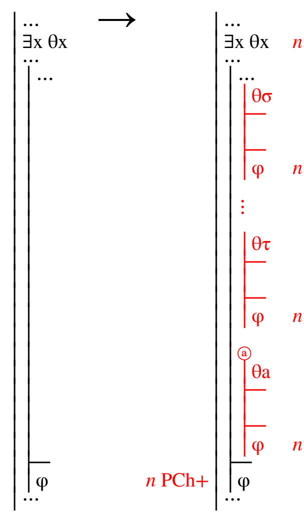
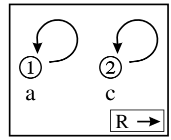


Fig. 8.5.3-4. Developing a derivation at stage *n* by exploiting an unrestricted or a restricted existential; the independent term a is new to the derivation and the terms σ, …, τ include at least one from each current alias set for the gap

The following derivation illustrates this rule. It shows that a claim of general exemplification need not imply uniformity. That is, it finds a counterexample to the entailment ∀x ∃y Rxy ⊨ ∃y ∀x Rxy.



	∀x ∃y Rxy	a:2, c:9
	∀y ¬ ∀x Rxy	a:3, c:10
2 UI	∃y Ray	5
3 UI	¬ ∀x Rxa	4
	Raa	(7)
	●	
7 QED	Raa	6
	ⓒ	
	¬ Rca	(15)
9 UI	∃y Rcy	12
10 UI	¬ ∀x Rxc	11
	Rca	(15)
	(unfinished but closes)	
	∀x Rxc	12
	Rcc	
	¬ Rac	
	○	Raa, ¬ Rca, Rcc, ¬ Rac ⊭ ⊥
	⊥	14
14 IP	Rac	13
	(unfinished but closes)	
	Rcc	13
	ⓔ	
	(unfinished)	
	Rec	13
13 UG+	∀x Rxc	12
	ⓓ	
	Rcd	
	(unfinished)	
	∀x Rxc	12
12 PCh+	∀x Rxc	11
11 CR	⊥	8
8 IP	Rca	6
6 UG+	∀x Rxa	5
	ⓑ	
	Rab	
	(unfinished)	
	∀x Rxa	5
5 PCh+	∀x Rxa	4
4 CR	⊥	1
1 NcP	∃y ∀x Rxy	

Although this is long and cumbersome, the development of the dead-end gap goes through the kinds steps you would need to go through in your own thinking to arrive the same counterexample (the bracketed numbers link steps in this thinking with the stages of the derivation):

> The premise says that everything stands in relation R to something or other. And, to make the conclusion false, we need to know that there is nothing that has everything standing in the relation R to it [1]. So we must have an object a such that a stands in R to something but not everything stands in R to a [2-4]. Now we can't be sure that a is related to itself; but we can still consider this possibility as one way of making it true that a is related to something by R, so let's suppose that Raa [5]. But we also need to make it false that everything stands in R to a, so let's suppose we have an object c that doesn't stand in R to a [6-8]. Now c must stand in R to something and it can't have everything standing in R to it [9-11, cf. 2-4]. Let's try supposing it stands in R to itself (though it might be something else that it is related to) [12]. Now, we must also be sure that not everything stands in R to c, but nothing keeps us from supposing that a does not [13-14]. Summing up, we've described a possible world containing objects a and c where Raa, ¬ Rca, Rcc, ¬ Rac; and that's enough to make the premise true and the conclusion false.

Developing the unfinished gaps would lead to other counterexamples. For example, the last open gap in this derivation explores the possibility of making the premise true by having a stand in R to another object b and it would, among other things, lead us to a counterexample in which each of a and b is stands in R to the other but neither stands in R to itself.

