4. Disjunctions

4.1. Or: taking common content

4.1.0. Overview

The third connective we will study, *disjunction*, might be thought of as a logical mirror image conjunction; more precisely, the relation between them is another example of duality.

4.1.1. Hedging
While the components of a conjunction contribute their content to the whole, a disjunction asserts only the content its components have in common.

4.1.2. Inclusive and exclusive disjunction
The distinction between implications and implicatures is especially important when assessing the meaning of *or* in English.

4.1.3. Disjunction in English
Many of the other issues that arise for disjunction are like those that arise for conjunction; and one of the ways of expressing disjunction in English suggests a use of connectives to express certain numerical claims.

4.1.4. Further examples
We now have the means to give natural analyses to a wide variety of patterns in English, including a more natural analysis of sentences involving neither-nor.

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4.1.1. Hedging

Although, as was noted in 3.1.4, conjunction and negation are, by themselves enough to give us the effect of any connective for which has a truth table, these two are not the only connectives that are marked by special vocabulary in English. We will introduce special notation for two further connectives. The first is expressed by the English word or. This word has a range of grammatical uses comparable to those of and. It can join words and phrases with various grammatical functions, and the force of most of these uses can be captured by a use of or to join sentences. For example,

The weight is at or near the limit

can be paraphrased as

The weight is at the limit or the weight is near the limit

and we will study all uses of or by way of its use to join sentences.

The connective corresponding to or is called *disjunction;* we will use the symbol ∨ (the *logical or*) for it and represent it also with the English notion either … or (in which either plays a role like that of both). As in the case of conjunction we will sometimes use a special term for the components of a disjunction: they are *disjuncts*.

The effect of disjoining a sentence with another is to back off from a definite claim by leaving open a second alternative. The sentence above, instead of asserting The weight is at the limit in an unqualified way, adds the alternative The weight is near the limit to leave open a further range of possibilities. In general, we can regard a sentence $\varphi \vee \psi$ as leaving open all possibilities left open by φ as well as all those left open by ψ. As a result, a disjunction $\varphi \vee \psi$ says no more—and usually less—than either of the components φ and ψ, and the difference can be extreme, as in the cowardly weather forecast It will rain tomorrow, or else it won't. Since $\varphi \vee \psi$ leaves open as many possibilities as either φ or ψ, it rules out no more and has no more content. In particular, it rules out only those possibilities that are ruled out by both φ and ψ; and we can say that the content of $\varphi \vee \psi$ is the common content of φ and ψ, the content shared by the two. For example, the following sentences are roughly equivalent

The temperature was very hot or very cold
The temperature was extreme

and the second expresses the common content of The temperature was very hot and The temperature was very cold, the two components of the first.

Disjunction, then, adds the possibilities left open by one component to those left open by the other and selects as the possibilities ruled out those that are ruled out by both components. This is shown in Figure 4.1.1-1 below. The pictures of dice have the same significance as in Figure 2.1.2-1: they indicate regions consisting of the possible worlds in which a certain die shows one or another number. The proposition shown in 4.1.1-1B is The number shown by the die is odd ∨ the number shown by the die is less than 4 and 4.1.1-1A illustrates its two components.

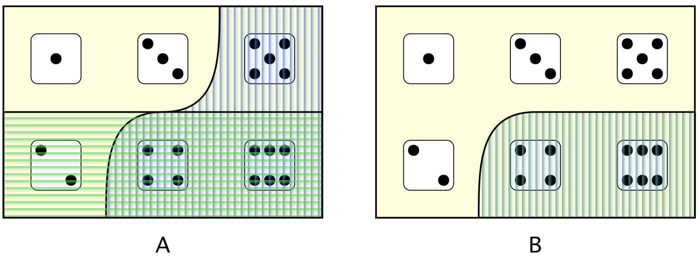

Fig. 4.1.1-1. Propositions expressed by two sentences (A) and their disjunction (B).

The possibilities ruled out by the components are shown in 4.1.1-1A shaded in different colors. 4.1.1-1B then shows the reduced set of possibilities ruled out by the disjunction and the enlarged set that are left open.

We can use these ideas to describe the truth conditions of disjunctions. If $\varphi \vee \psi$ is to leave open all possibilities left open by φ as well as all those left open by ψ, it must be true in all cases where φ is true and also in all cases where ψ is true. And if $\varphi \vee \psi$ captures the content common to φ and ψ—if it rules out the possibilities ruled out by both—it must be false whenever both φ and ψ are false. This is enough to tell us that disjunction is a connective with the table below. That is, $\varphi \vee \psi$ is true whenever at least one of φ and ψ is true and is false only when both are false.

φ	ψ	$\varphi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

This table should be compared to the diagram above; the worlds covered by the four rows of the table appear in 4.1-1A as the four regions at the top left and right and bottom left and right, respectively when φ rules out world at the bottom of the rectangle and ψ rules out worlds at the right.

Disjunction shares many of its logical properties with conjunction. In particular, analogues of the laws stated for conjunction at the end of 2.1.2 hold for it, too:

COMMUTATIVITY. *The order of disjuncts in a disjunction does not affect the content.* That is, $\varphi \vee \psi \simeq \psi \vee \varphi$.

ASSOCIATIVITY. *When a disjunction is a disjunct of a larger disjunction, the way components are grouped does not affect the content.* That is, $\varphi \vee (\psi \vee \chi) \simeq (\varphi \vee \psi) \vee \chi$.

