5. Conditionals

5.1. If: trimming content

5.1.0. Overview

The last connective we will consider is an asymmetric one whose asymmetry gives it an important role in deductive reasoning.

5.1.1. Conditions

In its simplest form, the conditional trims the content of one component by limiting the worlds it rules out to ones that the other component leaves open.

5.1.2. The conditional as a truth-functional connective

The trimming of content is naturally described by an asymmetric truth table.

5.1.3. Doubts about truth-functionality

The truth table just associated with the condition has been controversial since antiquity because the conditional is closely associated with certain implicatures that can seem to add further content.

5.1.4. Examples

The chief task in analyzing the English conditionals marked by if alone is to assign the correct order to the components.

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5.1.1. Conditions

The use of or is not the only way of hedging what we say. Instead of hedging a claim by offering an alternative, we can limit what we rule out to a certain range of possibilities. For example, instead of saying It will rain tomorrow, a forecaster might say It will rain tomorrow if the front moves through. The subordinate clause if the front moves through limits the forecaster's commitment to rain tomorrow to cases where the front does move through. If it does not move through, the forecaster's prediction cannot be faulted even if it does not rain.

We will refer to the connective marked by if as the *(if-)conditional* and to sentences of the form ψ if φ as *(if-)conditionals*. The qualification if- is used here to distinguish this connective from connectives associated with only if and unless that we will consider in 5.2. The three connectives are closely related, we will refer to all three as *conditionals*. However, the if-conditional is the most important of the three we will consider, and a reference to "the conditional" without qualification will be to it. Outside of contexts where we are discussing several sorts of conditional sentence, a reference to "conditionals" will be to the various compounds formed using it rather than to the three sorts of connective. In fact, we will analyze the other two connectives in a way that makes the if-conditional the main component of the result, so compounds formed using the other two connectives will count as special sorts of if-conditionals.

Although we take the word if, like the words and and or, to mark a two-place connective, it raises somewhat different grammatical issues. Since it is used mainly to join full clauses, there is less often a need to fill out the expressions it joins to get full sentences (though, of course, pronominal reference from one component to another must still be removed). And there are special problems associated with it. The conditional is an asymmetric connective: it makes a difference which component is having its content trimmed and which expresses the condition used to trim the content. For example, there is a considerable difference between the following sentences:

Mike entered the contest if he won the prize
Mike won the prize if he entered the contest.

The first is a truism about contests and merely rules out cases of Mike winning the prize without entering the content. On the other hand, the second suggests confidence in Mike's success and rules out cases where he entered the contest without winning.

Still, no fixed order between the two clauses of a conditional is imposed by English syntax. Like other subordinate clauses, if-clauses can be moved to the beginning of the sentence. Thus the two sentences above could be rephrased, respectively, as the following:

If Mike won the prize, he entered the contest
If Mike entered the contest, he won the prize

Sometimes the word then will precede the main clause when conditionals are stated in this order; but, as the examples above show, this is not necessary.

We will use the asymmetric notation → (the *rightwards arrow*) or ← (the *leftwards arrow*) for the conditional. The subordinate if-clause will contribute the component at the tail of the arrow, and the main clause of a conditional sentence will contribute the component at the head. We will refer to these two components, respectively, as the *antecedent* (i.e., what comes before, in the direction of the arrow) and the *consequent* (what comes after, again in the direction of the arrow).

Since the difference between the conditioned claim and what it is conditional on is marked by the difference between the two ends of the arrow, the order in which we write these components makes no difference provided that the arrow points from the antecedent to the consequent. For example, Adam opened the package if it had his name on it could be written as either of the following:

Adam opened the package ← the package had Adam's name on it
The package had Adam's name on it → Adam opened the package

This means that the reordering of clauses in English can be matched by our symbolic notation, with $\varphi \rightarrow \psi$ corresponding to If φ then ψ and $\psi \leftarrow \varphi$ corresponding to ψ if φ. When we are not attempting to match the word order of English sentence, the rightwards arrow will be the preferred notation, and generalizations about conditionals will usually be stated only for the form $\varphi \rightarrow \psi$.

We will use if φ then ψ as English notation for $\varphi \rightarrow \psi$. Here the word if plays the role of a left parenthesis (as both and either do). We will not often use English notation for the leftwards arrow, but it can help in understanding the relation of the two to have some available. If we are to have anything corresponding to the form $\psi \leftarrow \varphi$, we will put if between the two components, so we need another word to the role of left parenthesis. English usage provides no natural choices, so we will have to be a bit arbitrary. The interjection yes does not disturb the grammar of the surrounding sentence, so it can be easily placed where we want it. So we will write yes ψ if φ as our English notation for the form $\psi \leftarrow \varphi$. This way of tying the words yes and if is not backed up by an intuitive understanding of English, so the yes in the form yes ψ if φ does not help in understanding the symbolic form. On the other hand, it does not interfere with the help that if provides; and, as an interjection, yes can help to mark breaks in a sentence in much the way punctuation does.

On the other hand, the leftwards arrow $\leftarrow$ is the easier of the two to accommodate if we look for a simple English substitute to use along with parentheses, for $\leftarrow$ corresponds directly to if. We will not often need to use English notation with parentheses in the case of conditionals, so finding something for the rightwards arrow $\rightarrow$ is not a pressing practical problem. However, the way this problem is typically solved emphasizes an important point about the conditional

Of course, we cannot use if also for the rightwards arrow. And, even if we were not using if for the leftwards arrow, it would not work for $\rightarrow$ since if in English must precede rather than follow the subordinate clause. And then will not do either since it is if that bears the meaning of the connective in English. The usual approach is to look further afield and employ the word implies. Lacking a better alternative, we will follow this practice and use the word implies (in this typeface) as an English version of $\rightarrow$ to use with parentheses.

There is some danger of confusion in doing this, for we have used implies as a synonym for entails in the case of a single premises, and the signs $\rightarrow$ and $\vDash$ have quite different meanings. In particular, the notation $\varphi \rightarrow \psi$ refers to a sentence that speaks only of the actual world while, in saying that $\varphi \vDash \psi$, we make a claim about all possible worlds. One way to avoid the confusion is to say that $\varphi \rightarrow \psi$ expresses *material implication* while, when saying that $\varphi \vDash \psi$, we express *logical implication*. We will discuss this distinction further in 5.3.1; but, for now, we can note that this terminology is intended to capture a distinction between a claim about what is a matter of fact on the one hand and a claim about logical necessity on the other. And, however we describe the difference, this is a case where the typeface definitely matters, for

φ implies ψ

is the use of an English word to provide an alternative notation for $\varphi \rightarrow \psi$ while

φ implies ψ

is our way of saying in ordinary English what is expressed in notation as $\varphi \vDash \psi$.

To give an example of some of this notation in action, let us return to the idea that a conditional serves to trim the content of its consequent. This can be expressed in symbolic notation as the entailment

$$\psi \vDash \varphi \rightarrow \psi$$

which says that the argument ψ / $\varphi \rightarrow \psi$ is a valid one. If we use English notation for the conditional, we might express the same entailment as either

$$\psi \vDash \text{if } \varphi \text{ then } \psi$$

or

$$\psi \vDash \varphi \text{ implies } \psi$$

and we express the relation in English, using implies to express entailment, by saying that ψ implies $\varphi \rightarrow \psi$, that ψ implies if φ then ψ, or that ψ implies φ implies ψ. Of course, because we have all these options, we have many ways of avoiding potentially confusing expressions; but trying to discern the meaning of a potentially confusing but really unambiguous expression is a good exercise in sorting out the range of concepts we are working with.

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5.1.2. The conditional as a truth-functional connective

We have looked at ψ if φ as a way of hedging the claim ψ by limiting our liability, leaving ourselves in danger of error only in cases where φ is true. If this perspective on the conditional is correct, we cannot go wrong in asserting ψ if φ except in cases where ψ is false while φ is true. Thus, the forecaster who predicts that it will rain tomorrow if the front goes through is wrong only if it does not rain even though the front goes through. That suggests that the truth conditions of the conditional are captured by the table below. The only cases where φ → ψ has a chance of being false are those where φ is true; and, in these cases, φ → ψ has the same truth value as ψ.

φ	ψ	φ → ψ
T	T	T
T	F	F
F	T	T
F	F	T

This can be seen in another way by diagramming the propositions expressed by conditionals, as in Figure 5.1.2-1. Adapting the example used with this sort of illustration before, 5.1.2-1B represents the proposition expressed by The number shown by the die is less than 4 if it is odd.

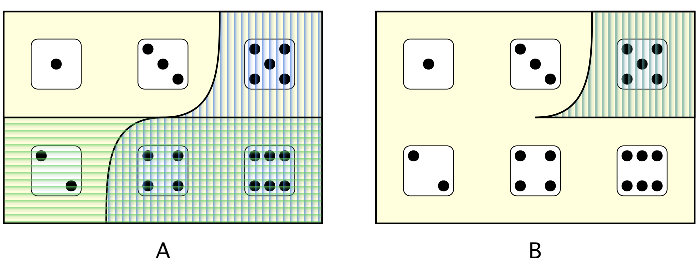

Fig. 5.1.2-1. Propositions expressed by two sentences (A) and a conditional (B) whose consequent rules out the possibilities at the right of A.

The possibilities ruled out by the main clause or consequent of the conditional form the hatched region at the right of 5.1.2-1A and those ruled out by the antecedent or condition form the lower half. In 5.1.2-1B, the region at the right is whittled down to the portion containing possibilities left open by the antecedent, showing how the conditional weakens the claim made by the consequent alone (in the example, The number shown by the die is less than 4). Since the consequent is the second component of the conditional φ → ψ, the rows of the truth table correspond to the top left and right and bottom left and right regions of 5.1.2-1A, respectively.

Apart from compositionality, the principles of implication and equivalence for the conditional are quite different from those we saw for conjunctions and disjunctions.

COVARIANCE WITH THE CONSEQUENT. *A conditional implies the result of replacing its consequent with anything that component implies.* That is, if $\psi \vDash \chi$, then $\varphi \to \psi \vDash \varphi \to \chi$.

CONTRAVARIANCE WITH THE ANTECEDENT. *A conjunction implies the result of replacing its antecedent with anything that implies that component.* That is, if $\chi \vDash \psi$, then $\psi \to \varphi \vDash \chi \to \varphi$.

CURRY'S LAW. *A conjunct of a conditional's antecedent may be made instead a condition on its consequent.* That is, $(\varphi \wedge \psi) \to \chi \simeq \varphi \to (\psi \to \chi)$.

COMPOSITIONALITY. *Conditionals are equivalent if their corresponding components are equivalent.* That is, if $\varphi \simeq \varphi'$ and $\psi \simeq \psi'$, then $\varphi \to \psi \simeq \varphi' \to \psi'$.

The asymmetry of the conditional (e.g., the fact that it is false in the second row of its table but true when the values of its components are reversed in the third) means that we would not expect it to obey a principle of commutativity. That asymmetry is also responsible for the fact that it obeys a principle of covariance for one component but contravariance for the other. It makes sense that a conditional varies in the same direction as its consequent since it's hedged assertion of that consequent. And it varies in the opposite direction from its antecedent because a condition that rules out more and will be harder to fulfill, so a commitment to the truth of the consequent will happen in fewer possibilties.

The asymmetry of the conditional also makes it no surprise that a principle of associativity does not hold because such a principle would involve several shifts between the roles of antecedent and consequent. A principle for regrouping that can be stated is here named after the logician Haskell Curry who made extensive use of an analgous operation on functions (and also directed people's attention to the analogy between certain operations on functions and principles governing conditionals). The operation on functions is sometimes called "currying," and you might think of the transition from the left to right of Curry's law as a matter of taking a pair of conditions clumped together as a conjunction in the antecedent of a conditional and combing them out into separate antecedents. The principle is also sometimes referred as an "import-export" principle because it tells us how to export a component of the antecedent to the consequent or import a component of the consequent into the antecedent. Curry's law holds because each side can be false only when both φ and ψ are true and χ is false. And this shows that both sides have the effect of hedging χ by the two conditions φ and ψ. Such a statement might then be called a *double conditional*.

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5.1.3. Doubts about truth-functionality

The account of the truth conditions of $\varphi \to \psi$ considered in the last subsection was proposed by the Greek logician Philo (who was active around 300 BCE). It was immediately subjected to criticisms by other logicians—Diodorus Cronus in particular—on the grounds that not having φ true along with ψ false is not sufficient for the truth $\varphi \to \psi$; some further connection between φ and ψ was felt to be necessary. The later report of this dispute by Sextus Empiricus contains the example

If it is day, I am conversing.