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8.5.4. First-order logic

Although we will go on in 8.6 to give some consideration to derivations for the description operator, our system of derivations is now essentially complete. It is intended to capture entailments that derive from truth-functional logic and the logical properties of identity, predication, and the quantifiers. This range of logical forms is the concern of *first-order logic*. (Usage varies a little, and sometimes identity is not included; in that case, our subject is "first-order logic with identity.") Beginning about a century ago, first-order logic came to replace the theory of syllogisms as the commonly accepted core of deductive logic. In the current practice of mathematics, for example, even very abstract general principles falling beyond its scope would be treated as special axioms (of the theory of sets for example) while principles of first-order logic would be accepted as background assumptions within the context of which the consequences of special axioms are assessed.

The theory of syllogisms itself appears as an account of a very special collections of arguments. The outlines of this collection were sketched in 7.5.6, but having the existential quantifier makes it possible to provide more detail in a compact way. The four moods, the logical forms recognized by the theory, are as follows (with the vowels that serve as their traditional labels):

A:	$(\forall x\colon \rho x)\ \theta x$	E:	$(\forall x\colon \rho x) \neg \theta x$
I:	$(\exists x\colon \rho x)\ \theta x$	O:	$(\exists x\colon \rho x) \neg \theta x$

and the four figures are the following patterns of occurence of the three predicates that can appear in a syllogism (where the predicate shown on the left is the resticting predicate of the sentence and the one on the right is its quantified predicate):

1st	*2nd*	*3rd*	*4th*
μ θ	θ μ	μ θ	θ μ
ρ μ	ρ μ	μ ρ	μ ρ
ρ θ	ρ θ	ρ θ	ρ θ

Here μ is the middle term.

Of the 64 syllogisms of the first figure, the following four are valid:

Barbara	*Celarent*	*Darii*	*Ferio*
$(\forall x\colon \mu x)\ \theta x$	$(\forall x\colon \mu x) \neg \theta x$	$(\forall x\colon \mu x)\ \theta x$	$(\forall x\colon \mu x) \neg \theta x$
$(\forall x\colon \rho x)\ \mu x$	$(\forall x\colon \rho x)\ \mu x$	$(\exists x\colon \rho x)\ \mu x$	$(\exists x\colon \rho x)\ \mu x$
$(\forall x\colon \rho x)\ \theta x$	$(\forall x\colon \rho x) \neg \theta x$	$(\exists x\colon \rho x)\ \theta x$	$(\exists x\colon \rho x) \neg \theta x$

Notice that the pattern of vowels in the traditional name shown below each argument matches the moods of its premises and conclusion. The proportion of valid arguments in the other figures is similar, and there are fifteen valid syllogisms all told.

One of the limitations of theory of syllogisms is an inability to consider logical relations between the restricitng and quantified predicates of a generalization or claim of exemplification. For example, using the resources of first-order logic, we can account for the fact that Every horse is a mammal implies Any head of a horse is a head of a mammal (an entailment mentioned in 7.1.1).

$$\frac{(\forall x\colon Hx)\ Mx}{(\forall x\colon (\exists y\colon Hy)\ Dxy)\ (\exists z\colon Mz)\ Dxz}$$

H: [_ is a horse]; M: [_ is a mammal]; D: [_ is a head of _]

	$\forall x\ (Hx \to Mx)$	b:5
	ⓐ	
	$\exists y\ (Hy \land Day)$	3
	ⓑ	
	$Hb \land Dab$	4
4 Ext	Hb	(6)
4 Ext	Dab	(7)
5 UI	$Hb \to Mb$	6
6 MPP	Mb	(7)
7 Adj	$Mb \land Dab$	X, (8)
8 EG	$\exists z\ (Mz \land Daz)$	X, (9)
	●	
9 QED	$\exists z\ (Mz \land Daz)$	3
3 PCh	$\exists z\ (Mz \land Daz)$	2
2 CP	$\exists y\ (Hy \land Day) \to \exists z\ (Mz \land Daz)$	1
1 UG	$\forall x\ (\exists y\ (Hy \land Dxy) \to \exists z\ (Mz \land Dxz))$	

(The use of adjunction and existential generalization at stages 7 and 8 saves us having to enter $\forall z\ \neg\ (Mz \land Daz)$ as a supposition to be reduced to absurdity.) Even though this argument is closely related to syllogisms—the active resources and goal after the use of CP at stage 2 form a valid syllogism of the third figure known as Disamis—its validity cannot be explained without an analysis of the restricting and quantified predicates of the conclusion, something the theory of syllogisms does not provide for.

