3. Negations

3.1. Not: contradicting content

3.1.0. Overview

In this chapter, we direct our attention to *negation*, the second of the logical forms we will consider.

3.1.1. Connectives

Negation is a way of forming sentences from sentences, so it is a connective even though it does not serve to connect sentences.

3.1.2. Contradictory propositions

The meaning of negation is closely tied to the idea of a pair of sentences being contradictory.

3.1.3. Negation in English

Although not is the chief way of expressing negation is English, there are others.

3.1.4. Negated conjunctions and conjoined negations

When we combine negation with conjunction, we obtain a wide range of further forms, some of them important enough to deserve names.

3.1.5. Some sample analyses

Analyzing sentences may involve recognizing not only the presence of negation and conjunction but also the way they are combined.

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3.1.1. Connectives

The connective we will study in this chapter is *negation*, which is associated with the English word **not**. As has been the case with *conjunction*, we will use the term *negation* also for the sentences produced by the operation of negation. We will represent the form of such sentences symbolically using ¬ (the *not sign*) as our sign for negation so that ¬ φ is the negation of φ. To indicate negations using English, we will use not as an alternative to ¬, writing it, too, in front of the negated sentence so that, in this notation, not φ is the negation of φ.

The use of the term connective for negation is standard but in some ways not very apt. The word not in English is not a combining operation; it is not a conjunction (in the grammatical sense) that serves to connect clauses but instead an adverb, a modifier of a single clause. Thus it would be a mistake to associate the term connective too closely with the ideas of connection or combination. A *connective* is better thought of as an operation that forms or generates a sentence from one or more sentences. This operation may combine or modify, and it may do both.

We will extend the terminology used for conjunction and refer, however inaptly, to any sentence generated by a connective as "compound" and refer to the one or more sentences it is generated from as "components." When analyzing English sentences, the ultimate components we encounter may not be parts, in any grammatical sense, of the sentences we analyze. They will rather be the sentences whose logical forms we do not describe; that is, they are the unanalyzed residue of our analysis.

3.1.2. Contradictory propositions

We could base the truth conditions of negation directly on the observation that the word false means '*not* true' and the word true means '*not* false.' But it will be more enlightening to base it instead on some understanding of the logical relations between a negation $\neg\varphi$ (or not φ) and its component φ.

One obvious generalization about negation is that a negative sentence is incompatible with the component that is negated. For example, in the traditional children's story, even before sitting down to her taste test, Goldilocks knew that The porridge is too hot and The porridge is not too hot could not both describe the same bowl. Each excludes the other; they are mutually exclusive (in the sense defined in 1.2.6). We can explain this fact about negation if we assume that the negation $\neg\varphi$ of a sentence φ is false whenever the sentence φ is true. And that settles the part of the truth table for negation shown below.

φ	$\neg\varphi$
T	F

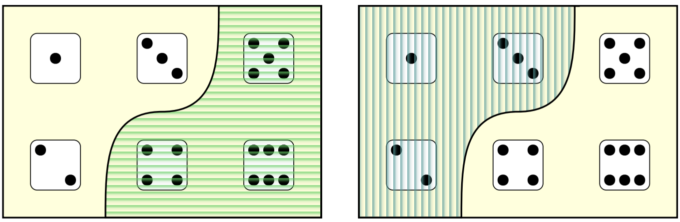
But it does not settle the rest. The sentences The porridge is too hot and The porridge is too cold are also mutually exclusive, but Goldilocks found two cases in which The porridge is too hot was false, one in which The porridge is too cold is true and another in which it was false. So the mutual exclusiveness of φ and $\neg\varphi$ is not enough to settle the truth value of $\neg\varphi$ when φ is false.

There is a second relation between a sentence and its negation that does settle this value. While the falsity of both The porridge is too hot and The porridge is too cold would leave open the possibility that the porridge is just right, The porridge is too hot and The porridge is not too hot allow no third case. That means the two sentences are jointly exhaustive of all possibilities (see 1.2.6 for this idea). This relation serves to settle the second row of the truth table for negation; if φ is false then $\neg\varphi$ must be true.

φ	$\neg\varphi$
T	F
F	T

A negation $\neg\varphi$ thus has a truth value that is always the opposite of the truth value of its component φ. In 1.2.6, we spoke of such sentences (that is, sentences that are both mutually exclusive and jointly exhaustive) as "contradictory." So a sentence and its negation are contradictory sentences; each contradicts the other. The negation of a sentence φ need not be the only sentence that contradicts φ, but any sentence that stands in this relation to φ will be logically equivalent to $\neg\,\varphi$.