IDEMPOTENCE. *Disjoining a sentence to itself does not change the content.* That is, $\varphi \vee \varphi \simeq \varphi$.

COVARIANCE. *A disjunction implies the result of replacing a component with anything that component implies.* That is, if $\psi \vDash \chi$, then $\varphi \vee \psi \vDash \varphi \vee \chi$ and $\psi \vee \varphi \vDash \chi \vee \varphi$.

COMPOSITIONALITY. *Disjunctions are equivalent if their corresponding components are equivalent.* That is, if $\varphi \simeq \varphi'$ and $\psi \simeq \psi'$, then $\varphi \vee \psi \simeq \varphi' \vee \psi'$.

There is nothing surprising in this. Conjunction shared analogues of these properties with both the minimum and the maximum operations on numbers, and conjunction and disjunction differ in the way the minimum and maximum operations do. In particular, a conjunction implies each of its components while a disjunction is implied by them (just as the minimum of two numbers is less than or equal to both while their maximum has both less than or equal to it).

4.1.2. Inclusive and exclusive disjunction

The fact that the table for $\varphi \lor \psi$ gives the value **T** when both φ and ψ are **T** may raise doubts about its correctness as an account of or. For we sometimes say things like

Al will go to France or Germany, or both;

and there are contexts where the expression and/or seems to capture our meaning better than or. But, if φ or ψ is already true when both φ and ψ are true, what does the alternative or both add? And, if φ or ψ is already true when φ and ψ is, why does and/or seem to differ from or?

Considerations like these have led logicians, from the Stoics on, to be interested in a connective with the table below.

φ	ψ	
T	T	F
T	F	T
F	T	T
F	F	F

This is the table of *exclusive disjunction*—so-called because it excludes the possibility that both components are true. The connective $\lor$ is known as *inclusive disjunction* because it leaves this possibility open. It has often been suggested that the English word or, in at least some of its uses, is a sign for exclusive rather than inclusive disjunction. If this were true, it would explain why we add the phrase or both or resort to and/or when we wish to express inclusive disjunction; for a sentence of the form Both φ and ψ is true in exactly the case in which inclusive and exclusive disjunction differ.

But in spite of this apparent evidence for regarding or as a sign of exclusive disjunction, there are strong reasons for thinking that it is always a sign for inclusive disjunction. That is, there are reasons for thinking that φ or ψ in English does not imply Not both φ and ψ (as it would if it were an exclusive disjunction of φ and ψ) but instead has the not-both claim an implicature in some contexts. The arguments we will look at touch on three features of a sentence that help to distinguish its implications among its implicatures: the effect of denying the sentence, yes-no questions concerning its truth, and the possibility of canceling implicatures.

Let us first look at the denial of the sentence Al will go to France or Germany. The most straightforward denial of this is Al will not go to France or Germany, but we could just as well say this:

Al will go to neither France nor Germany.

And we can paraphrase the latter as

Al will not go to France, and he will not go to Germany.

Now, we have seen that this sort of sentence can be analyzed as a not-and-not form, specifically, as $\neg F \wedge \neg G$ (F: Al will go to France; G: Al will go to Germany). And, it seems reasonable to suppose that the denial of φ or ψ can always be expressed as Neither φ nor ψ or, equivalently, as $\neg \varphi \wedge \neg \psi$.

But, if this is so, the word or must express inclusive disjunction. For the truth value of φ or ψ must be the opposite of the truth value of its denial, and we have seen reasons to believe that the truth value of its denial is given by the table below.

φ	ψ	$\neg \varphi \wedge \neg \psi$
T	T	F
T	F	F
F	T	F
F	F	T

If, on the other hand, the word or indicated exclusive disjunction, there would be two ways for a sentence φ or ψ to be false—i.e., when φ and ψ were both false and also when they were both true—and, therefore, two ways for its denial to be true. But the form Neither φ nor ψ, does not seem to leave open the possibility that both φ and ψ are true. In short, if the possibility that Al will go to both France and Germany must not be ruled out by the disjunction, because it is not left open by the corresponding neither-nor sentence.

A second argument concerns questions. Imagine that you intend to visit both France and Germany this summer and are filling out a questionnaire that includes the following:

Will you visit France or Germany this year? __ Yes __ No

The correct answer in this case seems to be yes. But this means that the sentence I will visit France or Germany this year is true if you will visit both.

A final argument concerns the following way of making it clear that Al might visit both France and Germany.

Al will visit France or Germany, and he may visit both.

Notice that instead of hedging the claim (as is done or both is added), this sentence uses and and thereby adds a second claim Al may visit both France and Germany. Now, if Al will visit France or Germany implied Al won't visit both France and Germany, the sentence displayed above would imply the following:

> Al won't visit both France and Germany, but he may visit both.

This sentence may not have fallen into self-contradiction, but it is teetering on the edge. On the other hand, Al will visit France or Germany, and he may visit both is neither a self-contradiction nor anything close to one.

If these arguments are correct, when a disjunction φ or ψ does convey the idea that φ and ψ are not both true, it does so by means of an implicature rather than an implication. Moreover, it seems possible to cancel any such implicature by adding a phrase like and maybe both. This possibility of cancellation is a sign that the implicature is of a special kind that Grice distinguished as a *conversational implicature*. A conversational implicature does not attach to a particular word as do the special implicatures that come with the use of even and but. Instead, it is produced by an interaction between the content of the claim being made and the conversational setting in which it is made. Conversational implicatures may be canceled while implicatures attaching to particular words typically cannot be canceled without lapsing into the sort incoherence exhibited by Even John was laughing, but John always laughs. Although it is not easy to say exactly how conversational implicatures arise in the case of disjunction, it does seem clear that any suggestion that the alternatives are not both true depends on the setting in which the disjunction is asserted. For example, if it was clear to everyone that the speaker's knowledge of Al's plans was derived from his responses on the kind of questionnaire described above, Al will visit France or Germany would carry no suggestion that Al would not visit both.