Although first-order logic forms the core of deductive logic, it is not the whole of it. One way to go beyond it is to study the sort of non-truth-functional connectives noted in 3.1.2. Another is to consider further sorts of quantifiers. The qualification first-order derives from the fact that we analyze quantification only over individuals and not over properties and relations. Thus we cannot analyze the sentence Objects a and b are identical if and only if every property of one is a property of the other, and we cannot ask whether this sentence is a tautology. The representation of such *higher-order* quantification symbolically would present few new problems. We would need bindable variables that functioned syntactically as predicates, notation for complex predicates of predicates (with our quantifiers serving as simple predicates of predicates), and quantifiers applying to such predicates of predicates. This would give us *second-order logic*. To go further, we might introduce quantification for predicates of predicates—and so on. If this process is continued to all (finite) orders, we end up with what is known as *higher-order logic* or *(simple) type theory*.

While higher-order logic introduces nothing really new in its syntax, the account of entailment for it is a completely different game, and the new problems appear already with second-order logic. In particular, there can be no sound system for settling questions of validity for second-order logic that is even complete, much less decisive. Indeed, a full understanding of validity for second-order logic would provide a full understanding of all truths concerning positive integers. But it was shown by Kurt Gödel in the early 1930s that these truths cannot be captured by anything like a system of derivations. (This is the result mentioned in 7.7.1 as the basis on which Church showed that there could be no system of derivations for first-order logic that was decisive as well as sound and complete.)

So there is reason to distinguish the theory of first-order quantification from higher-order logic. Frege's work did not make this distinction. The subject matter he addressed included the whole of what is now known as type theory because he was interested in connections with arithmetic, whose truths he wished to explain as logical tautologies. Although he provided what was essentially a complete account of validity for first-order logic, his treatment of other areas introduced inconsistencies. These were repaired shortly after (in the first decade of the 20th century) by Bertrand Russell, whose work led to the current conception of type theory. First-order logic came to be distinguished within type theory and was permanently set in its present form by Gödel when he showed that Frege's initial ideas provided a complete account of validity for this part of logic.

8.5.s. Summary

1 Existentials bear the kind of analogy to disjunctions that universals bear to conjunctions, and their role in entailment reflects this. Our principle for the unrestricted existential as premise says that the existential will support a proof by choice. This is a sort of proof by cases in which cases for each instance of the existential are handled not one by one but by using an independent term to make a general argument. This instance can be thought of as an example, chosen in ignorance of its identity, of the sort that the existential claims to exist.

2 There are two approaches to establishing an existential conclusion. Our general principle for the unrestricted existential as a conclusion uses the idea of non-constructive proof, in which a claim of exemplification is based on the reduction to absurdity of a corresponding negative universal. In a constructive proof using existential generalization, an existential conclusion is based on the proof of an instance, which thus "constructs" an example of the sort the existential claims to exist.

3 The laws for existential premises and conclusions are implemented in exploitation and planning rules using some ideas from the rules for universals. The principles for unrestricted existentials are implemented in the rules Proof by Choice (PCh), Non-constructive Proof (NcP), and Existential Generalization (EG). Also as was the case with the universal quantifier, to uncover counterexamples to invalid arguments using finite ranges (when such counterexamples exist), we would need a supplemented form of proof, in this case PCh+.

4 The system we have now completed accounts for the entailments of what is known as first-order logic. It has come to been seen as the core of deductive logic. Until a century ago that status was given to the theory of syllogims, which can be regarded as a portion of first-order logic which does not make use of the possibility of analyzing the restricting and quantified predicates of generalizations or claims of exemplification.