According to the table above, this is true whenever its speaker is engaged in conversation during the daytime as well as being true throughout the night under all conditions. On the other hand, according to the view of conditionals offered by Diodorus Cronus, this sentence is true at a given time only if its speaker is and always will be conversing from sunrise to sunset. If Diodorus' claim is correct, the truth of the sentence depends on more than the current truth values of its components and, since the current truth values are the only input in a truth table, no truth table is possible for a conditional as he understood it.

The controversy apparently became widespread enough in antiquity to be noticed by people other than logicians, and it has reappeared whenever the logic of conditionals has been given serious attention. In recent years, quite a bit of thought has been devoted to the issue, and a consensus may be emerging. It is widely granted that certain conditional sentences are in fact false in cases beyond those indicated in the table for $\to$. Other conditionals are held to obey the table but to carry implicatures that obscure this fact.

The clearest failures of the table occur with what are known variously as *subjunctive* or *counterfactual* conditionals. The difference in both form and content between these conditionals and ordinary *indicative* conditionals can be seen clearly in the following pair of examples (due to Ernest Adams):

If Oswald didn't shoot Kennedy, someone else did.
If Oswald hadn't shot Kennedy, someone else would have.

The first conditional, which grammarians would say is in the indicative mood, will be affirmed by anyone who knows Kennedy was shot by someone; but the second, which is in the subjunctive mood, would be asserted only by someone who believes there was a conspiracy to assassinate him (or who believes that his assassination was likely for other reasons). Notice also that the first suggests that the speaker is leaving open to question the identity of Kennedy's assassin while the second suggests the conviction that Oswald did shoot Kennedy. The antecedent of the second does not function simply as a hedge on what is claimed by the consequent; instead, it directs attention to possibilities inconsistent with what its speaker holds to be fact—in this case, possible worlds in which Oswald did not shoot Kennedy. That is the reason why conditionals like the second one are referred to as "contrary-to-fact" or counterfactual.

Now, if subjunctive conditionals are asserted primarily in cases where their antecedents are held to be false, it is clear that the table we have given is not appropriate for them. According to the table, a sentence of the form $\varphi \rightarrow \psi$ is bound to be true when its antecedent is false and therefore cannot provide any information about such cases; but subjunctive conditionals seem designed to provide information about just this sort of case.

We have to be a little careful here and remember that we can derive information from an assertion not only by considering what it implies (which is what a truth table is intended to capture) but also what it implicates. So we might consider the possibility that counterfactual conditionals really do not imply anything at all about the cases where their antecedents are false, and the information we get about such cases comes from their implicatures. But it is not hard to see that this is not so. Consider, for example, the following survey question (with X replaced by the name of a politician):

If the election were held today, would you vote for X?

This asks the respondent to evaluate the truth of the conditional If the election were held today, I would vote for X, and it makes sense to ask such a question only if a conditional like this can be false in cases where it has a false antecedent.

If the truth table above does not tell us the truth conditions of subjunctive conditionals, what are their truth conditions? A full discussion of this question would lead us outside the scope of this course, but I can outline what seems to be the most common current view. Like most good ideas, this account is hard to attribute; but two recent philosophers, Robert Stalnaker and David Lewis, did much to develop and popularize it (in slightly different versions). When evaluating the truth of a subjunctive conditional of the form If it were the case that φ, it would be the case that ψ in a given possible world, we do not limit our consideration to the truth values of φ and ψ in that world. We consider other possible worlds, too, and see whether we find φ true and ψ false in any of them. However, we do not consider all possible worlds (as we do when deciding whether φ entails ψ). Some possibilities are closer to the world in which we are evaluating the conditional than others are; and, as we broaden our horizons past a given possible world, we can move to more and more distant alternatives. When evaluating a subjunctive conditional, we extend our view just far enough to find possible worlds in which its antecedent is true and check to see whether its consequent is false in any of these. In short, a subjunctive conditional is true if its consequent is true in the nearest possible worlds in which its antecedent is true.

As an example, consider the following:

If we were in the Antarctic, we would have very cold summers.
If we were in the Antarctic, the Antarctic would have warm summers.

I take the first of these sentences to be true and the second false, because I take the nearest possibilities in which we are in the Antarctic to be ones in which it has retained its location and climate but we have traveled to it. There are, no doubt, possible worlds in which the Antarctic is a continent in the northern temperate zone (and perhaps even some in which we have stayed here and it has traveled to meet us) but they are much more distant possibilities.

This account of truth conditions of counterfactual conditionals cannot be stated in a truth table because, when judging the truth value of a subjunctive conditional in a given possible world, it forces us to consider the truth values of its components in other possible worlds. And the failure to have a truth table puts the logical properties of subjunctive or counterfactual conditionals outside the scope of this course.

But what about indicative conditionals? The argument just given to show that subjunctive conditionals do not have a truth table does not apply to indicate conditionals. However, we are still not prepared to assert indicative conditionals in all cases when Philo's table would count them as true. This can be seen by considering examples such as If Kennedy was west of the Mississippi when shot, he was shot in Texas. This sentence is true according to the table but suggests a belief on the part of the speaker that somehow ties being west of the Mississippi and Texas together in the matter of Kennedy's assassination, and it would be inappropriate for a speaker who did not have such a belief to utter the conditional. (Notice that the tie here can lie with the speaker as much as with the events. The sentence If Kennedy was west of the Mississippi when shot, he was shot in Texas would be appropriately asserted by someone who believed that Kennedy was shot while travelling in Florida and Texas but did not know the precise location.)

Still, inappropriateness as a result of false suggestions need not mean falsity through false implications, and there is reason for holding that a connection between Indiana and Texas is not implied by this example, only implicated. I hope you will grant that the following sentences are equivalent:

> If Kennedy was west of the Mississippi when shot, he was shot in Texas.
> Either Kennedy wasn't west of the Mississippi when shot or he was shot in Texas.
> Kennedy wasn't west of the Mississippi when shot without being shot in Texas.

And this suggests that the content of an indicative conditional can be captured by compounds that do have truth tables.

Indeed, the restrictions that we feel on the use of indicative conditionals are ones that can arise even if the truth table for → gives an accurate account of its truth conditions. They are found in the second and third sentences above, and the tables for ¬, ∧, and ∨ gives those sentences the truth conditions that are given to the first sentence by the table for →. Moreover, it is possible to see the restrictions on the appropriateness of indicative conditional as arising naturally from these truth conditions. A speaker who knows whether the components φ and ψ are true or false, generally ought to say so rather than assert the conditional (or disjunction or not-without form). For information about the truth values of at least one clause will usually be relevant to the conversation if the conditional is. As a result, someone who asserts only a conditional is assumed not to know the truth values of its components. But a speaker must have some basis for an assertion if it is to be appropriate. So we assume that anyone asserting a conditional is basing this assertion on some knowledge of φ and ψ that is sufficient to rule out the case where φ is true and ψ is false without settling the truth value of either φ or ψ. And this sort of knowledge concerning φ and ψ could only be knowledge of some connection between them. So assertion of a conditional will often be appropriate only when the speaker knows some connection between its two components, and the conditional will thus often carry the existence of such a connection as an implicature. An argument similar to this was one of Grice's chief applications of his idea of implicature.

We will pursue this a little further in 5.2.2 but, for now, we can say that one possible account of the indicative conditional is to say that its truth conditions and what it says or implies is captured by the truth table for → but that an the conditional suggests or implicates something more, and the content of this implicature cannot be captured by a truth table. Indeed, the corresponding subjunctive conditional often seems to roughly capture this implicature of an indicative conditional. However, it is hard to tell whether the correspondence is more than rough. Subjunctive conditionals have their own implicatures —e.g., that the antecedent is false—and these can make the comparison difficult. And the content of a subjunctive conditional depends on what possibilities are counted as nearer than others, something that can vary with the context in which a subjunctive conditional is asserted. So, while If Kennedy were west of the Mississippi when shot, he would have been shot in Texas may not seem to be an implicature of If Kennedy was west of the Mississippi when shot, he was shot in Texas, that may be because the relations among possibilities corresponding to the normal context of the first assertion are not the ones required to capture what the second implicates.

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5.1.s. Summary

1 One way to hedge a claim is to make it conditional on another one, limiting responsibility for the truth of the first claim to cases where the second is true. The English word if is used for this purpose. We will refer to a compound of this sort (and the connective used to form it) as a conditional. Its two components are distinguished as the antecedent (which expresses the condition placed on the claim and appears as a subordinate clause in English) and the consequent (which is the claim that is made conditional and appears as a main clause). Although, the two components have a different significance in the compound, they can be stated in either order in English, with the antecedent preceded by if.

The rightwards and leftwards arrows, $\rightarrow$ and $\leftarrow$, provide our signs for the if-conditional; the two components may be written in either order but the arrow should be chosen to point from the subordinate to the main clause. As English notation, we write if φ then ψ for $\varphi \rightarrow \psi$ and yes ψ if φ for $\psi \leftarrow \varphi$. When parentheses are to be used for grouping along with English for the connective itself, we can use if for $\leftarrow$ but we must resort to implies for $\rightarrow$ (understanding this to indicate material implication rather than the logical implication that is a special case of entailment).

2 In its truth table, a conditional as false only when its antecedent is true and its consequent is false. This asymmetry means that it says more as its consequent is strengthen but also as its antecedent is weakened.

3 The truth table of the conditional was first suggested in antiquity and has been controversial ever since. Current thinking distinguishes between indicative and subjunctive conditionals. The latter are held not to have truth tables (but to instead be true when their consequents are true in all the nearest worlds in which the antecedent is true). Indicative conditionals are held to have truth tables even though implicatures obscure this fact.

4 The rule of the thumb that if precedes the antecedent is the key to analyzing English conditionals, but it may not be obvious how much of the sentence is being made conditional on this antecedent. English conditionals about the future usually have antecedents in the present tense, so the tense must be changed to get an independent component with the correct meaning. When a branching conditional is stated in English, the term otherwise (which amounts to if that is not the case) is often used to state one of the antecedents.

5.1.x. Exercise questions

1. Analyze each of the following sentences in as much detail as possible.

a. If it was raining, the roads were slippery.

b. He was home if the light was on.

c. Ann and Bill helped if Carol was away

d. Sam will help—and Tom will, too, if we ask him.

e. If it was warm, they ate outside provided it didn't rain.

f. If the new project was approved, Carol started work on it and so did Dave if he was finished with the last one.

g. If he found the instructions, Tom set up the new machine; otherwise, he packed up the old one.

2. Restate each of the following forms, putting English notation into symbols and vice versa and indicating the scope of connectives in the result by underlining:

a. $A \wedge (B \rightarrow C)$ **c.** if A then both B and if C then D

b. $(A \wedge B) \rightarrow C$ **d.** both if A then B and if not A then not B

3. Synthesize idiomatic English sentences that express the propositions that are associated with the logical forms below by the intensional interpretations that follow them.

a. $\neg S \rightarrow \neg B$

S: I'll see it; B: I'll believe it

b. $S \rightarrow \neg (R \vee N)$

S: it was sunny; R: it rained; N: it snowed

c. $\neg W \leftarrow \neg (P \wedge \neg B)$

W: the set works; P: the set is plugged in; B: the set is broken

d. $\neg (A \vee B) \rightarrow (G \leftarrow \neg (C \vee D))$

A: Adams will back out; B: Brown will back out; G: the deal will go through; C: Collins will have trouble with financing; D: Davis will have trouble with financing

4. Calculate truth values for all components of the forms below on each possible extensional interpretation. Since the first two each have two unanalyzed components, there will be 4 interpretations and your table will have 4 rows of values; with three components, as in the third and fourth, there will be 8 interpretations giving 8 rows of values.

a. $(A \rightarrow B) \wedge (B \rightarrow A)$ **c.** $(A \rightarrow C) \wedge (B \rightarrow \neg C)$

b. $\neg (A \wedge B) \rightarrow (\neg B \vee A)$ **d.** $\neg (A \rightarrow C) \wedge (\neg B \rightarrow C)$

5.1.xa. Exercise answers

1. **a.** It was raining → the roads were slippery

R → S

if R then S

R: it was raining; S: the roads were slippery

b. He was home ← the light was on

H ← L

L → H

if L then H

H: he was home; L: the light was on

c. Ann and Bill helped ← Carol was away

(Ann helped ∧ Bill helped) ← Carol was away

(A ∧ B) ← C

C → (A ∧ B)

if C then both A and B

A: Ann helped; B: Bill helped; C: Carol was away

d. Sam will help ∧ Tom will help if we ask him

Sam will help ∧ (Tom will help ← we will ask Tom to help)

S ∧ (T ← A)

S ∧ (A → T)

both S and if A then T

A: we will ask Tom to help; S: Sam will help; T: Tom will help

e. it was warm → they ate outside provided it didn't rain

it was warm → (they ate outside ← it didn't rain)

it was warm → (they ate outside ← ¬ it rained)

