7.7. Soundness & completeness

7.7.0. Overview

While the positive use of derivations with generalizations is not too unlike what we had seen in previous chapters, their negative use involves some new ideas, and the change affects arguments for the adequacy of the system of derivations.

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7.7.1. Aspects of adequacy

What we have been asking of our system of derivations is that it always give us the right answer concerning the validity of a conclusion. But it was noted already in 2.3.7 that we would eventually have to retrench and ask only that the system be *complete* (in the sense of giving all correct affirmative answers) and *sound* (in the sense of never giving incorrect affirmative answers). A system that is complete and sound thus tells us that an argument is valid when and only when it is valid. Since completeness implies that we never get an incorrect negative answer, the two properties together also imply that, while we may not get all the right answers, all the answers we get will be right.

These two properties also imply that we can fail to get an answer only when the answer is negative. That sort of asymmetry is possible only if we can be in the position of not knowing whether we will ever receive an answer, for otherwise we could interpret silence as dissent. But that's just the position we are in if the process of developing a gap never ends. If a system of derivations is not decisive, we may not know in advance whether we will eventually get an answer. And, if not, we are in the position of someone waiting for a door or a phone to be answered: one more knock or ring may be enough, but no answer may come no matter how long the wait. Similarly, if we are working on a derivation that gives no answer, all we may ever know is that we have not received an answer yet.

We will look more closely later at why this can happen with derivations. But first, we will see what can be salvaged from the sort of argument that was given for the adequacy of the systems of previous chapters in order to show that our current system is at least sound and complete. In the approach taken in 2.3.4 and extended to the systems of later chapters, we argued for the soundness of a system solely on the basis of the soundness of its rules. If any interpretation that divides the initial gap of a derivation continues to divide some gap at each stage in its development, a derivation whose initial gap is divided by some interpretation can never have all its gaps close. It follows, then, that if all gaps close, the initial entailment holds. This argument can be carried over to the system of derivations for generalizations if we can show that the rules for universal quantifiers are sound. That is not hard to do, but we will need to refine our definition of soundness in order to accommodate rules that introduce new vocabulary into a derivation.

We saw in 2.3.7 how to base an argument for the completeness of the systems of chapters 2-6 on the properties of decisiveness, conservativeness, and sufficiency. It was noted there that, to show completeness, we do not need the full property of decisiveness: we need to know only that we receive an answer about validity whenever the argument is valid. For conservativeness and sufficiency imply that any answer we receive about validity is correct, so, if we always receive some answer when an argument is valid, we can be sure that our system will recognize the validity of any valid argument. In order to show this much of decisiveness, we need to show that, whenever a derivation develops infinitely without producing any dead-end gaps, a negative answer about validity is the correct one. This requires a different sort of argument from that used to show that any dead-end open gap establishes the existence of a counterexample, but the difference is not great, and the new argument will apply also to dead-end gaps.

It will be easier to state the latter argument if we extend the genealogical metaphor we have used to describe the development of derivations. Let us speak of a line of descent from parents to children to grandchildren, etc., as a *path*. A path always begins at the initial stage of the derivation and ends only when the last gap of the path has no children. A path at a given stage may be developed at the next stage by adding a child of its last gap; if there is more than one child it will divide into two or more paths as it develops. We will say that an interpretation *divides* a path when it divides each gap in the path.

If we think of a path as it develops through time, we can imagine a path within which any applicable rule is eventually applied, whether or not that path ever ends. Any way of developing such a path will be used at some point, but there may be no point at which there is no more to be done. So let us say that a path *develops fully* if the path never closes but we do anything that could be done to develop it at some point in its development. Such a path may end with a dead-end gap, but it need not. We can use the safety of rules and ideas from arguments for sufficiency to show that any fully developing gap is divided by an interpretation. And when it is true that any fully developing gap is divided, we will say that a system is *effectual*.