Of course, to assume that or in English always expresses inclusive disjunction is to not claim that exclusive disjunction cannot be expressed in English. We can, of course, always rule out the possibility that two alternatives are both true if we choose to do so. But, if this is to be done through the truth conditions of what we say (rather than through an implicature), we must rule out the possibility explicitly by, for example, saying something of the form φ or ψ but not both. And, in our notation, we have the following two forms:

Inclusive disjunction	*Exclusive disjunction*
$\varphi \lor \psi$	$(\varphi \lor \psi) \land \neg (\varphi \land \psi)$
either φ or ψ	both either φ or ψ and not both φ and ψ

But, for the remainder of this text, the term disjunction without qualification will always refer to inclusive disjunction—i.e., to the form $\varphi \lor \psi$.

4.1.3. Disjunction in English

Once we set aside controversies about the meaning of or, there are few special problems that arise in analyzing sentences as disjunctions. Of course, we must continue to be careful that the components we identify are independent sentences and that they really may be combined by disjunction to capture the content of the original sentence. This can keep us from analyzing a sentence as a disjunction even though it contains the word or. For example, Everyone stood at either the port or the starboard railing may not be analyzed as Everyone stood at the port railing ∨ everyone stood at the starboard railing.

The word or may be used in English to join a series of items and our approach to such serial disjunctions will similar to that used for serial conjunctions. We need to use two disjunctions and impose some grouping, but it will not matter which disjunction we take to have the wider scope. The parentheses indicating the grouping we impose may be suppressed when an analysis is written—so Al will visit England, France, or Germany could be analyzed using a run-on disjunction as

Al will visit England ∨ Al will visit France ∨ Al will visit Germany

However, we must recognize the grouping again in order to apply laws of entailment stated for two-component disjunctions.

There are few stylistic variants of or in English, but there is one especially clear way of stating an inclusive disjunction that deserves some comment. We might avoid any suggestion that Al will not visit both France and Germany by restating our earlier example as follows.

Al will visit at least one of France and Germany.

That we can have any chance at all of avoiding the implicature requires some explanation because, even though conversational implicatures are not part of the content of what we say, they derive from it. So it is hard to avoid them (in a given conversational context) by saying the same thing in different words. Perhaps we succeed in the case at hand because the phrase at least one is slightly stilted and would be appropriate only if the simpler form or could not be used. The stilted language could provide a clue to the audience that the speaker wants to avoid the implicatures ordinarily carried by a disjunction, and the implicature that is carried by the content of the assertion would then end up being canceled by the way that content was expressed.

The phrase at least one seems stilted in part because it presents a simple disjunction as if it was chosen from a whole family of similar claims, each saying something about how many alternatives from a list are true. For example, we might say that Al will not visit both countries by means of the following:

Al will visit at most one of France and Germany.

And we could state an exclusive disjunction as follows:

Al will visit exactly one of France and Germany.

Notice that this last sentence can be analyzed as the conjunction of the two preceding it.

With a list of more than two alternatives, there is a greater variety of claims of this sort; but, like the examples above, all of them can be expressed quite directly using conjunction, negation, and disjunction. For example, let us try to express the following sentence as a compound of the three abbreviated below it:

Exactly two of Dan, Ed, and Fred will make the finals

D: Dan will make the finals;
E: Ed will make the finals;
F: Fred will make the finals

As a first step in analyzing this sentence, we may note that it can be regarded as a conjunction of two claims, one saying that at least two of the three will make it and the other saying that at most two will.

A claim that at most two will make it denies that all three will make it and can be expressed as $\neg (D \wedge E \wedge F)$. The claim that at least two will make it tells us that there is at least one true sentence of the form a and b will make the finals where a and b are different names chosen from among Dan, Ed, and Fred. Now there are three non-equivalent sentences of this form—namely, $D \wedge E$, $D \wedge F$, and $E \wedge F$—so what we wish to say is that at least one of these three sentences is true. This can be expressed by the run-on disjunction $(D \wedge E) \vee (D \wedge F) \vee (E \wedge F)$. Putting the two analyses together, we get

$$((D \wedge E) \vee (D \wedge F) \vee (E \wedge F)) \wedge \neg (D \wedge E \wedge F)$$

as an analysis of the claim that exactly two will make it.

This analysis is admittedly complex, and no one would choose to carry out an analogous analysis for even a moderately long list of alternatives; but the fact that it would be theoretically possible to carry it out is interesting, for it shows that we can understand some implications that seem to depend on numerical reasoning—for example, the validity of

Exactly two of Dan, Ed, and Fred will make the finals

At least one of Dan, Ed, and Fred will make the finals

solely in terms of the logical properties of and, or, and not. In 8.3.2, we will see that this idea can be carried further by using other logical constants. The possibility of understanding numerical reasoning as an aspect of purely logical reasoning was one of the key reasons for Frege's interest in logic and one of the chief motivations for its development at the end of the 19th and beginning of the 20th centuries.

4.1.4. Further examples

The first example below illustrates the difference between not both and neither-nor, but it does so with an analysis of the latter that is closer to English than the one that was used in the examples of 3.1.5 .