W → (O ← ¬ R)

W → (¬ R → O)

if W then if not R then O

O: they ate outside; R: it rained; W: it was warm

f. the new project was approved → Carol started work on the new probject and so did Dave if he was finished with the last one

the new project was approved → (Carol started work on the new probject ∧ Dave started work on the new probject if he was finished with the last one)

the new project was approved → (Carol started work on the new probject ∧ (Dave started work on the new probject ←

Dave was finished with the last project))

$$A \to (C \land (D \leftarrow F))$$

$$A \to (C \land (F \to D))$$

if A then both C and if F then D

A: the new project was approved; C: Carol started work on the new probject; D: Dave started work on the new probject; F: Dave was finished with the last project

g. If he found the instructions, Tom set up the new machine ∧ if Tom didn't find the instructions, he packed up the old machine

(Tom found the instructions → Tom set up the new machine) ∧ (Tom didn't find the instructions → Tom packed up the old machine)

(Tom found the instructions → Tom set up the new machine) ∧ (¬ Tom found the instructions → Tom packed up the old machine)

$$(F \to S) \land (\neg F \to P)$$

both if F then S and if not F then P

F: Tom found the instructions; P: Tom packed up the old machine; S: Tom set up the new machine

2. **a.** both A and if B then C

b. if both A and B then C

c. $A \to (B \land (C \to D))$

d. $(A \to B) \land (\neg A \to \neg B)$

3. **a.** ¬ I'll see it → ¬ I'll believe it

I won't see it → I won't believe it

If I don't see it, I won't believe it

b. It was sunny → ¬ (it rained ∨ it snowed)

It was sunny → ¬ it rained or snowed

It was sunny → it didn't rain or snow

If it was sunny, it didn't rain or snow

c. ¬ the set works ← ¬ (the set is plugged in ∧ ¬ the set is broken)

¬ the set works ← ¬ (the set is plugged in ∧ the set isn't broken)

¬ the set works ← ¬ (the set is plugged in and isn't broken)

The set doesn't work if it isn't both plugged in and unbroken

d. ¬ (Adams will back out ∨ Brown will back out) → (the deal will go through ← ¬ (Collins will have trouble with financing ∨ Davis will have trouble with financing))

¬ Adams or Brown will back out → (the deal will go through ← ¬ (Collins or Davis will have trouble with financing))

¬ Adams or Brown will back out → (the deal will go through ← neither Collins nor Davis will have trouble with financing)

¬ Adams or Brown will back out → the deal will go through provided neither Collins nor Davis has trouble with financing

If neither Adams nor Brown backs out, the deal will go through provided neither Collins nor Davis has trouble with financing

4. Numbers below the tables indicate the order in which values were computed.

a.

A	B	(A →	B)	∧	(B →	A)
T	T		T	Ⓣ	T	
T	F		F	Ⓕ	T	
F	T		T	Ⓕ	F	
F	F		T	Ⓣ	T	
			1	2	1	

b.

A	B	¬	(A ∧ B)	→	(¬ B	∨ A)
T	T	F	T	Ⓣ	F	T
T	F	T	F	Ⓣ	T	T
F	T	T	F	Ⓕ	F	F
F	F	T	F	Ⓣ	T	T
		2	1	3	1	2

c.

A	B	C	(A	→	C)	∧	(B	→	¬	C)
T	T	T		T		Ⓕ		F	F	
T	T	F		F		Ⓕ		T	T	
T	F	T		T		Ⓣ		T	F	
T	F	F		F		Ⓕ		T	T	
F	T	T		T		Ⓕ		F	F	
F	T	F		T		Ⓣ		T	T	
F	F	T		T		Ⓣ		T	F	
F	F	F		T		Ⓣ		T	T	
				1		3		2	1	

d.

A	B	C	¬	(A	→	C)	∧	(¬	B	→	C)
T	T	T	F		T		Ⓕ	F		T	
T	T	F	T		F		Ⓣ	F		T	
T	F	T	F		T		Ⓕ	T		T	
T	F	F	T		F		Ⓕ	T		F	
F	T	T	F		T		Ⓕ	F		T	
F	T	F	F		T		Ⓕ	F		T	
F	F	T	F		T		Ⓕ	T		T	
F	F	F	F		T		Ⓕ	T		F	
			2		1		3	1		2	

5.2. Only if and unless

5.2.0. Overview

The simple conditional is one of a group of connectives whose other members can be expressed using it together with negation.

5.2.1. Only if

If the simple conditional trims the content of an unconditional assertion, a second sort of conditional offers a trimmed denial, ruling out the truth of the main clause in cases where the subordinate clause is false.

5.2.2. Necessary and sufficient conditions

The implicatures of an only-if conditional are associated with the idea of necessary conditions while the implicatures of an if conditional are associated with the idea of sufficient conditions.

5.2.3. Unless

Although it may be embellished with implicatures, the basic content of unless is provided by the phrase if not, a common dictionary definition for it.

5.2.4. Three forms compared

The implicatures associated with conditionals can make it difficult to distinguish the three conditionals but, once they are distinguished, some mnemonic devices point to their symbolic forms.

5.2.1. Only if

The bare word if is not the only way of making a conditional claim. Compare the following forecasts:

It will rain tomorrow if the front moves through.
It will rain tomorrow only if the front moves through.

The first was our original example of hedging a claim with an if-clause. The second differs in the substitution of only if for if. This makes quite a difference, though, for the second does not hedge the claim that it will rain but instead puts up a fence around it by placing a limit on the cases in which it might be true. While the first conditional leaves open some possibilities its main clause rules out, the second rules out some possibilities that its main clause leaves open. A forecaster who asserts the second sentence is committed to it *not* raining in cases where the front does not move through. That is, the force of only if is to offer a limited denial of the main clause rather than a limited assertion of it.

These considerations suggest the table below for sentences of the form ψ only if φ, sentences we will speak of as *only-if-conditionals*. This conditional form rules out ψ in cases where φ fails; that is, ψ only if φ is false only in a case where ψ is true even though φ is false. This means that, where the condition φ holds, the claim cannot go wrong. The form ψ only if φ thus provides information only about cases where φ fails and, in these, its truth value is opposite that of ψ. This is what makes it a limited denial of ψ; it rules out possibilities left open by ψ, but it rules out only those in which the condition φ does not hold. Or to put it in still other terms, it limits the truth of ψ to cases where φ is true; it does not assert ψ in those cases but excludes it in others.

φ	ψ	ψ only if φ
T	T	T
T	F	T
F	T	F
F	F	T

Diagrams of propositions may be of some help here, too. Figure 5.2.1-1 should be compared to Figure 5.1.2-1 and also to Figure 3.1.2-1. In the example we have been using, 5.2.1-1B represents the proposition expressed by The number shown by the die is less than 4 only if it is odd.

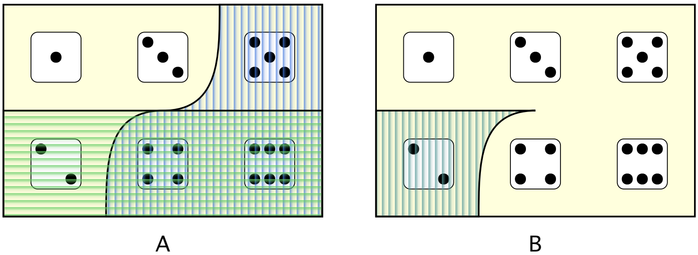


Fig. 5.2.1-1. Propositions expressed by two sentences (A) and an only-if-conditional (B) whose main clause leaves open the possibilities at the left in A.

Like the if-conditional, the only-if-conditional is a weak claim, leaving open possibilities in three of the four regions shown in Figure 5.2.1-1A; but it narrows the possibilities left open by the main clause (the area at the left in 5.2.1-1A) to those also left open by the subordinate clause. This is the reason for saying the function of an only-if-conditional is to fence in. Comparison with Figure 2.1.2-1 shows that it provides exactly the further information needed to move from the possibilities left open by the main clause ψ to the narrower range left open by $\psi \wedge \varphi$.

We will not introduce a new symbol for the connective marked by only if. A claim of the form ψ only if φ can be seen as a claim $\neg\, \psi$ hedged to be conditional on the truth of $\neg\, \varphi$; and that means we can express ψ only if φ as $\neg\, \psi \leftarrow \neg\, \varphi$. (You should check that this form has the correct table.)

Because the only-if-conditional is analyzed using the simple conditional and negation, there is no need to state further principles of implication or equivalence for it. But there is one consequence for it of the principles for the simple conditional and negation that is worth noting. While ψ if φ is covariant with its main clause ψ and contravariant with its subordinate clause φ, the conditional ψ only if φ will have the opposite relation to its component clauses (because they are negated before being combined with a simple conditional). This is in keeping with the role of ψ only if φ as a limited denial of ψ. It says more as ψ says less because a weaker claim is harder to deny; and it says more as φ says more because strengthening φ narrows the range of possibilities to which the truth of ψ has been limited. For example, The package will arrive in the next week only if you pay extra says more than does The package will arrive tomorrow only if you pay extra while The package will arrive in the next week says less than does The package will arrive tomorrow. And strengthening You will pay extra to You will pay a lot extra changes The package will arrive tomorrow only if you pay extra to the stronger The package will arrive tomorrow only if you pay a lot extra.

While the interpretation of English sentences stated using only if raises most of the same issues as if-sentences, these arise with different severity and in different ways. For example, it is possible to move an only-if-clause to the front of a sentence, but this is done only in rather formal contexts. The sentence Only if the front moves through will we have rain tomorrow is perfectly grammatical, but you would not expect it to be used by a television weather forecaster. And, while there are only-if-conditionals in the subjunctive that we must leave unanalyzed (for example, We would be able to see the eclipse only if we were near the equator), they are less common than subjunctive if-conditionals. Only-if-conditionals in English do have one special feature that is linked to the use of negations in their analysis. It is rare for any sort of conditional to be negated in English, perhaps because of the difficulty of knowing what to make of the implicatures in that sort of context. Now any conditional appearing as either component of an only-if-conditional would not negated on our analysis and, in fact, it is also very rare for a conditional to appear as a component of an only-if-conditional.

Glen Helman 07 Aug 2010

5.2.2. Necessary and sufficient conditions

Like if-conditionals, only-if-conditionals in the indicative voice carry implicatures, but their implicatures are different. This difference can be captured by the phrases *necessary condition* and *sufficient condition*. Consider the following sentences:

The match burned only if oxygen was present.
The match burned if it was struck.

Each carries, as an implicature, the suggestion of a connection between the burning of the match and some other state or event. In the first, the suggestion is that the presence of oxygen was required for the match to burn, that it was a necessary condition without which combustion could not occur. The suggestion of the second is that the striking of the match would have been enough for it to burn, that it would have been a sufficient condition. These necessary and sufficient conditions might be described as *causal*; they concern states whose absence can prevent an event from occurring or other events which are enough to bring it about.

Another kind of necessary and sufficient conditions could be described as *epistemic* since they concern grounds for reasonable belief. For example, we might say this.

If the match burned, oxygen was present.

In making this assertion, we do not mean to suggest that the burning of the match would have brought about the presence of oxygen but rather that the burning would be evidence of oxygen's presence. Combustion would give us sufficient grounds for believing that oxygen was present, so it is epistemically sufficient. On the other hand, we might say this:

The switch was thrown only if the light was on.

To see the force of this example, suppose it is known that the switch is in a different room from the light. The sentence would not suggest that the light was required for the switch to be thrown but rather that the light being on served as a test of the belief that the switch was thrown. That is, seeing that the light was not on would lead us to reject a belief that the switch was thrown, so it is an epistemically necessary condition for the belief. Epistemic conditions of both sorts are sometimes referred to as *signs* or *marks*.

Now, statements of necessary and sufficient conditions can themselves be understood as connectives, ones that we might express more explicitly in the following way:

The truth of φ is a necessary condition for the truth of ψ
The truth of φ is a sufficient condition for the truth of ψ.

A compound of either of these forms is plainly not truth-functional. Knowing, for example, that φ and ψ are both true will not tell us whether either is a necessary or a sufficient condition for the other. So necessary and sufficient conditions are not strictly within our purview. But, since they attach to indicative if- and only-if-conditionals as implicatures, we need to be aware of them because they can make certain ways of restating such conditionals more natural than others.

When checking that the form $\neg \psi \leftarrow \neg \varphi$ has that same truth table as ψ only if φ, you may have noticed that the simpler form $\psi \rightarrow \varphi$ also has the same table. This might suggest that as a first step in analyzing ψ only if φ we could rephrase it as If ψ then φ. However, to do so would often wreak such havoc on the implicatures that the paraphrase would sound crazy. In saying ψ only if φ, we suggest that the truth of φ is a necessary condition for the truth of ψ while in saying If ψ then φ, we suggest that the truth of ψ is a sufficient condition for the truth of φ. And sufficiency and necessity are not simple converse relations like *parent of* and *child of*.