Figure 3.1.2-1 shows the effect of negation on the proposition expressed; the possibilities ruled out by the sentence (A) and its negation (B) are shaded. The images of dice recall the example of Figure 2.1.2-1; if they are taken to indicate regions consisting of the possible worlds in which a certain die shows one or another number, the proposition shown in 3.1.2-1A is The number shown by the die is less than 4 and 3.1.2-1B illustrates the negation of this proposition.



A B

Fig. 3.1.2-1. Propositions expressed by a sentence (A) and its negation (B).

The possibilities left open by a sentence are ruled out by its negation—no possibilities are left open by both—because the two are mutually exclusive. And the possibilities ruled out by a sentence are left open by a sentence—none are ruled out by both—because the two are jointly exhaustive.

The reversal in the range of possibilities left open in moving from a sentence to its negation are the basis for what can be seen as the the key properties of negation.

CONTRAVARIANCE. *A negation implies the result of replacing its component with anything that component is implied by.* That is, if $\varphi \vDash \psi$, then $\neg\,\psi \vDash \neg\,\varphi$.

INVOLUTION. *To deny a negation is to assert what it negates.* That is, $\neg\,\neg\,\varphi \simeq \varphi$.

COMPOSITIONALITY. *Negations are equivalent if their components are equivalent.* That is, if $\varphi \simeq \varphi'$, then $\neg\,\varphi \simeq \neg\,\varphi'$.

The last of these follows from contravariance just as the compositionality of conjunction follows from covariance; and, as noted in 2.1.2, compositionality is something we would expect to hold of any connective. So the distinctive character of negation appears in the first two principles.

In particular, contravariance and involution together tell us that $\neg\,\psi$ implies $\neg\,\varphi$ if and only if φ implies ψ. Contravariance alone supplies the if part of this;

in the other direction, the two principles tell us that, if $\neg\,\psi$ implies $\neg\,\varphi$, then φ is equivalent to something (namely, $\neg\,\neg\,\varphi$) that implies something (namely, $\neg\,\neg\,\psi$) that is equivalent to ψ. In sum, the more said by a claim, the less said by its denial; and the less said by a claim, the more said by its denial. Compare The package won't arrive next Wednesday and The package won't arrive next week. The latter is the more informative, and it denies the less informative of the two positive sentences The package will arrive next Wednesday and The package will arrive next week. Notice in the diagrams above that, as the area ruled out by a sentence increases, the area ruled out by its denial decreases, and vice versa.

Connectives that have truth tables express truth functions and are therefore said to be *truth functional*, and this is something more than being propositional. Conjunction and negation are truth functional, but not all connectives have this property. The following simple example of a non-truth-functional connective should suggest a whole range of further examples. Compare these two sentences:

The bridge is not finished
The bridge will never be finished.

The truth value of the first is determined once we know the truth value of The bridge is finished, but this is not always enough information in the case of the second. When The bridge is finished is true, we know that The bridge will never be finished is false; but, when The bridge is finished is false, we need more information to determine the truth value of The bridge will never be finished. In particular, we need at least some information about the truth value of The bridge is finished at times in the future; and before we can know that The bridge will never be finished is true, we need to know the truth value of The bridge is finished at *all* times in the future. And this means that the connective marked by the English form It will never be the case that φ is not truth functional: the actual truth value of the compound formed by it is not settled by the actual truth value of the component φ. But we would still expect the proposition expressed by it to be settled if we knew everything about the proposition expressed by its component—i.e., if we knew the truth value of its component in all possible worlds. We simply cannot limit consideration to one possibility at a time in the way we can with truth-functional connectives.

We will limit our study of connectives to those that are truth-functional. The study of such connectives is *truth-functional logic* (a phrase that was mentioned in 1.1.7). The connective expressed by It will never be the case that φ would be studied by *tense logic*, the logic of tenses and other temporal modifiers. This is one part of the logic of connectives that lies beyond truth-functional logic. Another part is the logic of modal auxiliaries like must and can. These, too, are associated with non-truth-functional connectives, and the study of the logical properties of these connectives is referred to as *modal logic*, an ancient branch of logic that became an active area of research again in the 20th century.

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3.1.3. Negation in English

Many questions that arise concerning the use of conjunction to analyze English sentences do not apply to negation. In particular, since a compound formed by negation has only a single component, there is no need to worry identifying components that make independent contributions to the whole. It is important, though, to be sure that the component that we uncover is related to the whole compound in the way that negation indicates—that is, we need to make sure that the two are contradictory.