Ann and Bill didn't both enjoy the meal but neither complained about it

Ann and Bill didn't both enjoy the meal ∧ neither Ann nor Bill complained about the meal

¬ Ann and Bill both enjoyed the meal ∧ ¬ either Ann or Bill complained about the meal

¬ (Ann enjoyed the meal ∧ Bill enjoyed the meal) ∧ ¬ (Ann complained about the meal ∨ Bill complained about the meal)

¬ (A ∧ B) ∧ ¬ (C ∨ D)

not both A and B and not either C or D

A: Ann enjoyed the meal; B: Bill enjoyed the meal; R: Ann complained about the meal; S: Bill complained about the meal

The second example is a sample of the complexity of structure we are now in a position to find in even fairly ordinary sentences.

Either Smith went ahead without Jones or Hardy backing him, or else Brown knew of his wishes and carried them out without consulting him

Smith went ahead without Jones or Hardy backing him ∨ Brown knew of Smith's wishes and carried them out without consulting him

(Smith went ahead ∧ ¬ Jones or Hardy backed Smith) ∨ (Brown knew of Smith's wishes ∧ Brown carried out Smith's wishes without consulting him)

(Smith went ahead ∧ ¬ (Jones backed Smith ∨ Hardy backed Smith)) ∨ (Brown knew of Smith's wishes ∧ (Brown carried out Smith's wishes ∧ ¬ Brown consulted Smith))

(A ∧ ¬ (J ∨ H)) ∨ (K ∧ (C ∧ ¬ N))

either both A and not either J or H or both K and both C and not N

A: Smith went ahead; C: Brown carried out Smith's wishes; H: Hardy backed Smith; J: Jones backed Smith; K: Brown knew of Smith's wishes; N: Brown consulted Smith

Notice how often it was necessary to replace a pronoun by its antecedent in order to uncover components that were independent sentences. If this replacement changed the meaning, analysis would be impossible.

Consider a sentence like the one above but having a certain partner where that one has the name Smith.

Either a certain partner went ahead without Jones or Hardy backing him, or else Brown knew of his wishes and carried them out without consulting him

We can analyze this as a disjunction A certain partner went ahead without Jones or Hardy backing him ∨ Brown knew of a certain partner's wishes and carried them out without consulting him; but we can go no further with the analysis until we have other sorts of logical form at our disposal.

4.1.s. Summary

1 While the logical word or is grammatically similar to and, its logical role is to weaken claims by hedging them with a second alternative rather than to strengthen them by adding with a second assertion. This difference from conjunction is expressed by the truth table of the connective disjunction, according to which a disjunction is true when at least one true sentence among its components, which are called disjuncts. The symbol ∨ (logical or) is our notation for the operation of disjunction, and its scope is marked by parentheses. Alternatively, we can write a disjunction $\varphi \vee \psi$ as either φ or ψ, where either serves (like both with conjunction) to indicate scope.

2 The truth of a disjunction when both its components are true distinguishes inclusive disjunction from another logical form, exclusive disjunction, whose compounds are true only when exactly one component is true. While English sentences stated with or often convey the idea that two alternatives are not both true, it can be argued that this information is conveyed as an implicature rather than an implication and that, as far as its truth conditions are concerned, the English word or may be taken as a sign of inclusive disjunction.

3 As is true of conjunction, there are cases where a word like or marking disjunction appears in a sentence but the sentence cannot be analyzed as a disjunction due to our inability to replace pronouns by their antecedents. Also, English has serial disjunctions just as it has serial conjunctions; and serial disjunction in English can be mimicked to some degree by run-on disjunctions, which suppress parentheses. Disjunction can be expressed in English by the phrase at least one, one of the group of related phrases indicating numerical compounding operations. In some cases, sentences containing these phrases can be analyzed by employing disjunction along with conjunction and negation.

4 Finally, disjunction provides an alternative, and more natural, way of analyzing *neither-nor* claims.

4.1.x. Exercise questions

1. Analyze each of the following sentences in as much detail as possible.

- **a.** Either Tommy ate his vegetables or he didn't get any dessert.
- **b.** Mike heard neither the phone nor the doorbell.
- **c.** Either Mike wasn't home or he wasn't answering the phone.
- **d.** The package was sent, but either it's still on its way or it's been lost in the mail.
- **e.** Neither the House nor the Senate had acted on the bill, but the White House expressed confidence that it would pass.
- **f.** Sam won't pass through without either stopping by or calling.
- **g.** Either Davis or Edwards will take you or give you directions.
- **h.** We'll have either a can without an opener or an opener without a can.
- **i.** Neither Jan nor Ken had matches or a lighter.
- **j.** Both Ann and Bill were in town but neither knew the other was.
- **k.** Either Tom, Dick, or Harry will handle both the scheduling and the publicity.
- **l.** The scheduling will be handled by either Tom, Dick, or Harry—as will the publicity.

2. Restate each of the following forms, putting English notation into symbols and vice versa. Indicate the scope of connectives in the result by underlining.

- **a.** $A \land (B \lor C)$
- **b.** $(A \land B) \lor C$
- **c.** not either A or not B
- **d.** both either A or B and either A or C

3. Synthesize idiomatic English sentences that express the propositions associated with the logical forms below by the intensional interpretations that follow them.