The qualification first-order indicates that we consider quantification only over individuals and not over properties, properties of those properties, or any other second-order or higher-order entities. Although higher-order logic, or type theory, has attracted interest since Frege, it cannot be given a complete system of derivations.

8.5.x. Exercise questions

1. Use the system of derivations to establish each of the following:

a. $\exists x\, Fx, \forall x\, (Fx \to Gx) \vDash \exists x\, Gx$

b. $\exists x\, (Fx \land Gx), \forall x\, (Gx \to Hx) \vDash \exists x\, (Fx \land Hx)$ [this is the syllogism Darii]

c. $\forall x\, (Fx \to Ga) \simeq \exists x\, Fx \to Ga$

d. $Fa \simeq \exists x\, (x = a \land Fx)$

e. $\exists x\, (Fx \land \forall y\, Rxy) \vDash \forall x\, \exists y\, (Fy \land Ryx)$

f. $\exists x\, (Gx \land Fx), \neg Fa \vDash \exists x\, (\neg x = a \land Gx)$

g. $\forall x\, (Fx \to Ga), \forall x\, (Ga \to Fx), \exists x\, Fx \vDash \forall x\, Fx$

h. Everyone loves everyone who loves anyone, Someone loves someone ⊨ Everyone loves everyone

i. Something is such that nothing other than it is done ≃ At most one thing is done

2. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, describe a structure that divides an open gap. You need not worry about infinite derivations.

a. $\exists x\, Fx, \exists x\, Gx \vDash \exists x\, (Fx \land Gx)$

b. $\exists x\, (Fx \land Gx), \exists x\, (Fx \land Hx), \forall x\, (Fx \to \forall y\, (Fy \to x = y)) \vDash \exists x\, (Gx \land Hx)$

3. In the following, choose one of each bracketed pair of premises and one each bracketed pair of words or phrases in the conclusion so as to make a valid argument; then analyze the premises and conclusion and construct a derivation to show that the argument is valid.

a. Some road sign was colored

[Every stop sign was a road sign | Every road sign was a traffic marker]

[If anything was red, it was colored | If anything was colored, it was painted]

Some [stop sign | traffic marker] was [red | painted]

b. Someone who owns a snake was pleased

[Every cobra is a snake | Every snake is a reptile]

Someone who owns a [cobra | reptile] was pleased

For more exercises, use the exercise machine.

8.5.xa. Exercise answers

1. Some of the derivations below are given in two forms, one that does not use EG and another that does.

a.

	∃x Fx	1
	∀x (Fx → Gx)	a:2
	ⓐ	
	Fa	(3)
2 UI	Fa → Ga	3
3 MPP	Ga	
	∀x ¬ Gx	a:5
5 UI	¬ Ga	(6)
	●	
6 Nc	⊥	
4 NcP	∃x Gx	1
1 PCh	∃x Gx	

	∃x Fx	1
	∀x (Fx → Gx)	a:2
	ⓐ	
	Fa	(3)
2 UI	Fa → Ga	3
3 MPP	Ga	(4)
4 EG	∃x Gx	X, (5)
	●	
5 QED	∃x Gx	1
1 PCh	∃x Gx	

b.

	∃x (Fx ∧ Gx)	1
	∀x (Gx → Hx)	a:3
	ⓐ	
	Fa ∧ Ga	2
2 Ext	Fa	(7)
2 Ext	Ga	(4)
3 UI	Ga → Ha	4
4 MPP	Ha	(8)
	∀x ¬ (Fx ∧ Hx)	a:6
6 UI	¬ (Fa ∧ Ha)	7
7 MPT	¬ Ha	(8)
	●	
8 Nc	⊥	5
5 NcP	∃x (Fx ∧ Hx)	1
1 PCh	∃x (Fx ∧ Hx)	

	∃x (Fx ∧ Gx)	1
	∀x (Gx → Hx)	a:3
	ⓐ	
	Fa ∧ Ga	2
2 Ext	Fa	(5)
2 Ext	Ga	(4)
3 UI	Ga → Ha	4
4 MPP	Ha	(5)
5 Adj	Fa ∧ Ha	X, (6)
6 EG	∃x (Fx ∧ Hx)	X, (7)
	●	
7 QED	∃x (Fx ∧ Hx)	1
1 PCh	∃x (Fx ∧ Hx)	

c.

