5.4. Extreme measures

5.4.0. Overview

There are two further rules for the conditional that reflect its truth table in very direct ways.

5.4.1. Last resorts

We do not always have the opportunity to exploit a conditional by detachment, so we need means to exploit one in a *reductio*.

5.4.2. Optional extras

The principle of weakening for the conditional provides the basis for an attachment rule that is occasionally useful.

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5.4.1. Last resorts

The detachment rules for the conditional—and especially MPP—will be the ways of exploiting conditional resources that you will use the most. However, they cannot cover all cases because both require the presence of a second premise as an available resource. So we need a fully general way of taking account of conditional resources.

Since any open gap will eventually turn into a *reductio* argument, it is enough that we have a way of exploiting conditionals in such arguments. An entailment

$$\Gamma, \varphi \to \psi \vDash \bot$$

says that $\varphi \to \psi$ is inconsistent with Γ, and that will be so if and only if $\varphi \to \psi$ is false in every possible world in which all members of Γ are true. But the conditional $\varphi \to \psi$ is false only when ψ is false while φ is true. So the displayed entailment says that in any world in which all members of Γ are true, we will find φ true and ψ false—and that is to say both that φ is entailed by Γ and that ψ is inconsistent with it. This way of describing the requirements for the validity of a *reductio* with a conditional premise provides our account of the role of conditionals as premises:

LAW FOR THE CONDITIONAL AS A PREMISE. $\Gamma, \varphi \to \psi \vDash \bot$ if and only if both $\Gamma \vDash \varphi$ and $\Gamma, \psi \vDash \bot$.

In other words, a conditional $\varphi \to \psi$ is excluded by a set Γ if and only if its antecedent φ is entailed by Γ and its consequent ψ is excluded by Γ.

In terms of the metaphor of inference tickets, this law says that we can get to an absurd conclusion given Γ and the ticket $\varphi \to \psi$ if and only if Γ will get us to φ, the point of departure on our ticket, and then from its destination, ψ, on to the absurd conclusion. The "if" part of this holds also for conclusions that are not absurd, but the "only if" part does not. In particular, the fact that $\Gamma, \varphi \to \psi \vDash \chi$ does not insure that $\Gamma \vDash \varphi$ when χ is not absurd: we may be able to get to χ given Γ and the ticket $\varphi \to \psi$ without being able to get there via φ.

We will call the rule based on this principle, *Rejecting a Conditional* (RC). It is shown in Figure 5.4.1-2.

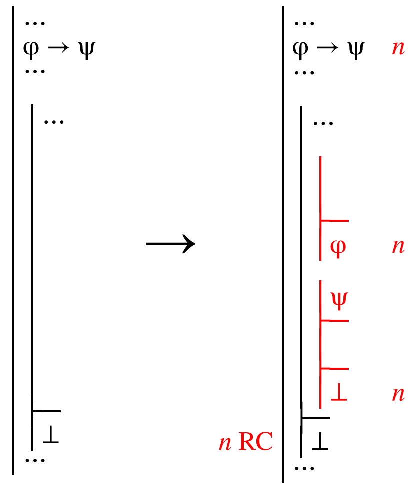


Fig. 5.4.1-2. Developing a *reductio* derivation at stage *n* by exploiting a conditional.

When we apply RC, we divide the gap into two, with the aim of showing that the antecedent of the conditional is entailed by our other resources and that its consequent is inconsistent with them. This is what is required to show that the conditional itself is inconsistent with our other resources, which is why we say that our aim is to *reject* the conditional. While this way of thinking about the rule is the most appropriate one given its place in the system of derivations, RC can also be thought of as a way of planning to use an inference ticket $\varphi \to \psi$ by planning to reach the point of departure φ and planning to get from the destination ψ to the goal $\bot$, and this perspective is the one that is most clearly displayed in the corresponding rule in tree form proofs:

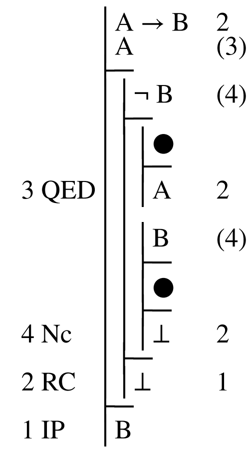
$$\text{RC}\ \frac{\varphi \to \psi \qquad \varphi \qquad \overset{\not{\psi}}{\bot}}{\bot}$$