- **a.** $B \lor N$

  B: it was the butler; N: it was the nephew
- **b.** $\neg (A \lor S)$

  A: the alarm worked; S: the sprinkler worked
- **c.** $\neg A \lor \neg P$

  A: the part arrived; P: the part was the problem
- **d.** $A \lor \neg (B \land C)$

  A: Ann has a large car; B: Bill will ride with us; C: Carol will ride with us

e. $(R \lor D) \land W$

D: there was a heavy dew; R: it rained over night; W: it is wet

f. $(A \land Z) \lor (F \land \neg (A \lor Z))$

A: AAA ∧ Co. will profit from the deal; F: the deal will fall through; Z: ZZZ Inc. will profit from the deal

4.1.xa. Exercise answers

1. **a.** Tommy ate his vegetables ∨ Tommy didn't get any dessert

Tommy ate his vegetables ∨ ¬ Tommy got dessert

$$V \lor \neg D$$

either V or not D

D: Tommy got dessert; V: Tommy ate his vegetables

b. ¬ (Mike heard either the phone or the doorbell)

¬ (Mike heard the phone ∨ Mike heard the doorbell)

$$\neg (P \lor D)$$

not either P or D

D: Mike heard the doorbell; P: Mike heard the phone

c. Mike wasn't home ∨ Mike wasn't answering the phone

¬ Mike was home ∨ ¬ Mike was answering the phone

$$\neg H \lor \neg P$$

either not H or not P

H: Mike was home; P: Mike was answering the phone

d. The package was sent ∧ either the package is still on its way or it's been lost in the mail

The package was sent ∧ (the package is still on its way ∨ the package has been lost in the mail)

$$S \land (W \lor L)$$

both S and either W or L

L: the package has been lost in the mail; S: the package was sent; W: the package is still on its way

e. Neither the House nor the Senate had acted on the bill ∧ the White House expressed confidence that the bill would pass

¬ either the House or the Senate had acted on the bill ∧ the White House expressed confidence that the bill would pass

¬ (the House had acted on the bill ∨ the Senate had acted on the bill) ∧ the White House expressed confidence that the bill would pass

$$\neg (H \lor S) \land W$$

both not either H or S and W

H: the House had acted on the bill; S: the Senate had acted on the bill; W: the White House expressed confidence that the bill would pass

f. ¬ Sam will pass through without either stopping by or calling

¬ (Sam will pass through ∧ ¬ Sam will either stop by or call)

¬ (Sam will pass through ∧ ¬ (Sam will stop by ∨ Sam will call))

$$\neg (P \wedge \neg (S \vee C))$$

not both P and not either S or C

C: Sam will call; P: Sam will pass through; S: Sam will stop by

g. Davis will take you or give you directions ∨ Edwards will take you or give you directions

(Davis will take you ∨ Davis will give you directions) ∨ (Edwards will take you ∨ Edwards will give you directions)

$$(D \vee G) \vee (E \vee V)$$

either either D or G or either E or V

D: Davis will take you; E: Edwards will take you; G: Davis will give you directions; V: Edwards will give you directions

h. We'll have a can without an opener ∨ we'll have an opener without a can

(we'll have a can ∧ we won't have an opener) ∨ (we'll have an opener ∧ we won't have a can)

(we'll have a can ∧ ¬ we'll have an opener) ∨ (we'll have an opener ∧ ¬ we'll have a can)

$$(C \wedge \neg O) \vee (O \wedge \neg C)$$

either both C and not O or both O and not C

C: we'll have a can; O: we'll have an opener

i. ¬ either Jan or Ken had matches or a lighter

¬ (Jan had matches or a lighter ∨ Ken had matches or a lighter)

¬ ((Jan had matches ∨ Jan had a lighter) ∨ (Ken had matches ∨ Ken had a lighter))

$$\neg ((M \vee L) \vee (K \vee G))$$

not either either M or L or either K or G

G: Ken had a lighter; K: Ken had matches; L: Jan had a lighter; M: Jan had matches

j. Both Ann and Bill were in town ∧ neither Ann nor Bill knew the other was in town

(Ann was in town ∧ Bill was in town) ∧ ¬ either Ann or Bill knew the other was in town

(Ann was in town ∧ Bill was in town) ∧ ¬ (Ann knew Bill was in town ∨ Bill knew Ann was in town)

$$(A \wedge B) \wedge \neg (K \vee N)$$

both both A and B and not either K or N

A: Ann was in town; B: Bill was in town; K: Ann knew Bill was in town; N: Bill knew Ann was in town

k. Tom will handle both the scheduling and the publicity ∨ Dick will handle both the scheduling and the publicity ∨ Harry will handle both the scheduling and the publicity

(Tom will handle the scheduling ∧ Tom will handle the publicity) ∨ (Dick will handle the scheduling ∧ Dick will handle the publicity) ∨ (Harry will handle the scheduling ∧ Harry will handle the publicity)

$$(T \wedge P) \vee (D \wedge B) \vee (H \wedge L)$$

(both T and S) or (both D and C) or (both T and S)

[B: Dick will handle the publicity; D: Dick will handle the scheduling; H: Harry will handle the scheduling; L: Harry will handle the publicity; P: Tom will handle the publicity; T: Tom will handle the scheduling]

Note: this sentence is ambiguous and could also be interpreted as equivalent to the following one.

l. The scheduling will be handled by either Tom, Dick, or Harry ∧ the publicity will be handled by either Tom, Dick, or Harry