	$\forall x\,(Fx \to Ga)$	b:3
	$\exists x\, Fx$	2
	ⓑ Fb	(4)
3 UI	$Fb \to Ga$	4
4 MPP	Ga	(5)
	●	
5 QED	Ga	2
2 PCh	Ga	1
1 CP	$\exists x\, Fx \to Ga$	

	$\exists x\, Fx \to Ga$	4
	ⓑ Fb	(8)
	$\neg Ga$	(4)
4 MTT	$\neg \exists x\, Fx$	5
	$\forall x\, \neg Fx$	b:7
7 UI	$\neg Fb$	(8)
	●	
8 Nc	$\bot$	6
6 NcP	$\exists x\, Fx$	5
5 CR	$\bot$	3
3 IP	Ga	2
2 CP	$Fb \to Ga$	1
1 UG	$\forall x\,(Fx \to Ga)$	

	$\exists x\, Fx \to Ga$	4
	ⓑ Fb	(3)
3 EG	$\exists x\, Fx$	X, (4)
4 MPP	Ga	(5)
	●	
5 QED	Ga	2
2 CP	$Fb \to Ga$	1
1 UG	$\forall x\,(Fx \to Ga)$	

d.

```
            │Fa                    (3)
            ├─
            ││∀x ¬ (x = a ∧ Fx)    a:2
            │├─
2 UI        ││¬ (a = a ∧ Fa)       3
3 MPT       ││¬ a = a              (4)
            ││●
            │├─
4 DC        ││⊥                    1
            ├─
1 NcP       │∃x (x = a ∧ Fx)
```

```
            │Fa                    (2)
            ├─
1 EC        │a = a                 X, (2)
2 Adj       │a = a ∧ Fa            X, (3)
3 EG        │∃x (x = a ∧ Fx)       X, (4)
            │●
            ├─
4 QED       │∃x (x = a ∧ Fx)
```

```
            │∃x (x = a ∧ Fx)       1
            ├─
            │ⓑ
            ││b = a ∧ Fb           2
            │├─
2 Ext       ││b = a                a—b
2 Ext       ││Fb                   (3)
            ││●
            │├─
3 QED=      ││Fa                   1
            ├─
1 PCh       │Fa
```

e.

```
            │∃x (Fx ∧ ∀y Rxy)          2
            ├─
            │ⓐ
            ││ⓑ
            │││Fb ∧ ∀y Rby             3
            ││├─
3 Ext       │││Fb                      (6)
3 Ext       │││∀y Rby                  a:7
            ││││∀y ¬ (Fy ∧ Rya)        b:5
            │││├─
5 UI        ││││¬ (Fb ∧ Rba)           6
6 MPT       ││││¬ Rba                  (8)
7 UI        ││││Rba                    (8)
            ││││●
            │││├─
8 Nc        ││││⊥                      4
            ││├─
4 NcP       │││∃y (Fy ∧ Rya)           2
            │├─
2 PCh       ││∃y (Fy ∧ Rya)            1
            ├─
1 UG        │∀x ∃y (Fy ∧ Ryx)
```

```
            │∃x (Fx ∧ ∀y Rxy)      2
            ├─
            │ⓐ
            ││ⓑ
            │││Fb ∧ ∀y Rby         3
            ││├─
3 Ext       │││Fb                  (5)
3 Ext       │││∀y Rby              a:4
4 UI        │││Rba                 (5)
5 Adj       │││Fb ∧ Rba            X, (6)
6 EG        │││∃y (Fy ∧ Rya)       X, (7)
            │││●
            ││├─
7 QED       │││∃y (Fy ∧ Rya)       2
            │├─
2 PCh       ││∃y (Fy ∧ Rya)        1
            ├─
1 UG        │∀x ∃y (Fy ∧ Ryx)
```

f.