Since every path stems from the initial gap of the derivation, if we are able to divide a fully developing path, we will know that we can divide the initial gap and that a negative answer to the question of validity is the correct one. This means that we will be able to establish completeness for an effectual system if we can show also that any derivation that does not close will have some path that develops fully. Let us say that a system for which this is true is *thorough*.

To recap, we may show that our system is sound by showing that its rules are sound. And we may show that it is complete by showing that it is thorough and effectual. This is summarized in Table 7.7.1-1.

rules are *sound:* they never drop interpretations that divide the initial gap	⊨	system is *sound:* if all gaps close, entailment holds
system is *effectual:* any fully developing path is divided by an interpretation	} ⊨	system is *complete:* if entailment holds, all gaps close
system is *thorough:* development is organized so that either all gaps close or at least one path develops fully		

Table 7.7.1-1. Some logical relations among properties of a system of derivations. (The brace indicates that the second entailment has two premises.)

Although our system of derivations for universals is not decisive, it is sound and complete. And that makes it pretty good, especially since a use of derivations to show validity is more important than its use to show invalidity. But why should a pretty good system be good enough? The answer is that we cannot do any better. There can be systems that answer questions in cases where ours is silent, but there is none that will answer in all cases and never answer incorrectly. This was shown in the mid-1930s by Alonzo Church (the logician who studied lambda abstraction) based on work a few years earlier by Kurt Gödel (who, a little earlier still, was the first to establish the completeness of an account of validity for arguments involving generalizations).

Further, there cannot even be a system that picks up where ours leaves off by giving all correct negative answers and never giving incorrect ones. The argument here is easy once it is shown that no system can be found that is both decisive and accurate: if there were a system that complemented ours, we could make a system that was decisive by using ours and its complement in tandem since, no matter what question we asked, one or the other system would eventually give us an answer.

We will return to these negative considerations in section 7.8, where we will look more closely at the reasons why decisiveness fails. For the rest of this section, we will look in more detail at the virtues our system does have. First we will re-define soundness and consider the soundness of rules for universals. Next we will see what is takes to insure thoroughness. After that we will look at the argument for effectuality.

7.7.2. Soundness

A strict rule, in the sense introduced in 2.3.4, does not throw away gap-dividing interpretations as it develops a gap. That is, any structure dividing a gap to which the rule is applied will divide at least one child gap produced by the rule. In applying this idea to the rules for universals, we are faced with a problem caused by rules that introduce new vocabulary. New vocabulary is introduced always by the planning rule UG, which introduces independent terms, and new vocabulary must be introduced by the exploitation rule UI if generalization would otherwise go unexploited.

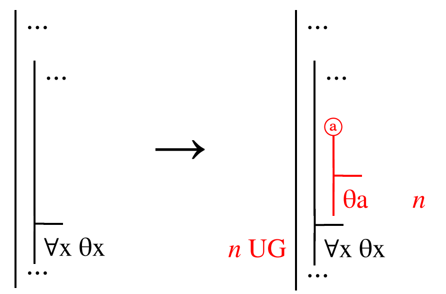
Now, a structure that divides a gap before a rule is applied may fail to divide a gap afterwards simply because it gives no interpretation at all to new vocabulary that the rule introduces. And, even if it does happen to interpret this new vocabulary, the interpretations it gives have played no role in dividing the gap before the vocabulary was introduced, so we may need to revise them as we go on. In short, if an interpretation dividing a gap is to divide any of its children, we may need to provide new interpretations of new vocabulary appearing in that child.