(the scheduling will be handled by Tom ∨ the scheduling will be handled by Dick ∨ the scheduling will be handled by Harry) ∧ (the publicity will be handled by Tom ∨ the publicity will be handled by Dick ∨ the publicity will be handled by Harry)

$$(T \vee D \vee H) \wedge (P \vee B \vee L)$$

both (T or D or H) and (P or B or L)

B: the publicity will be handled by Dick; D: the scheduling will be handled by Dick; H: the scheduling will be handled by Harry; L: the publicity will be handled by Harry; P: the publicity will be handled by Tom; T: the scheduling will be handled by Tom

2. **a.** both A and either B or C

b. either both A and B or C

c. ¬ (A ∨ ¬ B)

d. (A ∨ B) ∧ (A ∨ C)

3. **a.** It was the butler ∨ it was the nephew

It was either the butler or the nephew

b. ¬ (the alarm worked ∨ the sprinkler worked)

¬ (either the alarm or the sprinkler worked)

Neither the alarm nor the sprinkler worked

c. ¬ the part arrived ∨ ¬ the part was the problem

The part didn't arrive ∨ the part wasn't the problem

Either the part didn't arrive or it wasn't the problem

d. Ann has a large car ∨ ¬ (Bill will ride with us ∧ Carol will ride with us)

Ann has a large car ∨ ¬ Bill and Carol will ride with us

Ann has a large car ∨ Bill and Carol won't both ride with us

Either Ann has a large car or Bill and Carol won't both ride with us

Note: both is introduced here to help distinguish this sentence from A ∨ (¬ B ∧ ¬ C)

e. (it rained over night ∨ there was a heavy dew) ∧ it is wet

It rained over night or there was a heavy dew ∧ it is wet

It rained over night or there was a heavy dew but, either way, it is wet

Note: either way here serves to indicate that the scope of the disjunction has ended and that the final clause is unhedged and but reinforces this by marking the contrast between the indefinite disjunction and the definite final clause.

f. (AAA ∧ Co. will profit from the deal ∧ ZZZ Inc. will profit from the deal) ∨ (the deal will fall through ∧ ¬ (AAA ∧ Co. will profit from the deal ∨ ZZZ Inc. will profit from the deal))

AAA ∧ Co. and ZZZ Inc. will both profit from the deal ∨ (the deal will fall through ∧ ¬ (either AAA ∧ Co. or ZZZ Inc. will profit from the deal))

AAA ∧ Co. and ZZZ Inc. will both profit from the deal ∨ (the deal will fall through ∧ neither AAA ∧ Co. nor ZZZ Inc. will profit from the deal)

AAA ∧ Co. and ZZZ Inc. will both profit from the deal ∨ the deal will fall through and neither AAA ∧ Co. nor ZZZ Inc. will profit from it)

Either AAA ∧ Co. and ZZZ Inc. will both profit from the deal, or the deal will fall through and neither will profit from it

4.2. Arguing from and for alternatives

4.2.0. Overview

Because a disjunction normally says less than its components while a conjunction says more, the two connectives play very different roles in deductive inference.

4.2.1. Proofs by cases

Since a disjunction says only what is said by both its disjuncts, it entails only things that both of them entail.

4.2.2. Proving disjunctions

Since a disjunction makes a relatively weak claim, it is easy to state a sound rule to plan for it, but a safe rule that will cover all cases where it holds is more complex.

4.2.3. Further examples

There are now many choices to be regarding the order in which rules are applied and some differences in the length of derivations can result.

4.2.4. The duality of conjunction and disjunction

Conjunction and disjunction are, in a certain formal sense, mirror images of one another.

4.2.1. Proofs by cases

The validity of the argument

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam either praised or condemned the proposal

Sam granted the proposal's significance.

can be accounted for by the validity of the following two arguments:

Sam didn't praise the proposal without granting its significance Sam didn't condemn the proposal without granting its significance Sam praised the proposal ——— Sam granted the proposal's significance	Sam didn't praise the proposal without granting its significance Sam didn't condemn the proposal without granting its significance Sam condemned the proposal ——— Sam granted the proposal's significance

Each replaces the disjunctive third premise of the original argument by one of its two components. This way of establishing an entailment is sometimes called a *proof by cases*. In this example, the two cases are Sam having praised the proposal and Sam having condemned it. Since the disjunction says all and only what is common to these two claims, what follows from the disjunction in isolation or in addition to other premises is what follows from each of these claims under similar circumstances.

More formally, the idea behind proofs by cases is captured by this principle:

LAW FOR DISJUNCTION AS A PREMISE. $\Gamma, \varphi \vee \psi \vDash \chi$ if and only if both $\Gamma, \varphi \vDash \chi$ and $\Gamma, \psi \vDash \chi$ (for any set Γ and sentences φ, ψ, and χ).

To see why this law is true note that to divide the members of Γ and $\varphi \vee \psi$ on the one hand from χ on the other, a possible world must make $\varphi \vee \psi$ and all members of Γ true while making χ false. To do this it must make at least one of φ and ψ true, so it must divide at least one of the arguments Γ, φ / χ and Γ, ψ / χ. So, to say that the original argument is valid is to say that neither of these latter arguments can have its premises and alternatives divided—that is, that both are valid.

This idea appears in derivations by way of a rule we will call *Proof by Cases* (PC); it is shown in Figure 4.2.1-1.