Negative prefixes on adjectives (un-, in-, a-, etc.) sometimes function as stylistic variants for not. But the effect of such a prefix may not always be to negate since the result of adding it may not always be contradictory to the original sentence. For example, happy and unhappy seem to be used sometimes as synonyms for joyful and sad. In such usage, the sentence Hal is unhappy is not the negation of Hal is happy because both might be false. The only way to distinguish such cases from ones where the prefix is a sign of negation (as in The road is unfinished) is to ask yourself whether a sentence with a negative prefix and the corresponding sentence without it jointly exhaust all possibilities in the sense that at least one of the two is bound to be true.

When doing this, it is important to remember the difference between truth and appropriateness. That is, to show that Hal is happy and Hal is unhappy are not jointly exhaustive, it is not enough to find a case where it would not be appropriate to assert either—as when Hal's state of mind is neutral—for one of the two inappropriate assertions might still be true. It would even be possible for unhappy to be appropriate in exactly the same circumstances as some term like sad even though the two had different truth conditions. While it is not easy to rule out this sort of possibility, remember that we have one test to use. Imagine being asked the two questions Is Hal happy? and Is Hal unhappy? when you know his state of mind is neutral. Ask yourself if you would reply No to both or reply No to one and Yes, but … to the other.

So the presence of a stylistic variant of not is not sufficient to indicate negation—and it is also not necessary. Some sentences can be analyzed as negations even though they do not contain either not or a stylistic variant because they contain another logical expression that introduces a negative element. For example, The road was neither smooth nor straight can be analyzed as the negation of The road was either smooth or straight. In this case, we were able to simply remove the negative element in order to identify the component to which negation is applied; but, in other cases, some restatement may be needed to formulate a component that is contradictory to the whole compound.

That is often the case when negation is introduced by way of words or phrases containing no. For instance, No one bought the book is negative, but what is it the negation of? It is not the negation of Everyone bought the book, for to deny that would be to say only that there is at least one person who failed to buy it. No one bought the book must be the negation of At least one person bought the book or, more briefly, Someone bought the book. English is regular enough on this point that you could make it a rule of thumb to treat no as indicating the negation of some, but this is not a rule to be applied without thought. Again, the best general policy is to ask yourself whether the original sentence and the component you take to be negated are really contradictories—whether it really is the case that they cannot both be true and cannot both be false.

A related problem concerns the word any. This often appears in negative sentences—such as I didn't speak to anyone. Although this sentence is a negation, it cannot be analyzed as the negation of I spoke to anyone—a sentence that is hard to understand (except in contexts where it is elliptical for something like I spoke to anyone I wanted to). Instead, I didn't speak to anyone is the negation of I spoke to someone where this is understood to mean I spoke to at least one person. The problem with retaining any in the component of a negation is that it is generally used only in the presence of certain other words—not is one, but also if and some others—and it is hard, if not impossible, to understand the force of any when it is removed from such a context. But English is fairly regular here, too; and a sentence in which any is used with not can usually be regarded as a negation whose component can be stated using some in place of any.

For this approach to no and not … any to work, it is important that some mean 'at least one'. Now, in some contexts, the fact that some is used with a singular noun can lead to an implicature of 'only one'. For example, a sentence like I spoke to someone may implicate that *only* one person was spoken to. To see that this implicature is not an implication, imagine speaking to two people and being asked, "Did you speak to someone?" I think the natural answer would be Yes rather than No—though you might add In fact, I spoke to two people if this further information was relevant. If that is right, the suggested analysis of I spoke to no one and I didn't speak to anyone does work, but the best policy is still to ask yourself whether the component you identify is really contradictory to the original sentence.

Similar issues arise when we consider the result of negating a negation (that is, the form $\neg \neg \varphi$ or not not φ). Although we can capture some further English constructions by this form, the principle of involution in 3.1.3 tells us that we can find no new logical properties since the two forms $\neg \neg \varphi$ and φ are logically equivalent. That is, doubling a negation cancels it. The sentence The road is not unfinished is merely a roundabout way of saying that the road is finished. It is true that double negations do not always seem to have the same force as positive statements; but this is naturally ascribed to a difference in appropriateness without a difference in truth conditions, a difference in implicatures but not implications.

To get a sense of the play of implicatures here, consider the following dialogue (with underlining used to mark emphasis):

A: Hal is not unhelpful.
B: So, in other words, he's helpful.
A: Well, yes, but he's not *really* helpful.
B: You mean he just appears to be helpful?
A: No, he's really *helpful*. He's just not *really* helpful.