To begin to handle this problem, let us first be more explicit about the conditions under which a structure counts as an interpretation of a gap. In previous chapters we took it for granted that the interpretations we considered interpreted all vocabulary appearing anywhere in the derivation since all such vocabulary appeared in the initial premises and conclusion and we wanted all our interpretations to give truth values to these sentences. Now we need to be more flexible, so let us say that a structure *interprets* a gap if it assigns interpretations to all the non-logical vocabulary that appears in resources or goals of the gap or any of its ancestors. Such a structure must interpret all vocabulary in the initial premises and conclusion of the ultimate argument of the derivation and also interpret all independent terms introduced along the way to the gap in question, but it need not interpret independent terms whose occurrences are boxed off from the gap we are considering. Notice that we allow an interpretation of a gap to provide interpretations of vocabulary not appearing in a gap. This means that any interpretation of gap not only interprets the vocabulary of all its ancestor gaps but in fact counts as an interpretation of those gaps. Among the structures that interpret a gap, we distinguish those that divide it in the same way we have in the past—that is, as the structures that make its active resources true and its goal false.

In order to adapt the definition of soundness to the possibility of changing vocabulary, we can no longer require that, when an interpretation divides a gap, an identical interpretation divides at least one child since we may need to extend or modify the interpretation to accommodate new vocabulary. Let us that say that two interpretations *agree for* a gap when they have the same referential range and give the same interpretation to all vocabulary appearing in the gap and all its ancestors. This idea is motivated by a principle concerning structures that should seem plausible but that we will not argue for: if two structures have the same range and agree on the interpretation of all vocabulary in a sentence, then they each assign the same truth value to that sentence. It follows that if two interpretations agree for a gap, then one will divide the gap if and only if the other does (and this will be true also for all ancestors of the gap).

Given these ideas, we will redefine strictness and say that a rule is *strict* when, for any interpretation dividing a gap before the rule is applied, we can find an interpretation that agrees with the given interpretation for that gap and that divides at least one child gap resulting from the rule. According to this definition, an strict rule need not preserve gap-dividing structures unchanged; it must preserve what was essential to the function of such a structure in dividing a parent gap, but it may force it to be elaborated or altered in order to interpret a gap resulting from the rule. We will say that a rule is *sound* when it preserves (in this way) interpretations that divide both the gap to which the rule is applied and all of its ancestors. Equivalently, a rule is sound when for any interpretation that divides a path before the rule is applied to its final gap, we can find an interpretation that agrees with given interpretation on this path and that divides at least one path that results from applying the rule.

The rules UG and UI are strict in the new sense. The actual arguments showing this are not very surprising, and we will look only only the case of UG.



Suppose S is a structure dividing the gap on the left. Since S makes the goal ∀x θx false, it must assign θ an extension that does not include the whole referential range. Let S′ be like S except in assigning to the independent term a some value outside the extension of θ. Then S′ will agree with S for the gap at the left (since a does not appear before UG is applied), and it will make θa false. So S′ (like S) will make all active resources of the two gaps true, and it will make the new goal false (whether or not S does). So, given a structure S dividing the old gap, the essentials of the way it does so are preserved in a structure S′ dividing the new one; and that means that UG is strict.

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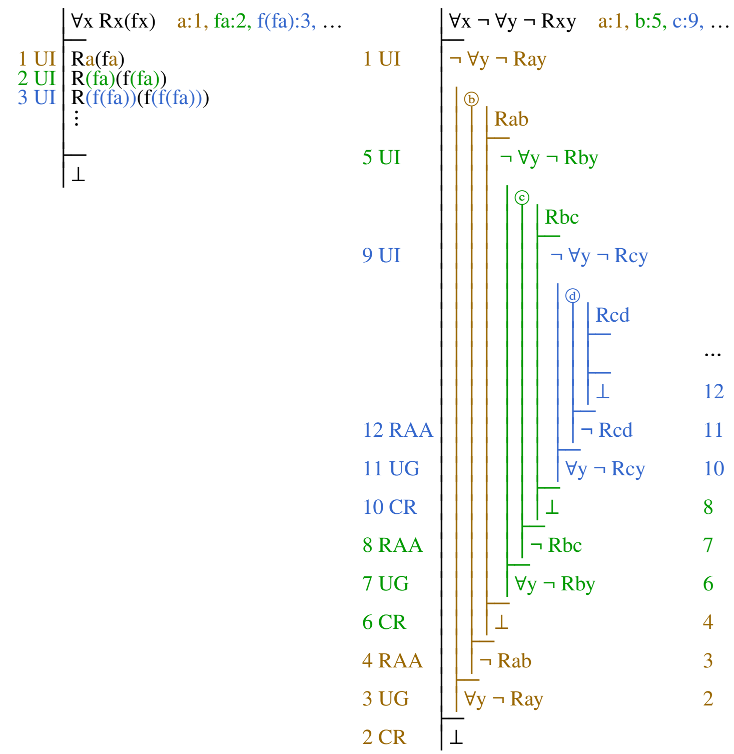
7.7.3. Thoroughness