	∃x (Gx ∧ Fx)	1
	¬ Fa	(6)
	ⓑ	
	│ Gb ∧ Fb	2
2 Ext	│ Gb	(5)
2 Ext	│ Fb	(6)
	│ │ ∀x ¬ (¬ x = a ∧ Gx)	b:4
4 UI	│ │ ¬ (¬ b = a ∧ Gb)	5
5 MPT	│ │ b = a	a—b
	│ │ ●	
6 Nc=	│ │ ⊥	3
3 NcP	│ ∃x (¬ x = a ∧ Gx)	1
1 PCh	∃x (¬ x = a ∧ Gx)	

g.

	∀x (Fx → Ga)	c:3
	∀x (Ga → Fx)	b:5
	∃x Fx	2
	ⓑ	
	│ ⓒ	
	│ │ Fc	(4)
3 UI	│ │ Fc → Ga	4
4 MPP	│ │ Ga	(6)
5 UI	│ │ Ga → Fb	6
6 MPP	│ │ Fb	(7)
	│ │ ●	
7 QED	│ │ Fb	2
2 PCh	│ Fb	1
1 UG	∀x Fx	

h.

(∀x: Px) (∀y: Py ∧ (∃z: Pz) Lyz) Lxy
(∃x: Px) (∃y: Py) Lxy

(∀x: Px) (∀y: Py) Lxy

```
          | ∀x (Px → ∀y ((Py ∧ ∃z (Pz ∧ Lyz)) → Lxy))        b:7, a:14
          | ∃x (Px ∧ ∃y (Py ∧ Lxy))                          5
          |----
          | @a
          | | | Pa                                           (15)
          | | |----
          | | | @b
          | | | | | Pb                                       (8), (19)
          | | | | |----
          | | | | | @c
          | | | | | | Pc ∧ ∃y (Py ∧ Lcy)                     6
          | | | | | |----
6 Ext     | | | | | | Pc                                     (12), (17)
6 Ext     | | | | | | ∃y (Py ∧ Lcy)                          10
7 UI      | | | | | | Pb → ∀y ((Py ∧ ∃z (Pz ∧ Lyz)) → Lby)  8
8 MPP     | | | | | | ∀y ((Py ∧ ∃z (Pz ∧ Lyz)) → Lby)       c:9
9 UI      | | | | | | (Pc ∧ ∃z (Pz ∧ Lcz)) → Lbc            13
          | | | | | |
          | | | | | | @d
          | | | | | | | Pd ∧ Lcd                             (11)
          | | | | | | |----
11 EG     | | | | | | | ∃z (Pz ∧ Lcz)                       X, (12)
12 Adj    | | | | | | | Pc ∧ ∃z (Pz ∧ Lcz)                  X, (13)
13 MPP    | | | | | | | Lbc                                 (17)
14 UI     | | | | | | | Pa → ∀y ((Py ∧ ∃z (Pz ∧ Lyz)) → Lay) 15
15 MPP    | | | | | | | ∀y ((Py ∧ ∃z (Pz ∧ Lyz)) → Lay)    b:16
16 UI     | | | | | | | (Pb ∧ ∃z (Pz ∧ Lyz)) → Lab         20
17 Adj    | | | | | | | Pc ∧ Lbc                            X, (18)
18 EG     | | | | | | | ∃z (Pz ∧ Lbz)                       X, (19)
19 Adj    | | | | | | | Pb ∧ ∃z (Pz ∧ Lbz)                  X, (20)
20 MPP    | | | | | | | Lab                                 (21)
          | | | | | | | ●
          | | | | | | |----
21 QED    | | | | | | | Lab                                 10
          | | | | | |----
10 PCh    | | | | | | Lab                                    5
          | | | | |----
5 PCh     | | | | | Lab                                      4
          | | | |----
4 CP      | | | | Pb → Lab                                   3
          | | |----
3 UG      | | | ∀y (Py → Lay)                              2
          | |----
2 CP      | | Pa → ∀y (Py → Lay)                           1
          |----
1 UG      | ∀x (Px → ∀y (Py → Lxy))
```

Note that stages 10 and 11 serve only to move us from ∃y (Py ∧ Lcy) to ∃z (Pz ∧ Lcz)—i.e., to change a bound variable. If sentences that differ only in the choice of a letter for a bound variable are regarded as the same (or if a different variable had been chosen when analyzing the second premise), the assumption Pc ∧ ∃y (Py ∧ Lcy) could be used as a premise for MPP and stages 10-12 would not be needed.