In this setting RC might be thought of as an abbreviation for the following combination of LFR and MPP:

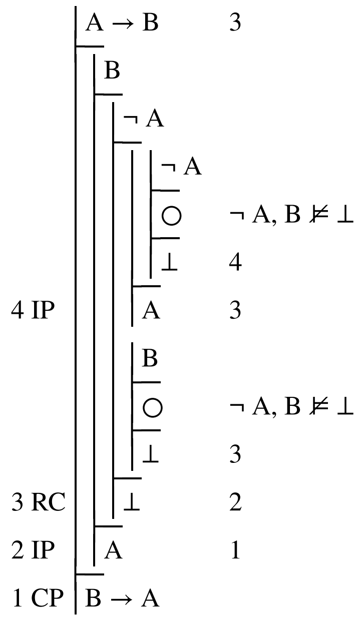
$$\text{LFR}\ \frac{\text{MPP}\ \dfrac{\varphi \to \psi \qquad \varphi}{\psi} \qquad \overset{\not{\psi}}{\bot}}{\bot}$$

There are three conclusions—$\varphi \to \psi$, φ, and $\bot$—that must be reached before going on in the way shown by this tree. In a derivation, on the other hand, we have already shown $\varphi \to \psi$ when we apply the rule. So we seek to complete only two arguments. The ticket $\varphi \to \psi$ serves to convert the proof of φ sought in the first of these arguments into a proof of ψ, the extra supposition used in the second, so that supposition may be discharged when we apply the rule.

Although MPP and MTT are more central to the deductive inference for the conditional than are MTP and MPT to inferences involving disjunction, negation, and conjunction, all detachment rules are dispensable. One role of RC is to exploit conditionals when detachment rules are not used, and one of the simplest example of its use is the following derivation which establishes the validity of *modus ponens* without use of MPP or MTT:



A more typical use of RC is a case we never have the second premise required in order to apply MPP or MTT, as in the following derivation, which shows that the conditional in not reversible:



A	B	A → B / B → A
F	T	Ⓣ Ⓕ

And, as is the case in this example, RC will serve us as a last resort for exploiting conditional resources before reaching a dead end in a derivation that fails.

5.4.2. Optional extras

The law for the conditional as a premise directly reflects the conditions under which a conditional is false. The two weakening principles for the conditional that were noted in 5.3.2 directly reflect the two cases under which a conditional is true—when its consequent is true and when its antecedent is false.

$$\psi \vDash \varphi \to \psi$$
$$\neg^{\pm} \varphi \vDash \varphi \to \psi$$

However, while the rule CR implementing the law for the conditional as a premise is vital if our set of rules is sufficient, the rule that implements these weakening principles is optional. Of course, that is true for all attachment rules, but this is probably the least important of them.

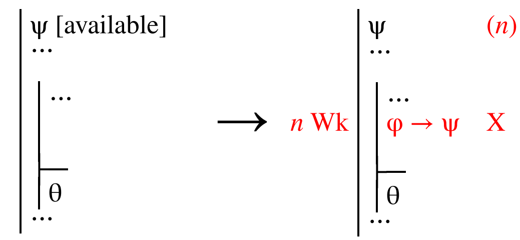


Fig. 5.4.2-1. Developing a derivation at stage *n* by adding an inactive conditional whose consequent is available.

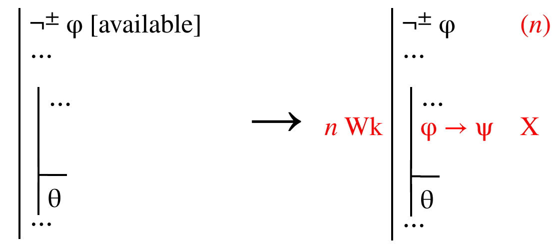


Fig. 5.4.2-2. Developing a derivation at stage *n* by adding an inactive conditional whose antecedent is negated or de-negated by an available resource.