We are going to be talking quite a bit about infinitely developing gaps, so it would be good to look at one. Two examples are shown below. The first illustrates a possibility mentioned in 7.5.4: the exploitation of a universal may introduce a term that immediately provides an opportunity for applying UI again. The second derivation shows that something similar can happen even without functors. Notice that the pattern of stages 1-4 in the second is repeated in stages 5-8 with the terms b and c playing the roles originally created by a and b, respectively; and this pattern is repeated again in 9-12 with the roles played by c and d. Clearly this process could continue forever.



We will see in 7.8.1 that a change in our rules would enable us to develop the initial gap in these derivations in a way that would produce dead-end open gaps, but we will see also (in 7.8.2) that in a derivation with certain additional premises such gaps would close, leaving only the ones shown here. So these gaps are good examples of the problem we face.

But they are also examples of the solution to this problem. For, if they are continued into infinity in the same way, anything that *can* be done to develop them *will* be done at some stage in their development—each resource that can be exploited will be exploited (and, in the case of universals, as often as possible using terms appearing in the gap) and each goal will be planned for. So these illustrate the sort of paths we describe as fully developing. Our aim is to show that our system is thorough, that any derivation will either close or generate a fully developing path, one which will either end in a dead-end open gap or continue as an infinitely developing path like the ones shown here.

In order to have a thorough system, we must rule out the possibility that a gap is developed infinitely without all possible rules being applied. For example, if either derivation above had A ∧ ¬ A as a second premise, it could be closed—but only if we got around to exploiting this resource and did not ignore the possibility of closing the gap as we exploited other resources. Nothing in the way our rules are stated prevents such oversights, so our system is not thorough as it stands. What we need is a way of organizing the application of the rules that will insure that we eventually apply every rule that we can.

Let us say that a sentence is *exploitable in a gap* when there is some exploitation rule for it that may be applied to develop that gap. To be exploitable, a sentence must first of all be among the active resources. But a universal resource may be active without being exploitable. This will happen when there are closed terms appearing in the gap and the universal is inactive for all of them. Other sorts of active resources may fail to be exploitable, either permanently or temporarily; examples are atomic sentences or sentences that can be exploited only in *reductio* arguments. Our aim is to manage the development of gaps so that no exploitable resource is left unexploited.

The only reason there is any difficulty here is that exploiting a universal can open up new ways of exploiting it and other universals, which in turn open up new ways of exploiting universals, with the result that we are never forced to turn our attention to other tasks. Accordingly, we will manage the development of a derivation by setting an arbitrary limit on the exploitation of universals and gradually relax it as a path develops. We begin by doing all we can to develop a derivation except that we exploit universal resources only for terms in our initial premises and conclusion. Then we take all the terms that have appeared in a path in the course of this development and add them as admissible terms. Again we do all we can to develop each path of a derivation using the enlarged group of terms admissible for it, and so on. Let us call each round of development before enlarging the group of admissible terms, a *cycle*. Although a universal may not be exploited for each term in the gap at the completion of any given cycle, it will be exploited for all such terms during the next cycle. And the limit on the terms that may be used to exploit universals in a given cycle insures that the current cycle will not continue forever.