As we saw in the examples above, a causally sufficient condition for an event may have the event as an epistemically necessary condition, and a causally necessary condition may have the event as an epistemically sufficient condition. However, in making such shifts we are changing the meaning of a sentence in a noticeable way, so a paraphrase of ψ only if φ by If ψ then φ will be at best awkward. This awkwardness becomes especially severe in the case of conditionals concerning the future, where causal and epistemic conditions tend to coincide. A meteorologist would certainly not be prepared to use the following interchangeably:

It will rain tomorrow only if the front moves through.
If it rains tomorrow, the front will move through.

We could do a bit better in this case by adjusting tenses to get If it rains tomorrow, then front will have moved through, but we would still have shifted from causal to epistemic implicatures.

The analysis only-if-conditionals that we do employ amounts to a paraphrase of ψ only if φ by It's not the case that ψ if it's not the case that φ. And this paraphrase tends to avoid such problems with implicatures. But it only *tends* to avoid them because our description of the implicatures of if- and only-if-conditionals in terms of necessary and sufficient conditions is still an oversimplified account of the relation between them.

For example, I might express my conviction that the temperature is high by using the sentence It's under 80° only if it's over 75°. Here the paraphrase If it's under 80°, it's over 75° works well even though it is the sort of paraphrase that failed in earlier examples; and a paraphrase of the sort we used in those examples—namely, If it isn't over 75°, then it isn't under 80°—sounds at least odd. The oddity here can be explained in a way that suggests it does not point to a widespread problem. It being over 75° could be a necessary condition for it being under 80° only if we take it for granted that it is hot. And the point of the initial sentence is more to commit the speaker to this presumption than to suggest the existence of a necessary condition. But the sentence If it isn't over 75°, then it isn't under 80° cannot play this role since it pointedly leaves open just the sort of case whose failure the original sentence is designed to suggest.

This sort of example shows that the implicatures of if- and only-if-conditionals can be sufficiently independent that the latter cannot be expressed in terms of the former. However, if we paraphrase using negation (rather than reversing main and subordinate clauses), the difference in implicatures will usually not be too great. The moral for our purposes is then that a paraphrase of ψ only if φ by $\neg\,\psi$ if $\neg\,\varphi$ will usually not be too jarring though if ψ then φ may be better in a few cases.

There is a final complication in dealing with if and only if that it is also a result of their implicatures. Conditionals of the two sorts can often be difficult to distinguish because a conditional of one sort carries a conditional of the other sort as an implicature. For example, imagine I were speaking of a farm in a year when corn yields have been affected by drought. If I were to assert the sentence

They will make a profit only if they get over $3.75 a bushel,

I would be understood to believe not only that this price was necessary for a profit but also that it was sufficient, and it seems that I would agree with the following:

They will make a profit if they get over $3.75 a bushel.

But this is only an implicature and, unlike the suggestion that the price is a necessary condition for making a profit, the suggestion that it is also sufficient is one that is easily canceled. If I wanted to avoid the implicature, I might have used the sentence

They will make a profit only if they get over $3.75 a bushel, and even that might not be enough

and I would not have contradicted myself by saying this.

Moreover, the implicature of an if-conditional by an only-if-conditional, or vice versa, does not always arise. We would usually take the forecast It will rain tomorrow only if the front moves through to suggest that the passing of the front would produce rain; but during a severe drought, when rain seems very unlikely, a forecaster might not need to add the canceling clause and it might stay dry even if the front does move through. So, while implicatures may conceal the difference between them, ψ if φ and ψ only if φ really are different in content from each other.

This means that the assertion of both conditionals, as in the form ψ if and only if φ, is not redundant. This sort of compound is known as the *biconditional*. Its analysis would lead us to the form

$$(\psi \leftarrow \varphi) \land (\neg\, \psi \leftarrow \neg\, \varphi)$$

or, with rightwards arrows,

$$(\varphi \rightarrow \psi) \land (\neg\, \varphi \rightarrow \neg\, \psi)$$

Biconditionals appear often in definitions, and calculating the truth table for this form will show why. A biconditional is true when the components φ and ψ are both true and also when they are both false, so this form enables us to say that two sentences have the same truth value without saying what that value is.

Glen Helman 03 Aug 2010

5.2.3. Unless

Yet another sort of conditional appears in this example:

They have run out of food unless they received new supplies.

Here the main clause is hedged, but in a different way than if the subordinate clause were introduced by if. The speaker's intent is to leave open some cases where the main clause fails (where they still have food) but to limit this failure to the sort of situation described in the subordinate clause. We can compare the function of unless to the function of only if by paraphrasing the sentence above as

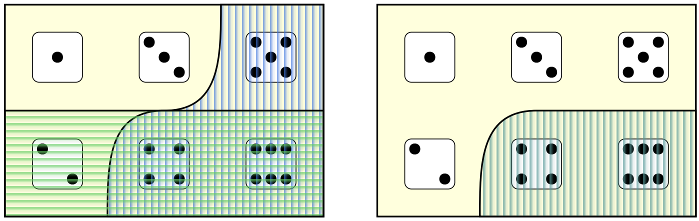
They still have food only if they received new supplies.

The second sentence limits the truth of They still have food to cases where The received new supplies is true. So it asserts the truth of They have run out of food with the possible exception of such cases. Similarly, the first sentence asserts They have run out of food but hedges this by allowing the exception expressed by the subordinate clause.

So, like an only-if-conditional, an *unless-conditional* is automatically true in cases where the subordinate clause is true; but unlike an only-if-conditional its truth value is the same as the main clause in cases where the subordinate clause is false. That is, the form ψ unless φ has the table below.

φ	ψ	ψ unless φ
T	T	T
T	F	T
F	T	T
F	F	F

This account of truth conditions appears also in Figure 5.2.3-1. Continuing the example of these diagrams, 5.2.3-1B represents the proposition expressed by The number shown by the die is less than 4 unless it is odd.



A B

Fig. 5.2.3-1. Propositions expressed by two sentences (A) and an unless-conditional (B) whose main clause rules out the possibilities at the right in A.

There are two ways of describing the proposition on the right. First of all, it fences in the failure of the main clause. That is, it rules out some of the possibilities in which the main clause ψ fails, those that are ruled out by the subordinate clause φ. This is to see the conditional as the proposition expressed by $\neg\,\psi$ only if φ. But the possibilities left open by the denial of the main clause are those ruled out by the main clause itself. So the conditional can be seen also to whittle down the possibilities ruled out by the main clause to those left open by the denial of the subordinate clause. And this is to see the conditional as the hedge of the main clause expressed by ψ if $\neg\,\varphi$.

The same restatements appear if we trace our way back to if-conditionals in order to get a way of expressing this conditional symbolically. The form ψ unless φ amounts to $\neg\,\psi$ only if φ and we are treating the latter as $\neg\,\neg\,\psi \leftarrow \neg\,\varphi$. If we use the principle of double negation to simplify this last expression, we get $\psi \leftarrow \neg\,\varphi$ as a rendering of ψ unless φ. The corresponding English paraphrase of ψ unless φ as ψ if it is not the case that φ is usually pretty good (good enough that if not is a common dictionary definition of unless).

The negation used to analyze the subordinate clause of only-if-conditionals means that they are covariant with both their clauses. That will be no surprise if you have noticed that they have the same truth conditions as disjunctions, but it is also to be expected if it is regarded as an assertion of the main clause with the subordinate clause as a possible exception. Such a claim will say more as the main clause says more, and it will say more also as the subordinate clause says more because a narrower exception will apply in fewer cases.

There are enough steps in the path from unless to $\leftarrow\,\neg$ to justify a fear that the implicatures are not all in order when we arrive, but this account of unless works better than using or. How far the synonymy of unless and or extends beyond truth conditions can be seen by considering a few examples. We might paraphrase the example above as

Either they have run out of food, or they received new supplies

and we would do so with reasonable success. But things do not work out as well in other cases, particularly with unless-conditionals concerning the future. The following two sentences have quite different implicatures:

We'll run out of gas unless we get to a town soon.
We'll either run out of gas or get to a town soon.

Disjunction is not symmetric when it comes to an implicated connection between its two components, and we could paraphrase the first sentence better by We'll either get to a town soon or run out of gas, but the need to change the order of the clauses reduces the advantages of or over if not as a paraphrase of unless.

The remaining issues regarding unless pretty well parallel those concerning if and only if. It is possible to find an unless-clause at the front of a sentence (e.g., Unless we get to town soon, we'll run out of gas). And the form ψ unless φ has, in addition to its core implicature that the truth of φ is necessary for the falsity of ψ, a secondary and easily canceled implicature of sufficiency. In our initial example (They have run out of food unless they received new supplies), this secondary implicature is rather weak if it is present at all, so there might be no need to add the canceling clause and they might have run out even if they got them. But, in other cases, the implicature is stronger. For example, in We'll go unless it rains, we would have to add and we might go even if it does if we did not want to suggest that rain would be enough to keep us from going.

Glen Helman 06 Oct 2010

5.2.4. Three forms compared

Before going on to work through some sample analyses, let us bring together the key points about the three connectives:

English forms	Symbolic analyses	Truth conditions	Core implicatures	Secondary implicatures
ψ if φ if φ, ψ	ψ ← φ φ → ψ	same value as ψ when φ is **T**; otherwise **T**	φ is sufficient for ψ's truth	φ is necessary for ψ's truth
ψ only if φ	¬ ψ ← ¬ φ ¬ φ → ¬ ψ	opposite value to ψ when φ is **F**; otherwise **T**	φ is necessary for ψ's truth	φ is sufficient for ψ's truth
ψ unless φ unless φ, ψ	ψ ← ¬ φ ¬ φ → ψ	same value as ψ when φ is **F**; otherwise **T**	φ is necessary for ψ's failure	φ is sufficient for ψ's failure

The core implicatures are the ones that can make an indicative conditional seem non-truth-functional. The secondary implicatures are the ones that can make it difficult to distinguish between different kinds of conditional. The latter implicatures are easily canceled.

It may help, when trying to recall the symbolic analysis of only if, that in response to the question Did they finish? the answers Only if the parts arrived and Not unless the parts arrived come to pretty much the same thing (give or take a few implicatures). Combining this idea with the paraphrase of unless as only if, we get the formula not if not for only if—that is, ψ only if φ amounts to Not ψ if not φ or ¬ ψ ← ¬ φ.

Here are some examples involving only if and unless.

If Dave didn't show up, they moved the piano only if it was a small one

Dave didn't show up → they moved the piano only if it was a small one

¬ Dave showed up → (¬ they moved the piano ← ¬ the piano was a small one)

¬ D → (¬ M ← ¬ S)

¬ D → (¬ S → ¬ M)

if not D then if not S then not M

D: Dave showed up; S: the piano was a small one; M: they moved the piano

Mike didn't hear from either Sue or Tom unless a call came through late
Mike didn't hear from either Sue or Tom ← ¬ a call came through late
¬ Mike heard from either Sue or Tom ← ¬ a call came through late
¬ (Mike heard from Sue ∨ Mike heard from Tom) ← ¬ a call came through late

$$\neg (S \vee T) \leftarrow \neg L$$
$$\neg L \rightarrow \neg (S \vee T)$$

if not L then not either S or T

L: a call came through late; S: Mike heard from Sue; T: Mike heard from Tom

Notice that the form assigned to the second example would do as well for Mike heard from either Sue or Tom only if a call came through late, a sentence that is a fair paraphrase of the one we analyzed. The first example shares its form with Unless Dave showed up, they moved the piano only if it was a small one, also a reasonable paraphrase.

In general, the forms marked by unless conditionals can also be expressed by simple conditionals, and the form marked by an only-if conditional can be expressed by any of the three English forms. That means that there can be a number of different ways of synthesizing an English sentence with a given form. For example, the truth conditions of the analyzed sentence

¬ they ate outside ← ¬ it was warm

can be expressed by any of the following:

They didn't eat outside if it wasn't warm.
They didn't eat outside unless it was warm.
They ate outside only if it was warm.

And the differences among implicatures in this case are limited enough that these sentences would be equally appropriate in many situations.

5.2.s. Summary

1 The simple if-conditional is not the only conditional in English. The phrase only if is used to mark a compound which limits the possibilities for the truth of its main clause. It does this by asserting a denial of the main clause that is conditional on the failure of the subordinate clause, so it can be thought of as a hedged denial. As this suggests, the only-if-conditional can be paraphrased using the if-conditional and negation, with ψ only if φ expressed symbolically as $\neg \psi \leftarrow \neg \varphi$.