This shows—if the point needed making—that truth conditions are often less the foundations of communication than walls to bounce things off. But even so, they make their presence felt—and that is what we are trying to capture. When logicians question the equivalence of a double negation and a positive statement, it is usually on different grounds.

And, surprising as it may be, the equivalence of φ and $\neg \neg \varphi$ is actually one of the more controversial principles among logicians. A small school of mathematics called *intuitionism* grew up around efforts in the early part of the 20th century by the Dutch mathematician L. E. J. Brouwer (1881-1966) to give what he took to be a philosophically satisfactory account of the continuum (the full range of real numbers including irrational numbers like π and the square root of 2). He came to reject certain ways of proving the existence of mathematical objects, and he also rejected certain logical principles—the equivalence of φ and $\neg \neg \varphi$ among them—which could be used to justify such proofs. Brouwer did not succeed in transforming mathematical practice or leading most logicians to doubt the equivalence of φ and $\neg \neg \varphi$, but his ideas have proved useful in the study of computation and have led to a deeper understanding of the significance of various logical principles concerning negation.

3.1.4. Negated conjunctions and conjoined negations

While the ability to negate a negation does not enable us to say any more—however much more we may suggest—we increase the range of propositions we can express considerably when we mix negation and conjunction. The variety of English sentences whose forms we can express naturally will still be somewhat limited, and we will go on to capture others in the next two chapters. But the variety of logical relations between compounds and their components that can be expressed using conjunction and negation will be as great as any we will see when we are considering connectives alone (that is, until chapter 6). Indeed, any connective that is truth-functional—i.e., any whose meaning can be captured in a truth table—can be expressed using conjunction and negation alone.

The real key to the power of expression of these two connectives lies in the ability to negate conjunctions, so let us look more closely at such forms. We will begin with the example It was not both hot and humid.

It was not both hot and humid
¬ it was both hot and humid
¬ (it was hot ∧ it was humid)

$\neg (T \wedge M)$
not both T and M

T: it was hot; M: it was humid

The parentheses and location of not before both record the fact that the sentence as a whole is a negation. That is, negation here has wider scope than conjunction and is thus the main connective.

We will refer to the way this sentence is related to its unanalyzed components as the *not-both form*. Our analysis together with the truth tables for negation and conjunction enable us to calculate a truth table for it. The table below follows the conventions for exhibiting the values of compounds that were introduced in 2.1.8. (That is, each of the two columns of values on the right is written under the sign for the connective whose table was the last used in calculating it.)

φ	ψ	¬	(φ	∧	ψ)
T	T	Ⓕ		T	
T	F	Ⓣ		F	
F	T	Ⓣ		F	
F	F	Ⓣ		F	

The plain roman Ts and Fs are the values for the conjunction $\varphi \wedge \psi$ in each case, and the circled values for the form as a whole come by following the table for negation and taking the opposite of the value of the conjunction in each row.

In the symbolic analysis of the not-both form, parentheses not only reflect the structure of the sentence analyzed but also make a significant difference in the proposition expressed. If we drop them and write $\neg \varphi \wedge \psi$ (i.e., both not φ and ψ), we will no longer be marking the conjunction as a component of a larger negation. The negation sign will instead apply (by default) to φ alone, and the main connective will be conjunction. That is, we will have a conjunction whose first component is a negation. The truth table for this form is as follows:

φ	ψ	$\neg$	φ	$\wedge$	ψ
T	T	F		Ⓕ	
T	F	F		Ⓕ	
F	T	T		Ⓣ	
F	F	T		Ⓕ	

In the example we began with, dropping the parentheses gives us $\neg$ T $\wedge$ M (that is, both not T and M), which can be put into English as follows:

$\neg$ it was hot $\wedge$ it was humid
It wasn't hot $\wedge$ it was humid

It wasn't hot, but it was humid

And we will refer to the general form $\neg \varphi \wedge \psi$ as the *not-but form*.

The not-but sentence above also could be expressed (though more awkwardly) as It was both not hot and humid. (If this does not seem to make sense, try reading not hot as if it was hyphenated and pause briefly after it; that is, read it as you would It was both not-hot—and humid.) A comparison of this last (awkward) expression of the not-but form with our original not-both example is revealing:

Sentence	*Analysis*
It was not both hot and humid	$\neg$ (T $\wedge$ M) ***or*** not both T and M
It was both not hot and humid	($\neg$ T $\wedge$ M) ***or*** both not T and M

(The whole of the second analysis is parenthesized to make the comparison easier.)