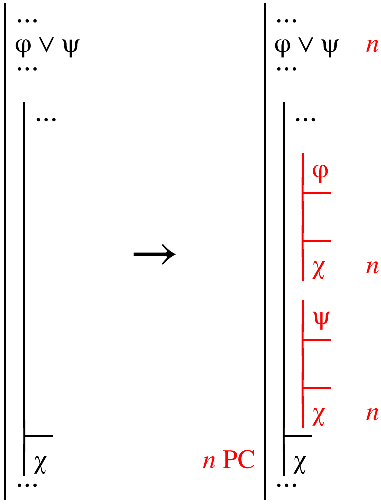

Fig. 4.2.1-1. Developing a derivation by exploiting a disjunction at stage *n*.

PC divides a gap into two new gaps. Each is a *case argument* that retains the original goal but adds one of the components of the disjunction as a supposition. The function of each supposition is to specify one of the two sorts of case in which the original disjunction is true. A supposition is required because, although our premises tell us that at least one of the disjuncts is true, we do not know which that is and the one that is true will vary among the possible worlds in which the premises are all true.

The safety and soundness (indeed, strictness) of this rule is shown by its effect on proximate arguments, which follows the pattern of law for disjunction as a premise understood as a rule for sequent proofs:

$$\text{disj. as prem.}\quad \frac{\Gamma, \varphi \vDash \chi \qquad \Gamma, \psi \vDash \chi}{\Gamma, \varphi \lor \psi \vDash \chi}$$

That is, moving from the root to the two branches, we exploit $\varphi \lor \psi$ and thus drop it from the active resources, and we add assumptions φ and ψ by introducing suppositions φ and ψ with separate scope lines. The goal of the parent gap is carried over to each of its two children. The rule is safe because any interpretation dividing one of the children is bound to make the resources of the parent true because $\varphi \lor \psi$ is implied by each of φ and ψ, and the requirement to make χ false remains unchanged as we move from the parent to the children. And the rule is strict because any interpretation dividing the parent must, in order to make $\varphi \lor \psi$ true, make true at least one of φ and ψ.

As in other cases, the use of numerical annotations in PC reflects the corresponding rule for tree-form proofs:

$$\text{PC}\ \frac{\varphi \vee \psi \qquad \begin{matrix} \not\varphi \\ \chi \end{matrix} \qquad \begin{matrix} \not\psi \\ \chi \end{matrix}}{\chi}$$

The conclusion χ is based on three premises (two with assumptions that are discharged when we draw this conclusion), so in derivations the stage number appears on the right of three lines, the disjunction that is exploited and the goals of the two new scope lines.

Here is a derivation which uses derivation rule to provide a proof for the example with which we began.

```
        │ ¬ (P ∧ ¬ G)        (4)
        │ ¬ (C ∧ ¬ G)        (7)
        │ P ∨ C              1
        ├─
        │ │ P                (3)
        │ ├─
        │ │ │ ¬ G            (3)
        │ │ ├─
3 Adj   │ │ │ P ∧ ¬ G        X,(4)
        │ │ │ ●
        │ │ ├─
4 Nc    │ │ │ ⊥              2
        │ ├─
2 IP    │ │ G                1
        │
        │ │ C                (6)
        │ ├─
        │ │ │ ¬ G            (6)
        │ │ ├─
6 Adj   │ │ │ C ∧ ¬ G        X,(7)
        │ │ │ ●
        │ │ ├─
7 Nc    │ │ │ ⊥              5
        │ ├─
5 IP    │ │ G                1
        ├─
1 PC    │ G
```

C: Sam condemned the proposal; G: Sam granted the proposal's significance; P: Sam praised the proposal

In the two case arguments, we suppose first that Sam praised the proposal and then that he condemned it and, in each case, we show that he granted the proposal's significance (by showing that he could not have failed to grant it). Since at least one of these two cases must be true whenever the premises are all true, we know that the conclusion must be true also.

The rule for tree-form proofs displayed above shows that PC represents a new function for suppositions. Like Lem (or the special case LFR) on the one hand and RAA and IP on the other, we use suppositions in PC to consider the consequences of claims without asserting them. But, while we did this in RAA and IP in order to show the suppositions were false and in Lem and LFR in order to separate the proof of a claim from the investigation of its consequences, we do it here to consider separately the consequences of two alternatives without deciding which of the two is true.

The rule for tree-form proofs also makes it clear that, apart from the separation of φ and ψ, the form of PC is much like that of Lem, but there is an important difference in the way they are employed in proofs. The rule Lem would be used to initiate the search for a proof of its first premise. But, while the tree-form rule PC might be used in this way, we use PC in derivations instead to derive consequences from a premise $\varphi \vee \psi$ that has already been established, and that aspect of the derivation rule is better reflected in the rule for sequent proofs.

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4.2.2. Proving disjunctions

Now let us look at disjunctions as conclusions. An entailment $\Gamma \vDash \varphi \vee \psi$ will hold if and only if $\varphi \vee \psi$ is true in every possible world in which all members of Γ are true. But this is to say that at least one of φ and ψ is true in every such world, and that is a way of saying that Γ renders φ and ψ jointly exhaustive. So we can state the following principle:

$$\Gamma \vDash \varphi \vee \psi \text{ if and only if } \Gamma \vDash \varphi, \psi$$

Since the right-hand side has two alternatives, this is not a law concerning entailment alone, and we will not take the principle in this form as our account of the role of disjunctions as conclusions. However, we can use the basic law for conditional exhaustiveness to restate the right-hand side as claim of entailment. Indeed we have two ways of doing that. If φ and φ' are contradictory, we can say

$$\Gamma \vDash \varphi \vee \psi \text{ if and only if } \Gamma, \varphi' \vDash \psi$$

and if ψ and ψ' are contradictory, we can say

$$\Gamma \vDash \varphi \vee \psi \text{ if and only if } \Gamma, \psi' \vDash \varphi$$

In short, a disjunction is a valid conclusion from premises Γ if and only if adding to our premises a sentence contradictory to one disjunct enables us to validly conclude the other disjunct.