2 Like the if-conditional the only-if-conditional has implicatures. It suggests that the truth of its subordinate clause is a necessary condition for the truth of its main clause (while the if-conditional suggests that the truth of the subordinate clause is a sufficient condition). There is a secondary implicature of each conditional in which it suggests the truth of the other conditional, and this can make each seem to say that same thing as a conjunction of the two, a compound known as a biconditional. However, these secondary implicatures are easily canceled. The biconditional ψ if and only if φ can be expressed symbolically as $(\psi \leftarrow \varphi) \wedge (\neg \psi \leftarrow \neg \varphi)$, or $(\varphi \rightarrow \psi) \wedge (\neg \varphi \rightarrow \neg \psi)$ when arrows are reversed.

3 A third sort of conditional is marked by the English word unless. It hedges the main clause by asserting a limitation on the possibility of its failure, saying this can happen only when the subordinate clause is true. The effect is to assert the main clause conditional on the denial of the subordinate clause, and the unless-conditional can be stated using the if-conditional and negation, with ψ unless φ expressed as $\psi \leftarrow \neg \varphi$. Like the other two conditionals, the unless-conditional carries implicatures, both core implicatures and easily canceled secondary ones.

4 The symbolic analyses of the conditionals can be captured by the rough formulas: only if = not unless (i.e., ψ only if φ = not ψ unless φ) and unless = if not. In these terms, only if = not if not.

Glen Helman 03 Aug 2010

5.2.x. Exercise questions

1. Analyze each of the following sentences in as much detail as possible.

a. Tom was late unless he left early.

b. You'll get a good picture only if you take the cap off the lens.

c. Neither Ann nor Bill knew of it unless they both did.

d. The bill will pass if the chairman supports it—unless public opinion runs heavily against it.

e. Unless Ed is late, we'll get started on time and finish early if there isn't a lot of business.

f. If Bob was under no obligation to help, he worked only if he was in a good mood and had nothing to do.

2. Synthesize idiomatic English sentences that express the propositions that are associated with the logical forms below by the intensional interpretations that follow them. These repeat **3 a**, **c**, and **d** of 5.1.x. This time, you should look for ways of stating the sentences using only if and unless.

a. $\neg S \rightarrow \neg B$

S: I'll see it; B: I'll believe it

b. $\neg W \leftarrow \neg (P \wedge \neg B)$

W: the set works; P: the set is plugged in; B: the set is broken

c. $\neg (A \vee B) \rightarrow (G \leftarrow \neg (C \vee D))$

A: Adams will back out; B: Brown will back out; G: the deal will go through; C: Collins will have trouble with financing; D: Davis will have trouble with financing

5.2.xa. Exercise answers

1. **a.** Tom was late ← ¬ Tom left early

$$L \leftarrow \neg E$$
$$\neg E \rightarrow L$$
if not E then L

E: Tom left early; L: Tom was late

b. ¬ you'll get a good picture ← ¬ you'll take the cap off the lens

$$\neg P \leftarrow \neg C$$
$$\neg C \rightarrow \neg P$$
if not C then not P

C: you'll take the cap off the lens; P: you'll get a good picture

c. Neither Ann nor Bill knew of it ← ¬ Ann and Bill both knew of it

¬ (Ann knew of it ∨ Bill knew of it) ← ¬ (Ann knew of it ∧ Bill knew of it)

$$\neg (A \lor B) \leftarrow \neg (A \land B)$$
$$\neg (A \land B) \rightarrow \neg (A \lor B)$$
if not both A and B then not either A or B

A: Ann knew of it; B: Bill knew of it

d. The bill will pass if the chairman supports it ← ¬ public opinion will run heavily against the bill

(the bill will pass ← the chairman will support the bill) ← ¬ A

$$(P \leftarrow S) \leftarrow \neg A$$
$$\neg A \rightarrow (S \rightarrow P)$$
if not A then if S then P

A: public opinion will run heavily against the bill; P: the bill will pass; S: the chairman will support the bill

e. ¬ Ed will be late → we'll get started on time and finish early if there isn't a lot of business

¬ L → (we'll get started on time ∧ we'll finish early if there isn't a lot of business)

¬ L → (T ∧ (we'll finish early ← there won't be a lot of business))

¬ L → (T ∧ (F ← ¬ there will be a lot of business))

$$\neg L \rightarrow (T \wedge (F \leftarrow \neg B))$$
$$\neg L \rightarrow (T \wedge (\neg B \rightarrow F))$$

if not L then both T and if not B then F

B: there will be a lot of business; F: we'll finish early; L: Ed will be late; T: we'll get started on time

It would be possible to understand the sentence to make the whole of we'll get started on time and finish early conditional on there won't be a lot of business. On that interpretation, the form would be ¬ L → (¬ B → (T ∧ F)). However, the interpretation used above fits better with common sense expectations concerning the content, and those are often the grounds on which ambiguous sentences are understood in a particular way.

f. Bob was under no obligation to help → Bob worked only if he was in a good mood and had nothing to do

¬ Bob was under an obligation to help → (¬ Bob worked ← ¬ Bob was in a good mood and had nothing to do)

¬ O→ (¬ W ← ¬ (Bob was in a good mood ∧ Bob had nothing to do))

¬ O→ (¬ W ← ¬ (G ∧ ¬ Bob had something to do))

$$\neg O \rightarrow (\neg W \leftarrow \neg (G \wedge \neg S))$$
$$\neg O \rightarrow (\neg (G \wedge \neg S) \rightarrow \neg W)$$

if not O then if not both G and not S then not W

O: Bob was under an obligation to help; G: Bob was in a good mood; S: Bob had something to do; W: Bob worked

2. a. ¬ I'll see it → ¬ I'll believe it

Unless I see it, I won't believe it

or: I'll believe it only if I see it

b. ¬ the set works ← ¬ (the set is plugged in ∧ ¬ the set is broken)

¬ the set works ← ¬ (the set is plugged in ∧ the set isn’t broken)

¬ the set works ← ¬ (the set is plugged in and isn’t broken)

The set works only if it is plugged in and isn’t broken

or: The set doesn’t work unless it is plugged in and isn’t broken

c. ¬ (Adams will back out ∨ Brown will back out) → (the deal will go through ← ¬ (Collins will have trouble with financing ∨ Davis will have trouble with financing))

¬ Adams or Brown will back out → (the deal will go through ← ¬ Collins or Davis will have trouble with financing)

Unless Adams or Brown backs out, the deal will go through if neither Collins nor Davis has trouble with financing

or: If neither Adams nor Brown backs out, the deal will go through unless Collins or Davis has trouble with financing

or: Unless Adams nor Brown backs out, the deal will go through unless Collins or Davis has trouble with financing

5.3. Conditional proofs: bottling inference

5.3.0. Overview

The use of implies for both the conditional and entailment suggests an analogy between the two, and this analogy figures in many of the deductive properties of conditionals.

5.3.1. Conditionalization

The basic grounds for concluding a conditional are the demonstrated ability to move from its antecedent as an assumption to its consequent as a goal.

5.3.2. Detachment

The chief significance of having a conditional as premise is the power to move from its antecedent as a resource to its consequent as a further resource.

5.3.1. Conditionalization

The truth conditions of the conditional, which count $\varphi \to \psi$ as true except when φ is **T** and ψ is **F**, may have reminded you of the definition of implication, which says that φ implies ψ if and only if there is no possible world in which φ is **T** and ψ is **F**. Even though similar, the two ideas are not the same, and the distinction between material implication on the one hand and logical implication on the other points to the difference between them. Saying that a conditional $\varphi \to \psi$ is true rules out only the actual occurrence of the values **T** for φ and **F** for ψ while saying that φ logically implies, or entails, ψ rules out the occurrence of this pattern in any possible world. The forecast It will rain tomorrow if the front moves through does not commit a meteorologist to the view that It will rain tomorrow is logically implied by The front will move through tomorrow.

This difference can be brought out in another way. In cases where a relation of entailment holds, the corresponding conditional is not only true but tautologous. For example, because It was hot and humid $\vDash$ It was hot, the conditional If it was hot and humid, it was hot tells us nothing; it is a tautology. And we can state this as a general principle: φ entails ψ if and only if $\varphi \to \psi$ is a tautology—in notation, $\varphi \vDash \psi$ if and only if $\vDash \varphi \to \psi$. Either way we are saying that we fail to have φ true and ψ false not merely in the actual world but in all possible worlds.

Since to be a tautology is to be a valid conclusion from no premises at all, the principle just stated provides a partial account of when a conditional is a valid conclusion. To cover cases where there are premises we can use the idea of *implication given* a set of additional premises. For example, a weather forecaster might say that the passing of a front "implies" rain, intending to rest this relation between the passing of the front and rain on certain assumptions about the conditions of the atmosphere and laws of meteorology. And when a scientific hypothesis is said to "imply" a certain result for an experimental test, this implication is based on certain assumptions about the behavior of the experimental set up. In such cases we say that a sentence ψ cannot be false when a sentence φ is true, provided that certain further assumptions Γ are true as well. But this is just to say that ψ is entailed by φ taken together with Γ—i.e., that $\Gamma, \varphi \vDash \psi$. So conditional implication is really just entailment with one premise singled out for special attention, something that it is quite reasonable to do when, as in the examples above, the set Γ of further premises is large or lacks definite boundaries.

Another way of separating one assumption from a group of others is to make the conclusion conditional upon it. For example, we might say that, based on certain assumptions about the weather, we can conclude that it will rain if the front passes or that, based on assumptions about the experimental set up, we can conclude that an experiment will yield a certain result if our hypothesis is true. But this way of giving special attention to one of a group of assumptions is equivalent to making a claim of conditional implication—that is, a conditional is a valid conclusion from given premises if and only if its antecedent implies its conclusion given those premises. And this gives us our account of conditional conclusions:

LAW FOR THE CONDITIONAL AS A CONCLUSION. $\Gamma \vDash \varphi \to \psi$ if and only if $\Gamma, \varphi \vDash \psi$ (for any set Γ and any sentences φ and ψ).

To see the truth of this law, note that an entailment $\Gamma \vDash \varphi \to \psi$ will hold if and only if there is no possible world in which $\varphi \to \psi$ is false while all members of Γ are true. But the sort of possible world that this rules out is one in which ψ is false while φ and the members of Γ are all true—i.e., one which divides the argument Γ, φ / ψ. And to rule out such a possibility is to say that $\Gamma, \varphi \vDash \psi$.

Reading the law above from right to left, we move a premise past the sign $\vDash$, making the conclusion conditional on it. We will use the term *conditionalization* for this operation. Any result of the process is a *conditionalization of* the argument, and we will sometimes say, more specifically, that it is a *conditionalization on* the premise that is moved.

The law for the conditional as a conclusion tells us that an argument Γ / $\varphi \to \psi$ is valid if and only if the argument Γ, φ / ψ is valid. Moving from the first argument to the second will lead us to consider the latter argument in cases where we do not know the premise φ to be true. In such cases, Γ, φ / ψ will be an argument concerning a hypothetical situation, a hypothetical argument in the sense introduced in 4.2.2. Modifying an example used there, we can see the validity of the argument at the left below by noting the validity of the one at the right.

Ann and Bill were not both home without the car being in the driveway The car was not in the driveway	Ann and Bill were not both home without the car being in the driveway The car was not in the driveway Ann was at home
If Ann was at home, Bill wasn't	Bill wasn't at home

The first argument is a conditionalization of the second, and the law for the conditional as a conclusion tells that the first is valid if and only if the second is. Someone who offers the first argument is unlikely to know whether or not

Ann was at home because there would then be no reason to assert a merely conditional conclusion. Consequently, Ann was at home describes a situation the arguer will regard as hypothetical, and the second argument can be described as a hypothetical argument. This means that we establish conditionals the way we established disjunctions in the last chapter, as compounds that serve to state categorically the upshot of a hypothetical argument.

In derivations, we can plan for a goal that is a conditional by setting out to reach it by a hypothetical argument. The rule embodying this approach, *Conditional Proof* (CP), is shown in Figure 5.3.1-1.

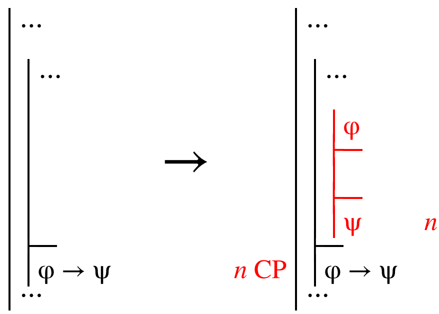


Fig. 5.3.1-1. Developing a derivation by planning for a conditional at stage *n*.

When we apply CP, we add the antecedent of the conditional goal as a supposition and set its consequent as a new goal. We thus plan to carry out, in a vertical direction, the transition indicated by the arrow in the conditional.