Now, if we survey the full development of a derivation, which may proceed to infinity, we have three possibilities: (i) all gaps close, (ii) at some point we find an open gap that cannot be either closed or developed further, or (iii) there is a path that is developed unceasingly. For, if the first two possibilities are not realized, we know that at each stage some gap is open and can be developed further; and it can be shown to follow that there is some path that is open at all stages. To know that our system is thorough, we must know that we have exploited resources and planned for goals as often as possible in cases (ii) and (iii) and have had no opportunity to close the gap. In case (ii) this is obvious, for otherwise the gap could have been developed further or closed. And the procedure above insures that there is a full application of the rules also in case (iii).

The way of organizing the application of rules that has been used here to establish thoroughness is not intended for actual use. It has been stated in a way that makes its effects are easy to see, but this does not make it easy to apply or make it an efficient way of completing derivations that do not go on forever. In practice, we will instead simply aim at the fullest planning for goals and the fullest exploitation of the broadest range of resources. So the system as we use it will be thorough, if it is thorough, not simply as a result of the rules governing its use but in part because of the way we in fact use it; that is, it will be thorough to the extent that we are thorough in our use of it.

7.7.4. Effectuality

All that remains in our argument for completeness is to show that any fully developing path is divided by an interpretation. This is in some ways like an argument that could be made for systems of earlier chapters. For them, it can be shown that any dead-end open gap is divided by an interpretation that also divides all ancestors of the gap. But, in the system we are looking at now, while a fully developing path might end with a dead-end gap, it might instead develop infinitely; and a path that develops infinitely is quite a different beast from a gap that has reached a dead end. The differences between the two will affect the way we argue for the existence of structures dividing them.

We need some new ways of talking about resources and goals. The *accumulated resources* of a path include all sentences that ever appear as active resources in the course of its development. Its *ultimate resources* are the accumulated resources that are not exploited (not even partially) at any stage in its development. In a fully developing path, the ultimate resources will consist solely of atomic sentences and negated atomic sentences. The *accumulated goals* of a path are all the sentences that ever appear as goals in the course of its development. In a fully developing gap, any such goal, apart from ⊥, will eventually be planned for. Since a structure divides a path if and only if it divides all gaps in the path, a path-dividing structure makes all of the accumulated resources of a path true and all of the accumulated goals false.

There are two parts to the argument that any fully developing path is divided. One involves considerations used to establish sufficiency in the old sense, and the other involves considerations related to the safety of rules. Specifically, we will show first that, given any fully developing path, we can find some structure that (i) makes the ultimate resources all true and (ii) assigns each value in the referential range of the structure to some term appearing in the ultimate resources. Secondly, we will show that such a structure divides the path. The first of these arguments really involves nothing we did not see already in 6.4.3. The concrete calculations we carried out there may no longer be possible since we may be dealing with infinitely many terms, but the definitions continue to apply and the arguments are essentially unchanged. However, we must make one stipulation that was left open there: each value of the referential range we set up must correspond to one of the alias sets derived from the ultimate resources. This handles our requirement (ii) that the structure assign each value in its range to some term—or, more briefly, that it associate a name with each value in the range.

There is also little that is new in the second part of the argument, although the form is different. Instead of arguing to the truth values a structure assigns at one stage from those it assigns at the next one, we argue to the truth values it assigns to a sentence from the truth values it assigns to the components (or instances) of the sentence. Since the chief difference between the resources and goal of one stage and those of the next lies in the introduction of components or instances at the new stage to replace or add to compounds that appear at the old one, the arguments both end up concerning the semantic relations between compounds and components, and we will not look at the new argument in much detail.

Why then do we need a new argument at all? One reason lies in the form. Suppose we have a structure making the ultimate resources of a path all true. We need to show that it divides the path. The old way was to begin with the final stage of the gap and work our way back stage by stage, with each step of this argument using the safety of the rules. The new way is to begin with the ultimate resources and work our way up to more and more complex sentences. The considerations will be much the same at each step. We have changed only the overall form of the argument, and we have changed it only because we have to: we have ultimate components to start from but there may not be a final stage to the path.