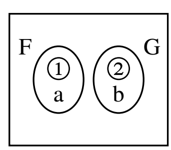
i. ∃x ¬ (∃y: ¬ y = x) Dy ≃ ¬ ∃x (∃y: ¬ y = x) (Dx ∧ Dy)

	∃x ¬ ∃y (¬ y = x ∧ Dy)	2
	∃x ∃y (¬ y = x ∧ (Dx ∧ Dy))	3
	ⓐ ¬ ∃y (¬ y = a ∧ Dy)	7
	ⓑ ∃y (¬ y = b ∧ (Db ∧ Dy))	4
	ⓒ ¬ c = b ∧ (Db ∧ Dc)	5
5 Ext	¬ c = b	(13)
5 Ext	Db ∧ Dc	6
6 Ext	Db	(10)
6 Ext	Dc	(12)
	∀y ¬ (¬ y = a ∧ Dy)	b:9, c:11
9 UI	¬ (¬ b = a ∧ Db)	10
10 MPT	b = a	a—b, c
11 UI	¬ (¬ c = a ∧ Dc)	12
12 MPT	c = a	a—b—c
	●	
13 DC	⊥	8
8 NcP	∃y (¬ y = a ∧ Dy)	7
7 CR	⊥	4
4 PCh	⊥	3
3 PCh	⊥	2
2 PCh	⊥	1
1 RAA	¬ ∃x ∃y (¬ y = x ∧ (Dx ∧ Dy))	

	¬ ∃x ∃y (¬ y = x ∧ (Dx ∧ Dy))	(12)
	∀x ∃y (¬ y = x ∧ Dy)	a:2, b:5
2 UI	∃y (¬ y = a ∧ Dy)	3
	ⓑ ¬ b = a ∧ Db	4
4 Ext	¬ b = a	
4 Ext	Db	(8)
5 UI	∃y (¬ y = b ∧ Dy)	6
	ⓒ ¬ c = b ∧ Dc	7
7 Ext	¬ c = b	(9)
7 Ext	Dc	(8)
8 Adj	Db ∧ Dc	X, (9)
9 Adj	¬ c = b ∧ (Db ∧ Dc)	X, (10)
10 EG	∃y (¬ y = b ∧ (Db ∧ Dy))	X, (11)
11 EG	∃x ∃y (¬ y = x ∧ (Dx ∧ Dy))	X, (12)
	●	
12 Nc	⊥	6
6 PCh	⊥	3
3 PCh	⊥	1
1 NcP	∃x ¬ ∃y (¬ y = x ∧ Dy)	

2. a.

	∃x Fx	1
	∃x Gx	2
	ⓐ Fa	(5)
	ⓑ Gb	(7)
	∀x ¬ (Fx ∧ Gx)	a:4, b:6
4 UI	¬ (Fa ∧ Ga)	5
5 MPT	¬ Ga	
6 UI	¬ (Fb ∧ Gb)	7
7 MPT	¬ Fb	
	○	Fa,¬ Fb,¬ Ga,Gb ⊭ ⊥
	⊥	3
3 NcP	∃x (Fx ∧ Gx)	2
2 PCh	∃x (Fx ∧ Gx)	1
1 PCh	∃x (Fx ∧ Gx)	



b.