The order of the words expressing negation and conjunction in the two English sentences corresponds exactly to their order in the analysis written using English notation. In particular, the word both can be seen to function in the English sentences, as it does in the analysis, to mark the beginning of the scope of a conjunction and thus to indicate whether the word not applies to the whole conjunction or only a part. Of course, things do not always work out this neatly in English, but the use of both after not is an important way of indicating exactly what is being denied. Emphasis is another way of indicating the scope of negation, and an emphasized both—as in It was not *both* hot and humid—can be particularly effective.

The real significance of negated conjunction lies in the way it modifies while combining, allowing us to say that at least one of the two components of the not-both form is false. The sentence It was not both hot and humid is false only when the components It was hot and It was humid are both true, so it leaves open every possibility in which at least one of them is false. And this is something we could not do by modifying the components separately and asserting each. On the other hand, a conjunction one or both of whose components is negative merely combines by adding content, and we could convey the same information by asserting the conjuncts separately.

While the not-both is the important new idea, conjunction of possibly negative components sometimes captures what we want to say; and there is a construction in English that seems designed to produce a logical form of this sort. The sentence It was humid but it wasn't hot could be rephrased as It was humid but not hot and thus as It was humid without being hot. So this last sentence, too, can be understood as a conjunction (i.e., as $M \land \neg T$ or both M and not T). Now without (in this use of the word) is a preposition, not a conjunction, so what follows it will not have the form of a sentence. But the object of without can be a nominalized predicate or nominalized sentence rather than an ordinary noun or noun phrase, and just about anything of the form $\varphi \land \neg \psi$ (which we will refer to as the *but-not form*) can be paraphrased using without. For example, Sue listened but didn't respond can be paraphrased as Sue listened without responding, and Ann walked in but Bill didn't see her could be paraphrased as Ann walked in without Bill seeing her. And, even when the object of without is an ordinary noun or noun phrase (rather than a nominalized predicate or sentence), the effect of without is often the same as that of a but-not form. Thus Tom left without his coat could be paraphrased as Tom left but didn't take his coat and thus analyzed as Tom left $\land \neg$ Tom took his coat. Of course, we have had to supply the verb take here, and we cannot expect any one pattern of paraphrase to work in all cases where without has an ordinary noun or noun phrase as its object.

Since this use of without is not a grammatical conjunction, it does not introduce a second main verb; and this makes it especially convenient when we want to negate a but-not form. For the easiest way to express the negation of a whole sentence is to apply not to a single main verb. Suppose we wish to say something with the following form:

¬ (it will fall ∧ ¬ it will be pushed)
not both it will fall and not it will be pushed

We might manage by expressing the three connectives one by one, ending with something like It won't both fall and not be pushed, where we have contrived a single conjoined predicate incorporating negation. But any such sentence is likely to be rather awkward. The natural way of making the claim analyzed above is to use It won't fall without being pushed. Accordingly, let us refer to the form $\neg (\varphi \wedge \neg \psi)$ as the *not-without form*.

Of course, it is also possible to conjoin sentences both of which are negations. Indeed, It was not hot and not humid is sometimes an accurate description of the weather. We would analyze this symbolically as $\neg T \wedge \neg M$ or both not T and not M. It will, at least for the time being, be convenient to have a label for the form $\neg \varphi \wedge \neg \psi$, too; and the natural one is *not-and-not form*. Although this is an important sort of truth-functional compound, we will see another way of expressing it in the next chapter that is closer to the grammatical form usually taken by such compounds in English. For the more idiomatic way of say that is not hot and also not humid is with the sentence It is neither hot nor humid. We noted earlier that this sentence can be seen as a negation of It is either hot or humid, and its analysis along those lines will await our account of the word or. But, until we have that, the not-and-not form can serve as an analysis of neither-nor sentences since it has the right truth conditions.

This way of analyzing neither-nor sentences is not the only case where conjunction and negation can be used to analyze sentences that we will be able to analyze in a different and more direct way later. For example, many if-then sentences can be analyzed using the not-without form (though doing so may be jarring due to differences in implicatures). But this is just a special case of something that was noted earlier: any truth-functional compound can be expressed using conjunction and negation alone.