There is one exception to the analogy between the two forms of argument, and it concerns the only part of the new argument we will consider. A universal resource is not exploited once and for all at a single stage in the development of a path, so the relation between a universal and its instances is not replicated by a transition from one stage of development to the next. So suppose we are arguing in the new way; that is, we have a structure making the ultimate resources of a path true and we are moving step by step from components (or instances) to compounds in order to show that this structure divides the path. How do we know that we can make the step we need to in the case of a universal $\forall x\ \theta x$ appearing among the accumulated resources?

Let us collect what we know (setting aside for the moment the possibility of non-trivial alias sets—i.e., ones that contain more than one term). Since the path is fully developing, the universal has been exploited for each term τ appearing in the gap. And this means that each instance $\theta\tau$ for such a term will appear among the accumulated resources. Moreover, in our step-by-step climb to more and more complex sentences, we will have already shown that the structure makes each of the instances $\theta\tau$ true. Now the structure assigns each value in its range to some term τ. So, since the structure makes every instance $\theta\tau$ true, it must assign θ an extension that includes the whole of the referential range, and that means the structure will make $\forall x\ \theta x$ true.

Now, notice that, for the structure to make $\forall x\ \theta x$ true, it is really only necessary that it make true an instance θτ for at least one term τ from each alias set, and that means that a fully developing gap need have only this many instances among is accumulated resources. Although it has been convenient for the purposes of these general arguments to think of fully developing gaps as exploiting universals for all terms appearing in them, this is not necessary to insure that the gap is divisible, and there is no need to render universals inactive for every term when constructing actual derivations.

It *is* crucial for this argument that the referential range of the structure dividing the gap contain no reference values beyond those used as the extensions of terms. That is why we limit the range to values that correspond to alias sets. And the reason for this is not at all mysterious. We can now state logical forms that are true only in ranges of limited size. To take an extreme case, the sentence ∀x ∀y x = y (i.e., Everything is identical to everything) is true if and only if the referential range has just one member. If this sentence is among the resources of a gap, the gap can be divided only by a structure whose range has a population of 1.

This need to limit the referential range of a structure makes it harder to duplicate structures by intensional interpretations and possible worlds. Clearly, we cannot always choose the actual world if the range of reference values must be severely limited, and it may not be clear what the extensions of ordinary English vocabulary are like in possible worlds that have very limited ranges. So it is hard to tell whether the constraints that we now face undermine the argument from the existence of a dividing structure to the failure of formal validity. If they do undermine that argument, we could redefine entailment so that we speak not simply of all possible worlds but of all worlds and all ways of choosing a referential range from each world. The device mentioned in 6.4.3 of regarding structures as partial accounts of a possible world would then be usable in accounts of entailment for arguments involving generalizations.

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7.7.s. Summary

1 Our system of derivations generalizations does not answer all questions concerning the validity of arguments; indeed, it has been shown that no system can answer all such questions (if its answers are all correct). However, our system is sound and complete. That is, it declares valid only arguments that are valid, and it does affirm the validity of all valid arguments. These properties make up more than half of what we might like a system to do: a sound and complete system always gives a correct answer concerning valid arguments and never gives an incorrect answer concerning arguments that are not valid (though it may give no answer at all in the case of such arguments). We can still establish the soundness of our current system much as before, and we can establish completeness by showing (i) that any derivation that does not close will contain a path that is fully developing (in the sense that every way of developing it is employed at some point) and (ii) that any fully developing path is divided by some interpretation. To show (i) is to show that a system is thorough, and to show (ii) is to show that it is effectual.