As an example, here is a derivation for the argument above.

	¬ ((A ∧ B) ∧ ¬ C)	2
	¬ C	(2)
	A	(3)
2 MPT	¬ (A ∧ B)	3
3 MPT	¬ B	(4)
	●	
4 QED	¬ B	1
1 CP	A → ¬ B	

Notice that the proximate argument of the gap after CP is applied is ¬ ((A ∧ B) ∧ ¬ C), ¬ C, A / ¬ B. That is, the ultimate argument of the derivation is a conditionalization on A of the proximate argument that results from CP. In short, when we apply CP, we plan to put ourselves in a position to conditionalize.

Of course, whenever we have premises, we are in a position to conditionalize, and the validity of the argument we have just considered establishes the validity of the result of conditionalization on its second premise: ¬ ((A ∧ B) ∧ ¬ C) / ¬ C → (A → ¬ B). This argument might be put into English as follows:

Ann and Bill were not both home without the car being in the driveway

---

Unless the car was in the driveway, Bill wasn't home if Ann was

A derivation for it will incorporate the derivation above, preceded by an initial use of CP.

	¬ ((A ∧ B) ∧ ¬ C)	3
	¬ C	(3)
	A	(4)
3 MPT	¬ (A ∧ B)	4
4 MPT	¬ B	(5)
	●	
5 QED	¬ B	2
2 CP	A → ¬ B	1
1 CP	¬ C → (A → ¬ B)	

After stage 2, we are making two suppositions—that the car is not in the driveway and that Ann is home—and we are thus considering a situation that is doubly hypothetical. And, in general, the most natural way of establishing the validity of a doubly conditional conclusion is by way of such a doubly hypothetical argument.

5.3.2. Detachment

The conditional was described by the philosopher Gilbert Ryle (1900-1976) as an *inference ticket*: it confers the right to travel from its antecedent to its consequent in an inference. It is the ability to make this trip that we demonstrate when we use a hypothetical argument to show that a conditional conclusion is valid. It is also true that, when we have a conditional as a resource, we have a ticket we can use to travel from its antecedent to its consequent.

The pattern of argument employing the latter idea, traditionally known as *modus ponens*, is perhaps the most well-known logical principle. The following instance of it was used by the Stoics as their standard example:

If it is day, it is light
It is day

It is light

The hedged character of the conditional means that, like disjunctions and not-both forms, it has no definite implications concerning the truth value of either of its components. *Modus ponens* tells us that if we add to the conditional the information that its antecedent is true, we can detach the consequent and assert it categorically.

In the traditional system of terminology we used for other detachment principles, this pattern of argument deserves the name *modus ponendo ponens*, and the more common form *modus ponens* is an abbreviated form of this. As was the case with disjunction and the not-both form, we have a pair of detachment principles for the conditional. However, due to the asymmetry of the conditional, these two principles take different forms and have different names:

MODUS PONENDO PONENS. $\varphi \to \psi, \varphi \vDash \psi$ (for any sentences φ and ψ).

MODUS TOLLENDO TOLLENS. $\varphi \to \psi, \neg^{\pm} \psi \vDash \neg^{\pm} \varphi$ (for any sentences φ and ψ).

The second is most often known by the abbreviated name *modus tollens*.

Notice that the conditional premise is used in very different ways in these two arguments. Often people who can agree about the truth of a conditional will disagree of the truth values of its components and will be ready to follow the different paths from the conditional that are laid out by these two principles, something that is reflected in the proverb One person's modus ponens is another person's modus tollens. Ann and Bill may agree that it will rain if the front moves through while Ann, who is convinced that the front will move through, concludes that it will rain and Bill, who is convinced that it will not rain, concludes that the front will not move through.

Also as was the case with the weak compounds considered in the last two chapters, there are weakening principles for the conditional; but again we have two different forms:

WEAKENING: $\psi \vDash \varphi \to \psi$ and $\neg^{\pm}\, \varphi \vDash \varphi \to \psi$ (for any sentences φ and ψ).

Although these weakening principles can be used directly as attachment rules (and we will consider this use in 5.4.2), their most important function is to combine with the detachment principles for the conditional and the law of lemmas to support the detachment rules *Modus Ponendo Ponens* (MPP) and *Modus Tollendo Tollens* (MTT) shown in Figures 5.3.4-1 and 5.3.4-2.

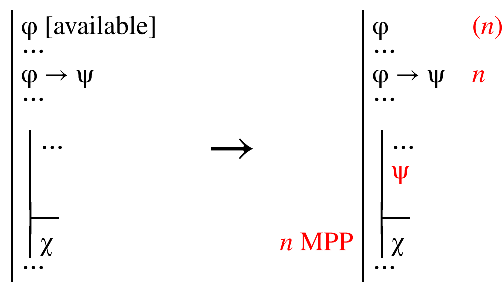


Fig. 5.3.2-1. Developing a derivation at stage *n* by exploiting a conditional whose antecedent is also an active resource.

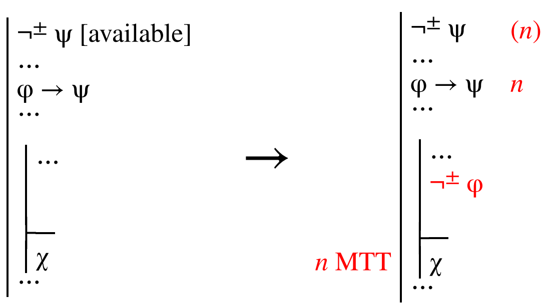
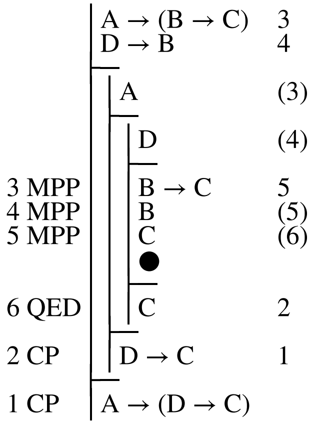


Fig. 5.3.2-2. Developing a derivation at stage *n* by exploiting a conditional when a sentence negating or de-negating its consequent is also an active resource.

The following example is typical of the way *modus ponens* functions along with CP.



This can be described, very roughly, as a process of cashing in some tickets in order to get a new one with a different itinerary. One of the respects in which this metaphor works only roughly is that the “point of departure” or “destination” are sometimes themselves indicated by conditionals—that is, the “ticket” in question is sometimes more like a voucher for a ticket or some other sort of more abstract right.

Glen Helman 03 Aug 2010

5.3.s. Summary

1 The truth conditions of the conditional recall the definition of implication. Indeed, an implication $\varphi \vDash \psi$ will hold if and only if the conditional $\varphi \rightarrow \psi$ is a tautology. We can apply similar ideas to conditionals that are conclusions from factual premises by considering a notion of conditional implication, implication depending on factual information. This idea appears in our law for the conditional as a conclusion. An entailment $\Gamma \vDash \varphi \rightarrow \psi$ holds when $\Gamma, \varphi \vDash \psi$—i.e., when ψ is implied by φ given the further premises Γ. The first of these entailments is a conditionalization of the second, and the second asserts the validity of a hypothetical argument. So an argument with a conditional conclusion is valid if and only if the hypothetical argument it conditionalizes is also valid. The derivation rule implementing this idea is Conditional Proof (CP).

2 The detachment principles for the conditional include the well-known *modus ponendo ponens* (usually called *modus ponens*), which is implemented as a rule Modus Ponendo Ponens (MPP), and a second detachment principle *modus tollendo tollens* (usually called *modus tollens*), which is implemented as a rule Modus Tollendo Tollens (MTT). *Modus ponens* in particular can be understood as the use of a conditional as an inference ticket licensing transitions from its antecedent to its consequent.

5.3.x. Exercise questions

1. Use derivations to establish each of the following. Notice that several are claims of equivalence and require two derivations. All these derivations are designed for the use of detachment rules (especially MPP and MTT), and a number will be quite long if they are not used. Attachment rules from previous chapters will occasionally be useful, and (since we do not yet have a full set of rules for the conditional) they are required in one of the derivations for **k**. Finally, note the leftwards arrow in the second premise of **b**. Although rules like MPP are written using a rightwards arrow they also apply to conditionals written using a leftwards arrow since a conditional $\psi \leftarrow \varphi$ is just an alternative way of writing $\varphi \rightarrow \psi$ and plays the same role in derivations.

a. $B \rightarrow C, A \rightarrow B \vDash A \rightarrow C$

b. $A \rightarrow B, C \leftarrow B, C \rightarrow D \vDash A \rightarrow D$

c. $A \rightarrow (B \rightarrow C) \vDash (A \rightarrow B) \rightarrow (A \rightarrow C)$

d. $A \rightarrow (B \rightarrow C), A \rightarrow \neg C \vDash B \rightarrow \neg A$

e. $\neg A \simeq A \rightarrow \neg A$

f. $A \rightarrow B \simeq \neg B \rightarrow \neg A$

g. $A \rightarrow B \simeq \neg (A \wedge \neg B)$

h. $A \rightarrow (B \rightarrow C) \simeq (A \wedge B) \rightarrow C$

i. $(A \rightarrow B) \wedge (A \rightarrow C) \simeq A \rightarrow (B \wedge C)$

j. $(A \rightarrow C) \wedge (B \rightarrow C) \simeq (A \vee B) \rightarrow C$

k. $(A \rightarrow B) \wedge (B \rightarrow C) \simeq (A \vee B) \rightarrow (B \wedge C)$

2. Give English sentences illustrating **d**, **f**, **g**, and **k** of **1**. (Notice that **k** tells how to restate a particular sort of conjunction of conditionals, one that might be called a *linked conditional*.)

The exercise machine is not designed to produce exercises and answers involving only the limited set of rules you have at this point.

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5.3.xa. Exercise answers

1.

a.

```
        │ B → C   3
        │ A → B   2
        ├─
        │ │ A       (2)
        │ ├─
2 MPP   │ │ B       (3)
3 MPP   │ │ C       (4)
        │ │ ●
        │ ├─
4 QED   │ │ C       1
        ├─
1 PC    │ A → C
```

b.

```
        │ A → B   2
        │ C ← B   3
        │ C → D   4
        ├─
        │ │ A       (2)
        │ ├─
2 MPP   │ │ B       (3)
3 MPP   │ │ C       (4)
4 MPP   │ │ D       (5)
        │ │ ●
        │ ├─
5 QED   │ │ D       1
        ├─
1 CP    │ A → D
```

c.

```
        │ A → (B → C)          3
        ├─
        │ │ A → B              4
        │ ├─
        │ │ │ A                (3),(4)
        │ │ ├─
3 MPP   │ │ │ B → C            5
4 MPP   │ │ │ B                (5)
5 MPP   │ │ │ C                (6)
        │ │ │ ●
        │ │ ├─
6 QED   │ │ │ C                2
        │ ├─
2 CP    │ │ A → C              1
        ├─
1 CP    │ (A → B) → (A → C)
```

d.

```
        │ A → (B → C)   3
        │ A → ¬ C       4
        ├─
        │ │ B           (5)
        │ ├─
        │ │ │ A         (3),(4)
        │ │ ├─
3 MPP   │ │ │ B → C     5
4 MPP   │ │ │ ¬ C       (6)
5 MPP   │ │ │ C         (6)
        │ │ │ ●
        │ │ ├─
6 Nc    │ │ │ ⊥         2
        │ ├─
2 RAA   │ │ ¬ A         1
        ├─
1 CP    │ B → ¬ A
```

e.

```
        │ ¬ A         (2)
        ├─
        │ │ A
        │ ├─
        │ │ ●
        │ ├─
2 QED   │ │ ¬ A       1
        ├─
1 CP    │ A → ¬ A
```

```
        │ A → ¬ A   2
        ├─
        │ │ A       (2),(3)
        │ ├─
2 MPP   │ │ ¬ A     (3)
        │ │ ●
        │ ├─
3 Nc    │ │ ⊥       1
        ├─
1 CP    │ ¬ A
```

f.

```
        │ A → B         2
        ├─
        │ │ ¬ B         (2)
        │ ├─
2 MTT   │ │ ¬ A         (3)
        │ │ ●
        │ ├─
3 QED   │ │ ¬ A         1
        ├─
1 CP    │ ¬ B → ¬ A
```

```
        │ ¬ B → ¬ A     2
        ├─
        │ │ A           (2)
        │ ├─
2 MTT   │ │ B           (3)
        │ │ ●
        │ ├─
3 QED   │ │ B           1
        ├─
1 CP    │ A → B
```

g.

	A → B	3
	│ A ∧ ¬ B	2
2 Ext	│ A	(3)
2 Ext	│ ¬ B	(4)
3 MPP	│ B	(4)
	│ ●	
4 Nc	│ ⊥	1
1 RAA	¬ (A ∧ ¬ B)	

	¬ (A ∧ ¬ B)	2
	│ A	(2)
2 MPT	│ B	(3)
	│ ●	
3 QED	│ B	1
1 CP	A → B	

h.