Much of the value of attachment rules lies in their use to assemble the auxiliary resource required to apply detachment rules. And, in natural arguments, the auxiliary resources of detachment rules are less often conditionals than the other forms of sentence we can conclude by attachment rules. So we must look elsewhere for natural examples of the use of weakening for the conditional. As one example, consider the entailment ¬ A ∨ B ⊨ A → B. This can be established quickly by the use of CP and MTP, but if, instead, the disjunction is exploited to plan for a proof by cases, Wk for the conditional provides the most natural way to complete the case arguments.

	$\neg A \lor B$	1
	│ $\neg A$	(2)
2 Wk	│ $A \to B$	X, (3)
	│ ●	
3 QED	│ $A \to B$	1
	│ B	(4)
4 Wk	│ $A \to B$	X, (5)
	│ ●	
5 QED	│ $A \to B$	1
1 PC	$A \to B$	

A derivation showing that $\neg (A \to B) \vDash A \land \neg B$ would provide a similar example of the use of these rules.

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5.4.s. Summary

1 The law for the conditional as a premise applies only to *reductio* arguments and provides a way of rejecting a conditional by deriving its antecedent φ from the premises and reducing its consequent to absurdity given the premises. Rejecting a Conditional (RC) is the corresponding derivation rule.

2 This rule reflects the fact that a conditional is false when its antecedent is true and its consequent is false. The rules of Weakening (Wk) that have conditionals as conclusions reflect the fact that a conditional is true if its consequent is and also if its antecedent is false.

With these rules, the system of derivations for truth-functional logic is complete. It is shown in the table below.

Basic system

Rules for developing gaps		
logical form	*as a resource*	*as a goal*
atomic sentence		IP
negation $\neg \varphi$	CR (if φ is not atomic and the goal is ⊥)	RAA
conjunction $\varphi \land \psi$	Ext	Cnj
disjunction $\varphi \lor \psi$	PC	PE
conditional $\varphi \to \psi$	RC (if the goal is ⊥)	CP

Rules for closing gaps	
when to close	*rule*
the goal is also a resource	QED
sentences φ and ¬ φ are resources & the goal is ⊥	Nc
⊤ is the goal	ENV
⊥ is a resource	EFQ

Detachment rules (optional)		
main resource	*auxiliary resource*	*rule*
$\varphi \to \psi$	φ	MPP
$\varphi \to \psi$	$\neg^{\pm} \psi$	MTT
$\varphi \lor \psi$	$\neg^{\pm} \varphi$ or $\neg^{\pm} \psi$	MTP
$\neg (\varphi \land \psi)$	φ or ψ	MPT

Added rules (optional)

Attachment rules	
added resource	*rule*
$\varphi \land \psi$	Adj
$\varphi \to \psi$	Wk
$\varphi \lor \psi$	Wk
$\neg (\varphi \land \psi)$	Wk

Rule for lemmas	
prerequisite	*rule*
the goal is ⊥	LFR

At the top and left appears the basic system, all of whose rules are progressive. It consists of the fundamental rules for developing gaps by exploiting resources or planning for goals, two rules each for negations, conjunctions, disjunctions, and conditionals along with a rule to plan for atomic sentences. There are the same four rules for closing gaps we had as of 3.2, and we now also have a set of four detachment rules that provide alternative ways of exploiting weak truth-functional compounds. In addition to the basic system, there is a group of rules that are not necessarily progressive although they are sound and safe. These are the rules makred off at the lower right in the table—the attachment rules and the general rule LFR for introducing lemmas in *reductio* arguments. As in the earlier tables of this form, the names of the rules in the following are links to places where they are actually stated.