2 We must refine our notion of interpretation to recognize the possibility that the non-logical vocabulary of a derivation may increase as it develops, and we need to modify the definition of soundness, too. The rules for universals may introduce terms, and a structure dividing a gap to which these rules are applied may assign inconvenient values, or no values at all, to these terms. So, when stating conditions for soundness, we will ask only that we be able to find a structure dividing a child gap that agrees with the old structure on the vocabulary that appears before the rule was applied. This new approach to defining of strict and sound rules still implies the soundness of our system.

3 A derivation may develop forever due to continual input of new terms for which universals are exploitable. To establish thoroughness, we must insure that all approaches to closing the gap are explored in the course of this development. We can do this by imposing an order of procedure that rations the terms used to instantiate over the course of time, requiring a full cycle in the application of other rules before new terms are used in UI. While this restriction insures thoroughness, it makes more sense in practice simply to take on the responsibility for being thorough.

4 Infinite derivations are not static structures but growing lines of development. This leads to changes in the way we argue for the existence of structures dividing paths that never close off. We collect the active resources and

goals that appear in the course of a gap's development as accumulated resources and accumulated goals distinguishing as ultimate those resources that are never exploited. When a gap is fully developing, its ultimate resources are limited to atomic sentences and their negations. We can show that any fully developing gap leads us to a structure that makes its accumulated resources true and its accumulated goals false. Although there are thus enough structures to meet our needs, some of the flexibility we have had in choosing structures is now gone: we can no longer expect to add values freely to the range of a structure since some sentences are true only when the referential range has a limited size.

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7.7.x. Exercise questions

Use derivations to check each of the claims below; if a derivation indicates that a claim fails, describe a structure that divides an open gap.

1. Fa ⊨ ∀x Fx

2. ∀x Rxx ⊨ ∀x Rxa

3. ∀x ¬ Fx ≃ ¬ ∀x Fx

4. No widget is a gadget
No gizmo is a widget
―――――――――――――
No gizmo is a gadget

5. No widget is a gadget ≃ Not every widget is a gadget

6. Everything is either finished or unstarted
―――――――――――――
Either everything is finished or everything is unstarted

7. ¬ ∀x ¬ ∀y Rxy ⊨ ∀x ¬ ∀y ¬ Rxy

For more exercises, use the exercise machine.

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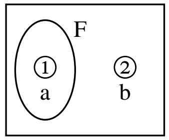
7.7.xa. Exercise answers

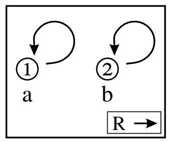
1.

	Fa	
	ⓑ	
	¬ Fb	
	○	Fa, ¬ Fb ⊭ ⊥
	⊥	2
2 IP	Fb	1
1 UG	∀x Fx	



2.

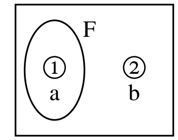
	∀x Rxx	a:1, b:3
1 UI	Raa	
	ⓑ	
3 UI	Rbb	
	¬ Rba	
	○	Raa, Rbb, ¬ Rba ⊭ ⊥
	⊥	4
4 IP	Rba	2
2 UG	∀x Rxa	



3.

	∀x ¬ Fx	a:3
	∀x Fx	a:2
2 UI	Fa	(4)
3 UI	¬ Fa	(4)
	●	
4 Nc	⊥	1
1 RAA	¬ ∀x Fx	

	¬ ∀x Fx	
	ⓐ	
	Fa	
	ⓑ	
	¬ Fb	
	○	Fa, ¬ Fb ⊭ ⊥
	⊥	5
5 IP	Fb	4
4 UG	∀x Fx	3
3 CR	⊥	2
2 RAA	¬ Fa	1
1 UG	∀x ¬ Fx	

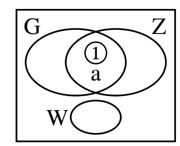


4.