	$\exists x\,(Fx \wedge Gx)$	1
	$\exists x\,(Fx \wedge Hx)$	3
	$\forall x\,(Fx \rightarrow \forall y\,(Fy \rightarrow x = y))$	a:3
	———	
	ⓐ $Fa \wedge Ga$	2
	———	
2 Ext	Fa	(6)
2 Ext	Ga	(11)
	ⓑ $Fb \wedge Hb$	4
	———	
4 Ext	Fb	(8)
4 Ext	Hb	(12)
5 UI	$Fa \rightarrow \forall y\,(Fy \rightarrow a = y)$	6
6 MPP	$\forall y\,(Fy \rightarrow a = y)$	b:7
7 UI	$Fb \rightarrow a = b$	8
8 MPP	$a = b$	a—b
	$\forall x\,\neg\,(Gx \wedge Hx)$	a:10
	———	
10 UI	$\neg\,(Ga \wedge Ha)$	11
11 MPT	$\neg\,Ha$	(12)
	●	
	———	
12 Nc=	$\bot$	9
	———	
9 NcP	$\exists x\,(Gx \wedge Hx)$	3
	———	
3 PCh	$\exists x\,(Gx \wedge Hx)$	1
	———	
1 PCh	$\exists x\,(Gx \wedge Hx)$	

3. **a.**

$(\exists x{:}\ Sx)\ Cx$
$(\forall x{:}\ Sx)\ Tx$
$\forall x\ (Cx \rightarrow Px)$

$(\exists x{:}\ Tx)\ Px$

	$\exists x\ (Sx \wedge Cx)$	1
	$\forall x\ (Sx \rightarrow Tx)$	a:3
	$\forall x\ (Cx \rightarrow Px)$	a:5
	ⓐ	
	$Sa \wedge Ca$	2
2 Ext	Sa	(4)
2 Ext	Ca	(6)
3 UI	$Sa \rightarrow Ta$	4
4 MPP	Ta	(7)
5 UI	$Ca \rightarrow Pa$	6
6 MPP	Pa	(7)
7 Adj	$Ta \wedge Pa$	X, (8)
8 EG	$\exists x\ (Tx \wedge Px)$	X, (9)
	●	
9 QED	$\exists x\ (Tx \wedge Px)$	1
1 PCh	$\exists x\ (Tx \wedge Px)$	

b.

$$\frac{\begin{array}{c}(\exists x\text{: } Px \wedge (\exists y\text{: } Sy)\ Oxy)\ Dx \\ (\forall x\text{: } Sx)\ Rx\end{array}}{(\exists x\text{: } Px \wedge (\exists y\text{: } Ry)\ Oxy)\ Dx}$$

	∃x ((Px ∧ ∃y (Sy ∧ Oxy)) ∧ Dx)	1
	∀x (Sx → Rx)	b:6
	ⓐ	
	(Pa ∧ ∃y (Sy ∧ Oay)) ∧ Da	2
2 Ext	Pa ∧ ∃y (Sy ∧ Oay)	3
2 Ext	Da	(11)
3 Ext	Pa	(10)
3 Ext	∃y (Sy ∧ Oay)	3
	ⓑ	
	Sb ∧ Oab	5
5 Ext	Sb	(7)
5 Ext	Oab	(8)
6 UI	Sb → Rb	7
7 MPP	Rb	(8)
8 Adj	Rb ∧ Oab	X, (9)
9 EG	∃y (Ry ∧ Oay)	X, (10)
10 Adj	Pa ∧ ∃y (Ry ∧ Oay)	X, (11)
11 Adj	(Pa ∧ ∃y (Ry ∧ Oay)) ∧ Da	X, (12)
12 EG	∃x ((Px ∧ ∃y (Ry ∧ Oxy)) ∧ Dx)	X, (13)
	●	
13 QED	∃x ((Px ∧ ∃y (Ry ∧ Oxy)) ∧ Dx)	4
4 PCh	∃x ((Px ∧ ∃y (Ry ∧ Oxy)) ∧ Dx)	1
1 PCh	∃x ((Px ∧ ∃y (Ry ∧ Oxy)) ∧ Dx)	

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