To see this, suppose the effect of some connective on the truth conditions of a sentence can be captured in a truth table—that is, suppose the connective is truth-functional. The force of a sentence formed by such a connective is to deny that the actual state (or history) of the world is described by any of the rows of the table in which the sentence is false. Now the description of the state of the world offered by a given row can be captured by a run-on conjunction that affirms or denies each component in turn. For example, knowing that φ is assigned **T** and ψ is assigned **F** comes to the same thing as knowing that the sentence $\varphi \wedge \neg \psi$ is true. As a result, the compound sentence as a whole is equivalent to a conjunction of the denials of the sentences corresponding to each row in which the sentence is false. (At least this is so, if there are any such rows; otherwise, the sentence is a formal tautology and is equivalent to any other formal tautology, for example, $\neg (\varphi \wedge \neg \varphi)$.) This argument applies no matter how many components the connective applies to and no matter what form the truth table takes. For this reason, conjunction and negation are said to form a *truth-functionally complete* set of connectives.

To take a particular case, a compound with the table below can be thought of as saying that φ and ψ are not both truth and also that they are not both false, so it will be equivalent to $\neg (\varphi \wedge \psi) \wedge \neg (\neg \varphi \wedge \neg \psi)$.

φ	ψ	
T	T	F
T	F	T
F	T	T
F	F	F

An English sentence whose grammatical form is close to this form—such as Sam didn't eat both pie and cake, but he also didn't eat neither—will be very cumbersome, and there are likely to be more idiomatic ways of saying the same thing whose most natural analyses would be different. But it is still important to note that it is possible to say this sort of thing by putting the sentences Sam ate pie and Sam ate cake together using conjunction and negation alone since it shows that the other expressions for this truth function do not introduce any fundamentally new logical ideas.

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3.1.5. Some sample analyses

We will conclude this discussion with several examples illustrating the issues we have discussed. First, consider a case that is entirely straightforward.

It isn't warm out
¬ it's warm out

¬ W
not W

W: it's warm out

A second example shows that uncovering even a simple form can require some thought and a paraphrase.

No one saw anyone enter the building
¬ someone saw someone enter the building

¬ S
not S

S: someone saw someone enter the building

Care is needed in distinguishing not-both forms from not-and-not forms. Everyone understands the distinction quite well intuitively, but it is easy to get tripped up when you are first learning to make this understanding explicit. Compare the following.

Britain and France won't both vote
¬ Britain and France will both vote
¬ (Britain will vote ∧ France will vote)

¬ (B ∧ F)
not both B and F

Britain and France both won't vote
Britain won't vote ∧ France won't vote
¬ Britain will vote ∧ ¬ France will vote

¬ B ∧ ¬ F
both not B and not F

B: Britain will vote; F: France will vote

The negation of a conjunction is not the same as a conjunction of negations. The second form is also the way we would analyze Neither Britain nor France will vote.

The scope of negation is one respect in which English sentences are often ambiguous, and it is not hard to find examples that people will interpret differently. For example, you may find it possible to understand the second sentence above as a denial of Britain and France will both vote—i.e., as equivalent to the first. The first seems unambiguous, but other sentences in which not appears before both are less clear. For example, it might be possible to understand Tom didn't like both the service and the price to say that he liked neither (if you have trouble understanding it to say anything *but* that, try reading it with an emphasis on both).

Finally, here is a somewhat longer example.

Al didn't get to both the meeting and the party without missing both the game and the movie
¬ Al got to both the meeting and the party without missing both the game and the movie
¬ (Al got to both the meeting and the party ∧ ¬ Al missed both the game and the movie)
¬ ((Al got to the meeting ∧ Al got to the party) ∧ ¬ (Al missed the game ∧ Al missed the movie))
¬ ((Al got to the meeting ∧ Al got to the party) ∧ ¬ (¬ Al got to the game ∧ ¬ Al got to the movie))

$$\neg ((T \land P) \land \neg (\neg G \land \neg V))$$

not both T and P and not both not G and not V

G: Al got to the game; P: Al got to the party; T: Al got to the meeting; V: Al got to the movie

The final step of analyzing X missed Y as contradictory to X got to Y is not crucial at this point in the course. While it is important to exhibit as much logical structure as possible, we end up with four logically independent sentences whether we carry out the final step or not. However, we will later go on to press analyses below the level of sentences, and this sort of step will then be of value since it leads us to four components that differ only in the object of the preposition to and therefore can be analyzed in a way that re-uses vocabulary.

3.1.s. Summary

1 Negation is an operation associated with the English word not. It generates a compound sentence from a single component, so it is a connective that serves to modify a sentence rather than to combine sentences. The not symbol $\neg$ is our notation for negation. As English notation for $\neg\varphi$, we use not φ.