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5.4.x. Exercise questions

1. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap. Since **d** is a claim of tautologousness, it is established by a derivation that begins with only a goal and no initial premises.

a. $A \to B \simeq \neg A \lor B$

b. $(A \land B) \to C \simeq A \to C$

c. $(A \to B) \land (B \to C) \simeq A \to C$

d. $\vDash ((A \to B) \to A) \to A$

2. Construct derivations for each of the following. These exercises are designed to make attachment rules often useful. The derivations can be constructed for the English sentences in **e-g** without first analyzing them since you generally need to recognize only the main connective and the immediate connectives in order to know what rules apply; however, the abbreviated notation provided by an analysis may be more convenient.

a. $(A \land B) \to C, (C \lor D) \to E, A, B \vDash E$

b. $(A \lor \neg B) \to C \vDash \neg C \to B$

c. $\neg (A \land B), B \lor C, D \to \neg C \vDash A \to \neg D$

d. $C \to \neg (A \lor B), E \lor \neg (D \land \neg C), D \vDash A \to E$

e. Tom will go through Chicago and visit Sue
Tom won't go through both Chicago and Indianapolis
Tom won't visit Ursula without going through Indianapolis

Tom will visit Sue but not Ursula

f. Either we spend a bundle on television or we won't have wide public exposure
If we spend a bundle on television, we'll go into debt
Either we have wide public exposure or our contributions will dry up
We'll go into debt if our contributions dry up and we don't have large reserves
We won't have large reserves

We'll go into debt

g. If Adams supports the plan, it will go though provided Brown doesn't oppose it
Brown won't oppose the plan if either Collins or Davis supports it

---

The plan will go through if both Adams and Davis support it

For more exercises, use the exercise machine.

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5.4.xa. Exercise answers

1. a.

	A → B	2
	│ A	(2)
2 MPP	│ B	(3)
	│ ●	
3 QED	│ B	1
1 PE	¬ A ∨ B	

	¬ A ∨ B	2
	│ A	(2)
2 MTP	│ B	(3)
	│ ●	
3 QED	│ B	1
1 CP	A → B	

b.

	(A ∧ B) → C	3
	│ A	(4)
	│ │ ¬ C	(3)
3 MTT	│ │ ¬ (A ∧ B)	4
4 MPT	│ │ ¬ B	
	│ │ ○	A, ¬ C, ¬ B ⊭ ⊥
	│ │ ⊥	2
2 IP	│ C	1
1 CP	A → C	

	A → C	3
	│ A ∧ B	2
2 Ext	│ A	(3)
2 Ext	│ B	
3 MPP	│ C	(4)
	│ ●	
4 QED	│ C	1
1 CP	(A ∧ B) → C	

A	B	C	(A ∧ B) → C / A → C
T	F	F	F Ⓣ Ⓕ

c.

	A → C	3,7
	│ │ A	(3)
3 MPP	│ │ C	
	│ │ │ ¬ B	
	│ │ │ ○	A, C, ¬ B ⊭ ⊥
	│ │ │ ⊥	4
4 IP	│ │ B	2
2 CP	│ A → B	1
	│ │ B	
	│ │ │ ¬ C	(7)
7 MTT	│ │ │ ¬ A	
	│ │ │ ○	B, ¬ C, ¬ A ⊭ ⊥
	│ │ │ ⊥	6
6 IP	│ │ C	5
5 CP	│ B → C	1
1 Cnj	(A → B) ∧ (B → C)	

	(A → B) ∧ (B → C)	1
1 Ext	A → B	3
1 Ext	B → C	4
	│ A	(3)
3 MPP	│ B	(4)
4 MPP	│ C	(5)
	│ ●	
5 QED	│ C	2
2 CP	A → C	

A	B	C	A → C / (A → B) ∧ (B → C)
T	F	T	Ⓣ F Ⓕ T
F	T	F	Ⓣ T Ⓕ F

The two rows divide the first and second gap, respectively.

d. The following are two approaches to this derivation, one without use of attachment rules and the other using one of the forms of Wk for the conditional.

```
        | | (A → B) → A          3
        | |---
        | | | ¬ A                (3),(7)
        | | |---
3 MTT   | | | ¬ (A → B)
        | | | | | A              (7)
        | | | | |---
        | | | | | | ¬ B
        | | | | | |---
        | | | | | | ●
        | | | | | |---
7 Nc    | | | | | | ⊥          6
        | | | | |---
6 IP    | | | | | B            5
        | | | |---
5 CP    | | | | A → B          4
        | | |---
4 CR    | | | ⊥                2
        | |---
2 IP    | | A                  1
        |---
1 CP    | ((A → B) → A) → A
```

```
        | | (A → B) → A          4
        | |---
        | | | ¬ A                (3),(5)
        | | |---
3 Wk    | | | A → B              X,(4)
4 MPP   | | | A                  (5)
        | | | ●
        | | |---
5 Nc    | | | ⊥                  2
        | |---
2 IP    | | A                    1
        |---
1 CP    | ((A → B) → A) → A
```