(∀x: Wx) ¬ Gx
(∀x: Zx) ¬ Wx

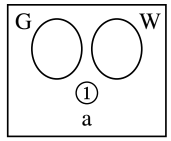
(∀x: Zx) ¬ Gx

	∀x (Wx → ¬ Gx)	a:6
	∀x (Zx → ¬ Wx)	a:3
	ⓐ	
	Za	(4)
3 UI	Za → ¬ Wa	4
4 MPP	¬ Wa	
	Ga	(7)
6 UI	Wa → ¬ Ga	7
7 MTT	¬ Wa	
	○	Za, ¬ Wa, Ga ⊭ ⊥
	⊥	5
5 RAA	¬ Ga	2
2 CP	Za → ¬ Ga	1
1 UG	∀x (Zx → ¬ Gx)	

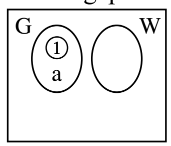


5. (∀x: Wx) ¬ Gx ≃ ¬ (∀x: Wx) Gx

	∀x (Wx → ¬ Gx)	a:2
	∀x (Wx → Gx)	a:3
2 UI	Wa → ¬ Ga	4
3 UI	Wa → Ga	6, 8
	¬ Wa	
	¬ Wa	
	○	¬ Wa ⊭ ⊥
	⊥	7
7 IP	Wa	6
	Ga	
	○	¬ Wa,Ga ⊭ ⊥
	⊥	6
6 RC	⊥	5
5 IP	Wa	4
	¬ Ga	(8)
8 MTT	¬ Wa	
	○	¬ Wa,¬ Ga ⊭ ⊥
	⊥	4
4 RC	⊥	1
1 RAA	¬ ∀x (Wx → Gx)	

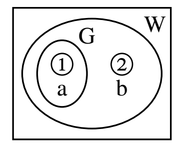


divides the 1st and 3rd gaps



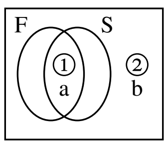
divides the 1st and 2nd gaps

	¬ ∀x (Wx → Gx)	4
	ⓐ	
	Wa	
	Ga	
	ⓑ	
	Wb	
	¬ Gb	
	○	Wa, Ga, Wb, ¬ Gb ⊭ ⊥
	⊥	7
7 IP	Gb	6
6 CP	Wb → Gb	5
5 UG	∀x (Wx → Gx)	4
4 CR	⊥	3
3 RAA	¬ Ga	2
2 CP	Wa → ¬ Ga	1
1 UG	∀x (Wx → ¬ Gx)	



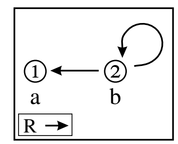
6.

	∀x (Fx ∨ ¬ Sx)	a:4, b:9
	¬ ∀x Fx	6
	ⓐ	
	Sa	(5)
4 UI	Fa ∨ ¬ Sa	5
5 MTP	Fa	
	ⓑ	
	¬ Fb	(10)
9 UI	Fb ∨ ¬ Sb	10
10 MTP	¬ Sb	
	○	Sa,Fa,¬ Fb,¬ Sb ⊭ ⊥
	⊥	8
8 IP	Fb	7
7 UG	∀x Fx	6
6 CR	⊥	3
3 RAA	¬ Sa	2
2 UG	∀y ¬ Sy	1
1 PE	∀x Fx ∨ ∀y ¬ Sy	



7.

	¬ ∀x ¬ ∀y Rxy	4
	ⓐ	
	∀y ¬ Ray	a:3, b:6
3 UI	¬ Raa	
	ⓑ	
6 UI	¬ Rab	
	∀y Rby	a:8, b:9
8 UI	Rba	
9 UI	Rbb	
	○	¬ Raa,¬ Rab,Rba,Rbb ⊭ ⊥
	⊥	7
7 RAA	¬ ∀y Rby	5
5 UG	∀x ¬ ∀y Rxy	4
4 CR	⊥	2
2 RAA	¬ ∀y ¬ Ray	1
1 UG	∀x ¬ ∀y ¬ Rxy	



Glen Helman 05 Nov 2011