2 A sentence and its negation cannot be both true (they are mutually exclusive) and cannot be both false (they are jointly exhaustive); in short, they must have different truth values (they are contradictory). Each leaves open the possibilities the other rules out and rules out the possibilities the other leaves open. This means that negation, like conjunction, has a truth table; in other words it is a truth-functional connective. Not all connectives are truth-functional. Truth-functional logic is the branch of logic which studies those that are, but there are branches of logic—such as tense logic and modal logic—in which non-truth-functional connectives are studied.

3 Negation appears in English not only in connection with the word not but also with negative prefixes (though such a prefix does not always mark negation because it does not always produce a sentence that is contradictory to the original). Negation also appears with uses of no in phrases of the form no X, uses that can often be treated as the negation of at least one or some. The same sort of treatment is usually what is required when not appears along with the word any (though such sentences usually must be rephrased when not is removed). By negating a negation, we can produce a double negation, but this undoes the negation rather than generating a logical form with new properties.

4 The really new ideas come with the negation of conjunctions, but conjunctions whose components may involve negation also provide important forms of expression. A number of forms are shown below, with labels that suggest the sort of English sentences they serve to analyze:

not-both form	$\neg(\varphi \land \psi)$	not both φ and ψ
not-but form	$\neg\varphi \land \psi$	both not φ and ψ
but-not form	$\varphi \land \neg\psi$	both φ and not ψ
not-and-not form	$\neg\varphi \land \neg\psi$	both not φ and not ψ
not-without form	$\neg(\varphi \land \neg\psi)$	not both φ and not ψ

That the last is the denial of the third reflects the fact that without can be used to express a but-not form. Also neither-nor can be used to express a not-and-not form. More generally, negation and conjunction form a truth-functionally complete set of connectives in the sense that any truth-functional compound can be expressed using them alone.

Glen Helman 01 Aug 2011

3.1.x. Exercise questions

1. Analyze each of the following sentences in as much detail as possible.

a. The soup was hot but not too hot, and thick but not too thick.

b. The equipment isn't here and it's unlikely to arrive soon.

c. No one answered the phone even though it rang 10 times.

d. The alarm must have gone off, but Ted didn't hear anything.

e. They won't both meet the deadline and stay within the budget.

f. They won't meet the deadline, but they will stay within the budget.

g. They won't meet the deadline, and they won't stay within the budget.

h. Tod shut off the alarm without waking up.

i. They won't meet the deadline without going over the budget.

j. Larry joined in, but not without being coaxed.

k. Ann liked the movie, but neither Bill nor Carol did.

2. Restate each of the forms below, putting English notation into symbols and vice versa. Indicate the scope of connectives in the result by underlining.

a. $\neg \neg (A \land B)$

b. $\neg (\neg A \land B)$

c. both not A and both not B and C

d. both not both A and B and not C

3. Synthesize idiomatic English sentences that express the propositions that are associated with the logical forms below by the intensional interpretations that follow them.

a. $C \land \neg F$

C: it was cold; F: there was frost

b. $\neg S \land (H \land I)$

H: Sue heard a crash; I: Sue went to investigate; S: someone saw the accident

c. $(D \land N) \land \neg P$

D: it was a design; N: it was new; P: it pleased someone

d. $\neg (I \land N)$

I: we'll win in Iowa; N: we'll win in New York

e. $\neg I \wedge N$

I: we'll win in Iowa; N: we'll win in New York

f. $\neg (I \wedge \neg L)$

I: we'll win in Iowa; L: we'll lose in New York

4. Complete the following truth tables. That is, calculate truth values for all components of the forms below using the extensional interpretation provided on the left in each case.

a.

A	B	C	A ∧ ¬ (B ∧ C)
T	F	F	

b.

A	B	C	A ∧ (¬ B ∧ C)
T	F	F	

c.

A	B	C	D	(¬ A ∧ ¬ B) ∧ (¬ (A ∧ C) ∧ D)
F	T	T	T	

3.1.xa. Exercise answers

1. a. The soup was hot but not too hot ∧ the soup was thick but not too thick

(the soup was hot ∧ the soup was not too hot) ∧ (the soup was thick ∧ the soup was not too thick)