2. a.

```
          | (A ∧ B) → C     2
          | (C ∨ D) → E     4
          | A               (1)
          | B               (1)
          |---
1 Adj     | A ∧ B           X,(2)
2 MPP     | C               (3)
3 Wk      | C ∨ D           X,(4)
4 MPP     | E               (5)
          | ●
          |---
5 QED     | E
```

b.

```
          | (A ∨ ¬ B) → C        2
          |---
          | | ¬ C                (2)
          | |---
2 MTT     | | ¬ (A ∨ ¬ B)        (5)
          | | | ¬ B              (4)
          | | |---
4 Wk      | | | A ∨ ¬ B          X,(5)
          | | | ●
          | | |---
5 Nc      | | | ⊥                3
          | |---
3 IP      | | B                  1
          |---
1 CP      | ¬ C → B
```

c.

```
          | ¬ (A ∧ B)     2
          | B ∨ C         3
          | D → ¬ C
          |---
          | | A           (2)
          | |---
2 MPT     | | ¬ B         (3)
3 MTP     | | C           (4)
4 MTT     | | ¬ D         (5)
          | | ●
          | |---
5 QED     | | ¬ D         1
          |---
1 CP      | A → ¬ D
```

d.

```
          | C → ¬ (A ∨ B)         3
          | E ∨ ¬ (D ∧ ¬ C)       5
          | D                     (4)
          |---
          | | A                   (2)
          | |---
2 Wk      | | A ∨ B               X,(3)
3 MTT     | | ¬ C                 (4)
4 Adj     | | D ∧ ¬ C             X,(5)
5 MTP     | | E                   (6)
          | | ●
          | |---
6 QED     | | E                   1
          |---
1 CP      | A → E
```

e.

```
          | Tom will go through Chicago and visit Sue                    1
          | Tom won't go through both Chicago and Indianapolis           2
          | Tom won't visit Ursula without going through Indianapolis    3
          |---
1 Ext     | Tom will go through Chicago                                  (2)
1 Ext     | Tom will visit Sue                                           (4)
2 MPT     | Tom won't go through Indianapolis                            (3)
3 MPT     | Tom won't visit Ursula                                       (4)
4 Adj     | Tom will visit Sue but not Ursula                            X,(5)
          | ●
          |---
5 QED     | Tom will visit Sue but not Ursula
```

f.

	Either we spend a bundle on television or we won't have wide public exposure	1
	If we spend a bundle on television, we'll go into debt	2
	Either we have wide public exposure or our contributions will dry up	4
	We'll go into debt if our contributions dry up and we don't have large reserves	6
	We won't have large reserves	(5)
	We'll spend a bundle on television	(2)
2 MPP	We'll go into debt	(3)
	●	
3 QED	We'll go into debt	1
	We won't have wide public exposure	(4)
4 MTP	Our contributions will dry up	(5)
5 Adj	Our contributions dry up and we won't have large reserves	X,(6)
6 MPP	We'll go into debt	(7)
	●	
7 QED	We'll go into debt	1
1 PC	We'll go into debt	

g.

	If Adams supports the plan, it will go though provided Brown doesn't oppose it	3
	Brown won't oppose the plan if either Collins or Davis supports it	5
	Both Adams and Davis will support the plan	2
2 Ext	Adams will support the plan	(3)
2 Ext	Davis will support the plan	(4)
3 MPP	The plan will go though provided Brown doesn't oppose it	6
4 Wk	Either Collins or Davis will support the plan	X,(5)
5 MPP	Brown won't oppose the plan	(6)
6 MPP	The plan will go through	(7)
	●	
7 QED	The plan will go through	1
1 CP	The plan will go through if both Adams and Davis support it	