	A → (B → C)	3
	│ A ∧ B	2
2 Ext	│ A	(3)
2 Ext	│ B	(4)
3 MPP	│ B → C	4
4 MPP	│ C	(5)
	│ ●	
5 QED	│ C	1
1 CP	(A ∧ B) → C	

	(A ∧ B) → C	4
	│ A	(5)
	│ │ B	(6)
	│ │ │ ¬ C	(4)
4 MTT	│ │ │ ¬ (A ∧ B)	5
5 MPT	│ │ │ ¬ B	(6)
	│ │ │ ●	
6 Nc	│ │ │ ⊥	3
3 IP	│ │ C	2
2 CP	│ B → C	1
1 CP	A → (B → C)	

i.

	(A → B) ∧ (A → C)	1
1 Ext	A → B	3
1 Ext	A → C	4
	│ A	(3),(4)
3 MPP	│ B	(6)
4 MPP	│ C	(7)
	│ │ ●	
6 QED	│ │ B	5
	│ │ ●	
7 QED	│ │ C	5
5 Cnj	│ B ∧ C	2
2 CP	A → (B ∧ C)	

	A → (B ∧ C)	3,7
	│ │ A	(3)
3 MPP	│ │ B ∧ C	4
4 Ext	│ │ B	(5)
4 Ext	│ │ C	
	│ │ ●	
5 QED	│ │ B	2
2 CP	│ A → B	1
	│ │ A	
7 MPP	│ │ B ∧ C	8
8 Ext	│ │ B	
8 Ext	│ │ C	(9)
	│ │ ●	
9 QED	│ │ C	6
6 QED	│ A → C	1
1 Cnj	(A → B) ∧ (A → C)	

j. Stages 3-5 and 7-11 in the derivation at the right could have taken analogous forms; they are varied here to show two approaches, one using attachment rules and the other without them.

	$(A \to C) \land (B \to C)$	1
1 Ext	$A \to C$	4
1 Ext	$B \to C$	6
	$A \lor B$	3
	A	(4)
4 MPP	C	(5)
	●	
5 QED	C	3
	B	(6)
6 MPP	C	(7)
	●	
7 QED	C	3
3 PC	C	2
2 CP	$(A \lor B) \to C$	

	$(A \lor B) \to C$	4,8
	A	(3)
3 Wk	$A \lor B$	X,(4)
4 MPP	C	(5)
	●	
5 QED	C	2
2 CP	$A \to C$	1
	B	(11)
	$\neg C$	(8)
8 MTT	$\neg (A \lor B)$	9
	$\neg A$	
	●	
11 QED	B	10
10 PE	$A \lor B$	9
9 CR	$\bot$	7
7 IP	C	6
6 CP	$B \to C$	1
1 Cnj	$(A \to C) \land (B \to C)$	

k. Parallel arguments are again completed differently in the two gaps of each derivation—in the first, to show approaches with attachment rules and without them and, in the second, to show two ways of using attachment rules.

	│ (A → B) ∧ (B → C)	1
1 Ext	│ A → B	4
1 Ext	│ B → C	5,10
	│ │ A ∨ B	3
	│ │ │ A	(4)
4 MPP	│ │ │ B	(5)
5 MPP	│ │ │ C	
6 Adj	│ │ │ B ∧ C	X,(7)
	│ │ │ ●	
7 QED	│ │ │ B ∧ C	3
	│ │ │ B	(9),(10)
	│ │ │ │ ●	
9 QED	│ │ │ │ B	8
10 MPP	│ │ │ │ C	(11)
	│ │ │ │ ●	
11 QED	│ │ │ │ C	8
8 Cnj	│ │ │ B ∧ C	3
3 PC	│ │ B ∧ C	2
2 CP	│ (A ∨ B) → (B ∧ C)	

	│ (A ∨ B) → (B ∧ C)	4,10
	│ │ │ A	(3)
3 Wk	│ │ │ A ∨ B	X,(4)
4 MPP	│ │ │ B ∧ C	5
5 Ext	│ │ │ B	(6)
5 Ext	│ │ │ C	
	│ │ │ ●	
6 QED	│ │ │ B	2
2 CP	│ │ A → B	1
	│ │ │ B	(11)
	│ │ │ │ ¬ C	(9)
9 Wk	│ │ │ │ ¬ (B ∧ C)	(10)
10 MTT	│ │ │ │ ¬ (A ∨ B)	(12)
11 Wk	│ │ │ │ A ∨ B	(12)
	│ │ │ │ ●	
12 Nc	│ │ │ │ ⊥	8
8 IP	│ │ │ C	7
7 CP	│ │ B → C	1
1 Cnj	│ (A → B) ∧ (B → C)	

2. **d.** If Ann was there, then Carol was there if Bill was
Carol wasn't there if Ann was

____

Ann wasn't there if Bill was

f. If Ann was there, Bill was, too

____

If Bill wasn't there, Ann wasn't either

g. If Ann was there, Bill was there

____

Ann wasn't there without Bill being there

k. If Ann was there, Bill was there; and if Bill was there, Carol was there

____

If either Ann or Bill was there, then both Bill and Carol were there

5.4. Extreme measures

5.4.0. Overview

There are two further rules for the conditional that reflect its truth table in very direct ways.

5.4.1. Last resorts

We do not always have the opportunity to exploit a conditional by detachment, so we need means to exploit one in a *reductio*.

5.4.2. Optional extras

The principle of weakening for the conditional provides the basis for an attachment rule that is occasionally useful.

5.4.1. Last resorts

The detachment rules for the conditional—and especially MPP—will be the ways of exploiting conditional resources that you will use the most. However, they cannot cover all cases because both require the presence of a second premise as an available resource. So we need a fully general way of taking account of conditional resources.

Since any open gap will eventually turn into a *reductio* argument, it is enough that we have a way of exploiting conditionals in such arguments. An entailment

$$\Gamma, \varphi \to \psi \vDash \bot$$

says that $\varphi \to \psi$ is inconsistent with Γ, and that will be so if and only if $\varphi \to \psi$ is false in every possible world in which all members of Γ are true. But the conditional $\varphi \to \psi$ is false only when ψ is false while φ is true. So the displayed entailment says that in any world in which all members of Γ are true, we will find φ true and ψ false—and that is to say both that φ is entailed by Γ and that ψ is inconsistent with it. This way of describing the requirements for the validity of a *reductio* with a conditional premise provides our account of the role of conditionals as premises:

> LAW FOR THE CONDITIONAL AS A PREMISE. $\Gamma, \varphi \to \psi \vDash \bot$ if and only if both $\Gamma \vDash \varphi$ and $\Gamma, \psi \vDash \bot$.

In other words, a conditional $\varphi \to \psi$ is excluded by a set Γ if and only if its antecedent φ is entailed by Γ and its consequent ψ is excluded by Γ.

In terms of the metaphor of inference tickets, this law says that we can get to an absurd conclusion given Γ and the ticket $\varphi \to \psi$ if and only if Γ will get us to φ, the point of departure on our ticket, and then from its destination, ψ, on to the absurd conclusion. The "if" part of this holds also for conclusions that are not absurd, but the "only if" part does not. In particular, the fact that Γ, $\varphi \to \psi \vDash \chi$ does not insure that $\Gamma \vDash \varphi$ when χ is not absurd: we may be able to get to χ given Γ and the ticket $\varphi \to \psi$ without being able to get there via φ.

We will call the rule based on this principle, *Rejecting a Conditional* (RC). It is shown in Figure 5.4.1-2.

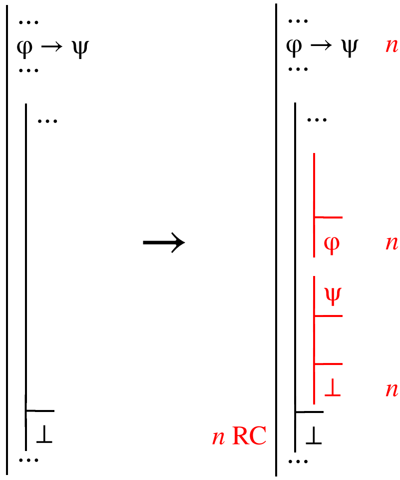


Fig. 5.4.1-2. Developing a *reductio* derivation at stage *n* by exploiting a conditional.

When we apply RC, we divide the gap into two, with the aim of showing that the antecedent of the conditional is entailed by our other resources and that its consequent is inconsistent with them. This is what is required to show that the conditional itself is inconsistent with our other resources, which is why we say that our aim is to *reject* the conditional. While this way of thinking about the rule is the most appropriate one given its place in the system of derivations, RC can also be thought of as a way of planning to use an inference ticket $\varphi \to \psi$ by planning to reach the point of departure φ and planning to get from the destination ψ to the goal $\bot$, and this perspective is the one that is most clearly displayed in the corresponding rule in tree form proofs:

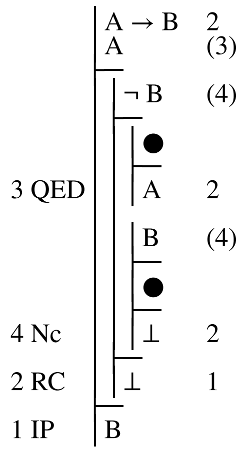
$$\text{RC}\;\dfrac{\varphi \to \psi \qquad \varphi \qquad \begin{array}{c}\not\psi\\ \\ \bot\end{array}}{\bot}$$

In this setting RC might be thought of as an abbreviation for the following combination of LFR and MPP:

$$\text{LFR}\;\dfrac{\text{MPP}\;\dfrac{\varphi \to \psi \qquad \varphi}{\psi} \qquad \begin{array}{c}\not\psi\\ \bot\end{array}}{\bot}$$

There are three conclusions—$\varphi \to \psi$, φ, and $\bot$—that must be reached before going on in the way shown by this tree. In a derivation, on the other hand, we have already shown φ → ψ when we apply the rule. So we seek to complete only two arguments. The ticket φ → ψ serves to convert the proof of φ sought in the first of these arguments into a proof of ψ, the extra supposition used in the second, so that supposition may be discharged when we apply the rule.

Although MPP and MTT are more central to the deductive inference for the conditional than are MTP and MPT to inferences involving disjunction, negation, and conjunction, all detachment rules are dispensable. One role of RC is to exploit conditionals when detachment rules are not used, and one of the simplest example of its use is the following derivation which establishes the validity of *modus ponens* without use of MPP or MTT:



A more typical use of RC is a case we never have the second premise required in order to apply MPP or MTT, as in the following derivation, which shows that the conditional in not reversible:

```
       │A → B         3
       ├─
       ││B
       │├─
       │││¬ A
       ││├─
       ││││┃¬ A
       ││││├─
       ││││┃○          ¬ A, B ⊭ ⊥
       ││││├─
       ││││┃⊥          4
       │││├─
4 IP   ││││A           3
       │││
       ││││B
       │││├─
       ││││○           ¬ A, B ⊭ ⊥
       │││├─
       ││││⊥           3
       ││├─
3 RC   │││⊥            2
       │├─
2 IP   ││A             1
       ├─
1 CP   │B → A
```

A	B	A → B	/	B → A
F	T	Ⓣ		Ⓕ

And, as is the case in this example, RC will serve us as a last resort for exploiting conditional resources before reaching a dead end in a derivation that fails.

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5.4.2. Optional extras

The law for the conditional as a premise directly reflects the conditions under which a conditional is false. The two weakening principles for the conditional that were noted in 5.3.2 directly reflect the two cases under which a conditional is true—when its consequent is true and when its antecedent is false.

$$\psi \vDash \varphi \rightarrow \psi$$
$$\neg^{\pm} \varphi \vDash \varphi \rightarrow \psi$$

However, while the rule CR implementing the law for the conditional as a premise is vital if our set of rules is sufficient, the rule that implements these weakening principles is optional. Of course, that is true for all attachment rules, but this is probably the least important of them.

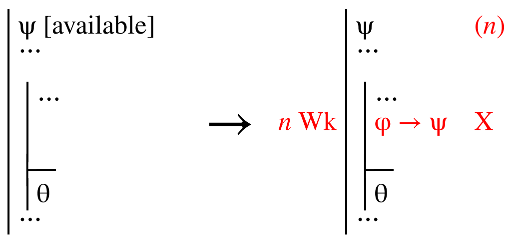

Fig. 5.4.2-1. Developing a derivation at stage *n* by adding an inactive conditional whose consequent is available.