(the soup was hot ∧ ¬ the soup was too hot) ∧ (the soup was thick ∧ ¬ the soup was too thick)

$$(H \land \neg T) \land (K \land \neg O)$$

both both H and not T and both K and not O

H: the soup was hot; K: the soup was thick; O: the soup was too thick; T: the soup was too hot

b. The equipment isn't here ∧ the equipment is unlikely to arrive soon

¬ the equipment is here ∧ ¬ the equipment is likely to arrive soon

$$\neg H \land \neg S$$

both not H and not S

H: the equipment is here; S: the equipment is likely to arrive soon

c. No one answered the phone ∧ the phone rang 10 times

¬ someone answered the phone ∧ the phone rang 10 times

$$\neg A \land R$$

both not A and R

A: someone answered the phone; R: the phone rang 10 times

d. The alarm must have gone off ∧ Ted didn't hear anything

The alarm must have gone off ∧ ¬ Ted heard something

$$A \land \neg H$$

both A and not H

A: the alarm must have gone off; H: Ted heard something

e. ¬ they will both meet the deadline and stay within the budget

¬ (they will meet the deadline ∧ they will stay within the budget)

$$\neg (D \land B)$$

not both D and B

B: they will stay within the budget; D: they will meet the deadline

f. They won't meet the deadline ∧ they will stay within the budget

¬ they will meet the deadline ∧ they will stay within the budget

$$\neg D \land B$$

both not D and B

B: they will stay within the budget; D: they will meet the deadline

g. They won't meet the deadline ∧ they won't stay within the budget

¬ they will meet the deadline ∧ ¬ they will stay within the budget

$$\neg D \land \neg B$$

both not D and not B

B: they will stay within the budget; D: they will meet the deadline

h. Tod shut off the alarm ∧ ¬ Tod woke up

$$A \land \neg W$$

both A and not W

A: Tod shut off the alarm; W: Tod woke up

i. ¬ they will meet the deadline without going over the budget

¬ (they will meet the deadline ∧ ¬ they will go over the budget)

$$\neg (D \land \neg G)$$

not both D and not G

D: they will meet the deadline; G: they will go over the budget

j. Larry joined in ∧ Larry did not join in without being coaxed

Larry joined in ∧ ¬ Larry joined in without being coaxed

Larry joined in ∧ ¬ (Larry joined in ∧ ¬ Larry was coaxed)

$$J \land \neg (J \land \neg C)$$

both J and not both J and not C

C: Larry was coaxed; J: Larry joined in

This is equivalent to $J \wedge \neg\neg C$ and also to $J \wedge C$, but the analysis shown is closer to the form of the English.

k. Ann liked the movie ∧ neither Bill nor Carol liked the movie

Ann liked the movie ∧ (¬ Bill liked the movie ∧ ¬ Carol liked the movie)

$$A \wedge (\neg B \wedge \neg C)$$

both A and both not B and not C

A: Ann liked the movie; B: Bill liked the movie; C: Carol liked the movie

The alternative (and logically equivalent) analysis as $A \wedge \neg E$ (where E is either Bill or Carol liked the movie) is closer to the English but it is less satisfactory because it displays less structure. The next chapter will give us the means carry this sort of analysis further by analyzing E as a compound of B and C.

2. a. not not both A and B

b. not both not A and B

c. $\neg A \wedge (\neg B \wedge C)$

d. $\neg (A \wedge B) \wedge \neg C$

3. a. It was cold ∧ ¬ there was frost

It was cold ∧ there was no frost

It was cold, but there was no frost

b. ¬ someone saw the accident ∧ (Sue heard a crash ∧ Sue went to investigate)

No one saw the accident ∧ Sue heard a crash and went to investigate

No one saw the accident, but Sue heard a crash and went to investigate

c. (it was a design ∧ it was new) ∧ ¬ it pleased someone

It was a new design ∧ it pleased no one

It was a new design, and it pleased no one

d. ¬ (we'll win in Iowa ∧ we'll win in New York)

¬ (we'll win in both Iowa and New York)

We won't win in both Iowa and New York

e. ¬ we'll win in Iowa ∧ we'll win in New York

We won't win in Iowa ∧ we'll win in New York

We won't win in Iowa, but we'll win in New York

f. ¬ (we'll win in Iowa ∧ ¬ we'll lose in New York)

¬ (we'll win in Iowa without losing in New York)

We won't win in Iowa without losing in New York

4. Numbers below the tables indicate the order in which values were computed.

a.

A	B	C	A	∧	¬	(B	∧	C)
T	F	F		Ⓣ	T		F	
				3	2		1	

b.

A	B	C	A	∧	(¬	B	∧	C)
T	F	F		Ⓕ	T		F	
				3	1		2	

[Note that, while in **a**, it is the value under the ¬ that is used in calculating the value of the main conjunction, in **b** it is the value under the second ∧; this is due to the change in relative scope of these two connectives.]

c.

A	B	C	D	(¬	A	∧	¬	B)	∧	(¬	(A	∧	C)	∧	D)
F	T	T	T	T		F	F		Ⓕ	T		F		T	
				1		2	1		4	2		1		3	