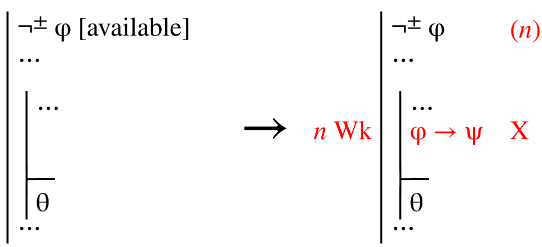

Fig. 5.4.2-2. Developing a derivation at stage *n* by adding an inactive conditional whose antecedent is negated or de-negated by an available resource.

Much of the value of attachment rules lies in their use to assemble the auxiliary resource required to apply detachment rules. And, in natural arguments, the auxiliary resources of detachment rules are less often conditionals than the other forms of sentence we can conclude by attachment rules. So we must look elsewhere for natural examples of the use of weakening for the conditional. As one example, consider the entailment

$\neg A \vee B \vDash A \rightarrow B$. This can be established quickly by the use of CP and MTP, but if, instead, the disjunction is exploited to plan for a proof by cases, Wk for the conditional provides the most natural way to complete the case arguments.

	$\neg A \vee B$	1
	$\neg A$	(2)
2 Wk	$A \rightarrow B$	X, (3)
	●	
3 QED	$A \rightarrow B$	1
	B	(4)
4 Wk	$A \rightarrow B$	X, (5)
	●	
5 QED	$A \rightarrow B$	1
1 PC	$A \rightarrow B$	

A derivation showing that $\neg (A \rightarrow B) \vDash A \wedge \neg B$ would provide a similar example of the use of these rules.

5.4.s. Summary

1 The law for the conditional as a premise applies only to *reductio* arguments and provides a way of rejecting a conditional by deriving its antecedent φ from the premises and reducing its consequent to absurdity given the premises. Rejecting a Conditional (RC) is the corresponding derivation rule.

2 This rule reflects the fact that a conditional is false when its antecedent is true and its consequent is false. The rules of Weakening (Wk) that have conditionals as conclusions reflect the fact that a conditional is true if its consequent is and also if its antecedent is false.

With these rules, the system of derivations for truth-functional logic is complete. It is shown in the table below.

Basic system

Rules for developing gaps	*for resources*	*for goals*
atomic sentence		IP
negation ¬ φ	CR (if φ is not atomic and the goal is ⊥)	RAA
conjunction φ ∧ ψ	Ext	Cnj
disjunction φ ∨ ψ	PC	PE
conditional φ → ψ	RC (if the goal is ⊥)	CP

Rules for closing gaps: *when to close*	*rule*
the goal is also a resource	QED
sentences φ and ¬ φ are resources & the goal is ⊥	Nc
⊤ is the goal	ENV
⊥ is a resource	EFQ

Detachment rules (optional): *main resource*	*auxiliary resource*	*rule*
φ → ψ	φ	MPP
φ → ψ	¬± ψ	MTT
φ ∨ ψ	¬± φ or ¬± ψ	MTP
¬ (φ ∧ ψ)	φ or ψ	MPT

Added rules (optional)

Attachment rules: *added resource*	*rule*
φ ∧ ψ	Adj
φ → ψ	Wk
φ ∨ ψ	Wk
¬ (φ ∧ ψ)	Wk

Rule for lemmas: *prerequisite*	*rule*
the goal is ⊥	LFR

At the top and left appears the basic system, all of whose rules are progressive. It consists of the fundamental rules for developing gaps by exploiting resources or planning for goals, two rules each for negations, conjunctions, disjunctions, and conditionals along with a rule to plan for atomic sentences. There are the same four rules for closing gaps we had as of 3.2, and we now also have a set of four detachment rules that provide alternative ways of exploiting weak truth-functional compounds. In addition to the basic system, there is a group of rules that are not necessarily progressive although they are sound and safe. These are the rules makred off at the lower right in the table—the attachment rules and the general rule LFR for introducing lemmas in *reductio* arguments. As in the earlier tables of this form, the names of the rules in the following are links to places where they are actually stated.

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5.4.x. Exercise questions

1. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap. Since **d** is a claim of tautologousness, it is established by a derivation that begins with only a goal and no initial premises.

a. $A \to B \simeq \neg A \vee B$

b. $(A \wedge B) \to C \simeq A \to C$

c. $(A \to B) \wedge (B \to C) \simeq A \to C$

d. $\vDash ((A \to B) \to A) \to A$

2. Construct derivations for each of the following. These exercises are designed to make attachment rules often useful. The derivations can be constructed for the English sentences in **e-g** without first analyzing them since you generally need to recognize only the main connective and the immediate connectives in order to know what rules apply; however, the abbreviated notation provided by an analysis may be more convenient.

a. $(A \wedge B) \to C, (C \vee D) \to E, A, B \vDash E$

b. $(A \vee \neg B) \to C \vDash \neg C \to B$

c. $\neg (A \wedge B), B \vee C, D \to \neg C \vDash A \to \neg D$

d. $C \to \neg (A \vee B), E \vee \neg (D \wedge \neg C), D \vDash A \to E$

e. Tom will go through Chicago and visit Sue
Tom won't go through both Chicago and Indianapolis
Tom won't visit Ursula without going through Indianapolis

Tom will visit Sue but not Ursula

f. Either we spend a bundle on television or we won't have wide public exposure
If we spend a bundle on television, we'll go into debt
Either we have wide public exposure or our contributions will dry up
We'll go into debt if our contributions dry up and we don't have large reserves
We won't have large reserves

We'll go into debt

g. If Adams supports the plan, it will go though provided Brown doesn't oppose it
Brown won't oppose the plan if either Collins or Davis supports it

---

The plan will go through if both Adams and Davis support it

For more exercises, use the exercise machine.

Glen Helman 03 Aug 2010

5.4.xa. Exercise answers

1. **a.**

```
          | A → B        2
          |---
          | | A          (2)
          | |---
2 MPP     | | B          (3)
          | | ●
          | |---
3 QED     | | B          1
          |---
1 PE      | ¬ A ∨ B
```

```
          | ¬ A ∨ B      2
          |---
          | | A          (2)
          | |---
2 MTP     | | B          (3)
          | | ●
          | |---
3 QED     | | B          1
          |---
1 CP      | A → B
```

b.

```
          | (A ∧ B) → C      3
          |---
          | | A              (4)
          | |---
          | | | ¬ C          (3)
          | | |---
3 MTT     | | | ¬ (A ∧ B)    4
4 MPT     | | | ¬ B
          | | | ○            A, ¬ C, ¬ B ⊭ ⊥
          | | |---
          | | | ⊥            2
          | |---
2 IP      | | C              1
          |---
1 CP      | A → C
```

```
          | A → C            3
          | | A ∧ B          2
          | |---
2 Ext     | | A              (3)
2 Ext     | | B
3 MPP     | | C              (4)
          | | ●
          | |---
4 QED     | | C              1
          |---
1 CP      | (A ∧ B) → C
```

```
A B C | (A ∧ B) → C / A → C
------+----------------------
T F F |      F    Ⓣ       Ⓕ
```

c.

```
          | A → C               3,7
          |---
          | | | A               (3)
          | | |---
3 MPP     | | | C
          | | | | ¬ B
          | | | |---
          | | | | ○             A, C, ¬ B ⊭ ⊥
          | | | |---
          | | | | ⊥             4
          | | |---
4 IP      | | | B               2
          | |---
2 CP      | | A → B             1
          | | | B
          | | |---
          | | | | ¬ C           (7)
          | | | |---
7 MTT     | | | | ¬ A
          | | | | ○             B, ¬ C, ¬ A ⊭ ⊥
          | | | |---
          | | | | ⊥             6
          | | |---
6 IP      | | | C               5
          | |---
5 CP      | | B → C             1
          |---
1 Cnj     | (A → B) ∧ (B → C)
```

```
          | (A → B) ∧ (B → C)   1
          |---
1 Ext     | A → B               3
1 Ext     | B → C               4
          | | A                 (3)
          | |---
3 MPP     | | B                 (4)
4 MPP     | | C                 (5)
          | | ●
          | |---
5 QED     | | C                 2
          |---
2 CP      | A → C
```

```
A B C | A → C / (A → B) ∧ (B → C)
------+---------------------------
T F T |   Ⓣ         F    Ⓕ    T
F T F |   Ⓣ         T    Ⓕ    F
```

d. The following are two approaches to this derivation, one without use of attachment rules and the other using one of the forms of Wk for the conditional.

```
       | | (A → B) → A            3
       | |----
       | | | ¬ A                  (3),(7)
       | | |----
3 MTT  | | | ¬ (A → B)
       | | | | | A                (7)
       | | | | |----
       | | | | | | ¬ B
       | | | | | |----
       | | | | | | ●
       | | | | | |----
7 Nc   | | | | | | ⊥            6
       | | | | |----
6 IP   | | | | | B              5
       | | | |----
5 CP   | | | | A → B            4
       | | |----
4 CR   | | | ⊥                  2
       | |----
2 IP   | | A                    1
       |----
1 CP   | ((A → B) → A) → A
```

```
       | | (A → B) → A          4
       | |----
       | | | ¬ A                (3),(5)
       | | |----
3 Wk   | | | A → B              X,(4)
4 MPP  | | | A                  (5)
       | | | ●
       | | |----
5 Nc   | | | ⊥                  2
       | |----
2 IP   | | A                    1
       |----
1 CP   | ((A → B) → A) → A
```

2. a.

```
       | (A ∧ B) → C   2
       | (C ∨ D) → E   4
       | A             (1)
       | B             (1)
       |----
1 Adj  | A ∧ B         X,(2)
2 MPP  | C             (3)
3 Wk   | C ∨ D         X,(4)
4 MPP  | E             (5)
       | ●
       |----
5 QED  | E
```

b.

```
       | (A ∨ ¬ B) → C     2
       |----
       | | ¬ C             (2)
       | |----
2 MTT  | | ¬ (A ∨ ¬ B)     (5)
       | | | ¬ B           (4)
       | | |----
4 Wk   | | | A ∨ ¬ B       X,(5)
       | | | ●
       | | |----
5 Nc   | | | ⊥             3
       | |----
3 IP   | | B               1
       |----
1 CP   | ¬ C → B
```

c.

	$\neg (A \wedge B)$	2
	$B \vee C$	3
	$D \rightarrow \neg C$	
	A	(2)
2 MPT	$\neg B$	(3)
3 MTP	C	(4)
4 MTT	$\neg D$	(5)
	●	
5 QED	$\neg D$	1
1 CP	$A \rightarrow \neg D$	

d.

	$C \rightarrow \neg (A \vee B)$	3
	$E \vee \neg (D \wedge \neg C)$	5
	D	(4)
	A	(2)
2 Wk	$A \vee B$	X,(3)
3 MTT	$\neg C$	(4)
4 Adj	$D \wedge \neg C$	X,(5)
5 MTP	E	(6)
	●	
6 QED	E	1
1 CP	$A \rightarrow E$	

e.

	Tom will go through Chicago and visit Sue	1
	Tom won't go through both Chicago and Indianapolis	2
	Tom won't visit Ursula without going through Indianapolis	3
1 Ext	Tom will go through Chicago	(2)
1 Ext	Tom will visit Sue	(4)
2 MPT	Tom won't go through Indianapolis	(3)
3 MPT	Tom won't visit Ursula	(4)
4 Adj	Tom will visit Sue but not Ursula	X,(5)
	●	
5 QED	Tom will visit Sue but not Ursula	

f.

	Either we spend a bundle on television or we won't have wide public exposure	1
	If we spend a bundle on television, we'll go into debt	2
	Either we have wide public exposure or our contributions will dry up	4
	We'll go into debt if our contributions dry up and we don't have large reserves	6
	We won't have large reserves	(5)
	| We'll spend a bundle on television	(2)
2 MPP	| We'll go into debt	(3)
	| ●	
3 QED	| We'll go into debt	1
	| We won't have wide public exposure	(4)
4 MTP	| Our contributions will dry up	(5)
5 Adj	| Our contributions dry up and we won't have large reserves	X,(6)
6 MPP	| We'll go into debt	(7)
	| ●	
7 QED	| We'll go into debt	1
1 PC	We'll go into debt	

g.

	If Adams supports the plan, it will go though provided Brown doesn't oppose it	3
	Brown won't oppose the plan if either Collins or Davis supports it	5
	| Both Adams and Davis will support the plan	2
2 Ext	| Adams will support the plan	(3)
2 Ext	| Davis will support the plan	(4)
3 MPP	| The plan will go though provided Brown doesn't oppose it	6
4 Wk	| Either Collins or Davis will support the plan	X,(5)
5 MPP	| Brown won't oppose the plan	(6)
6 MPP	| The plan will go through	(7)
	| ●	
7 QED	| The plan will go through	1
1 CP	The plan will go through if both Adams and Davis support it	