In stating a principle for disjunction we will limit ourselves to cases where a sentence and its negation are the pair of contradictory sentences. But, when the disjuncts are already negative, that leaves us with two choices for each of the pairs φ and φ' and ψ and ψ' since each of φ' and ψ' might be the result of either adding or dropping a negation. To avoid stating four principles to cover each of these possibilities, we will introduce some notation to capture the general idea of obtaining a contradictory sentence by either adding or dropping a negation. Let the sentence $\neg^{\pm} \varphi$ be the result of negating φ with an optional added step of deleting a double negation if φ was already negative. Then $\neg^{\pm} \varphi$ will stand for $\neg \varphi$ when φ is not a negation and, when φ is the negation $\neg \chi$, it will stand for either $\neg \neg \chi$ or χ. That is, $\neg^{\pm} \varphi$ is the result of either negating or, perhaps, de-negating φ, which means that $\neg^{\pm} \varphi$ will either be the negation of φ or have φ as its negation.

Then $\neg^{\pm} \varphi$ and φ form a contradictory pair consisting of a sentence and its negation in one order or the other, so we may formulate a principle to account for conclusions that are disjunctions with only two statements:

LAW FOR DISJUNCTION AS A CONCLUSION. (i) $\Gamma \vDash \varphi \lor \psi$ if and only if $\Gamma, \neg^{\pm} \varphi \vDash \psi$, and (ii) $\Gamma \vDash \varphi \lor \psi$ if and only if $\Gamma, \neg^{\pm} \psi \vDash \varphi$ (for any set Γ and sentences φ, ψ, and χ).

When these are implemented as derivation rules, they give us two ways of planning for a disjunctive goal.

The two rules are shown as alternative developments in Figure 4.2.2-1. We will refer to both forms of the rule as *Proof of Exhaustion* (PE) since it is a way of showing that φ and ψ, taken together, exhaust all possibilities left open by the premises.

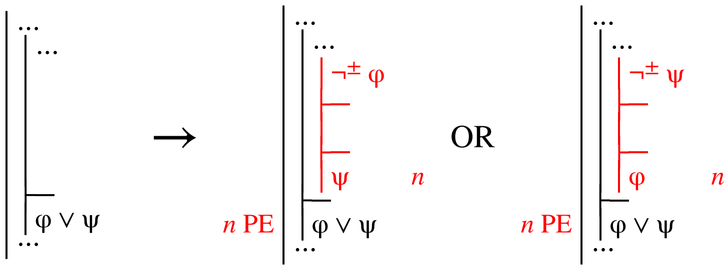


Fig. 4.2.2-1. Alternative ways of developing a derivation by planning for a disjunction at stage *n*.

In each way of developing a gap, we set one of the components of the disjunction as a new goal and add the negation or de-negation of the other component as a supposition. In each way of developing a gap, we set one of the components of the disjunction as a new goal.

Both forms of planning will lead to the same answer in the end, but one or the other may be more efficient in a particular case. There is no simple way of predicting which choice is best but the following rules of thumb may help:

(i) if only one component is a negation, choose it to form the supposition (by dropping its negation);
(ii) if only one component is a non-negative compound choose it as the goal;
(iii) if only one component seems likely to figure in closing the gap and it is not a negation, choose it as the goal.

In many cases none of these suggestions will apply; but, in most such cases, neither one of the two forms of the rule is better than the other.

As an example of this rule, consider the argument below, understanding X was out to be the denial of X was home. The validity of this argument can be established by the English derivation whose first stage is shown at the right.

Ann and Bill were not both home without the car being in the driveway
The car was not in the driveway

Either Ann or Bill was out

```
     │ ¬ ((A ∧ B) ∧ ¬ C)
     │ ¬ C
     ├─
     ││ A
     │├─
     ││
     │├─
     ││ ¬ B              1
     ├─
1 PE │ ¬ A ∨ ¬ B
```

The overall form is that of a argument that we will call "hypothetical" (for reasons discussed below) in which we suppose that Ann was at home (a supposition that is one of the two possibilities for ¬± ¬ A) and establish under this supposition that Bill was out. This shows the connection between Ann being out and Bill being out that we claim when we state, outside the scope of the supposition, that at least one was out.

Notice that if we continue the derivation

```
      │ ¬ ((A ∧ B) ∧ ¬ C)
      │ ¬ C
      ├─
      ││ A
      │├─
      │││ B
      ││├─
      │││
      ││├─
      │││ ⊥               2
      │├─
2 RAA ││ ¬ B              1
      ├─
1 PE  │ ¬ A ∨ ¬ B
```

we plan for the goal ¬ B by supposing B for *reductio*. And this example illustrates the different functions of the two sorts of supposition. We suppose that Ann is home in order to show that ¬ B (i.e., Bill is out) is true in all possible worlds in which ¬ A (i.e., Ann is out) is false. We go on to show that ¬ B is true in these cases by showing that to suppose further that B would rule out all possibilities—i.e., that this supposition would be absurd when added to our premises and the supposition A. From one point of view, both suppositions are merely added assumptions. But we add the first in order to show that to accept the second would be to go too far. That is, we add the second in order to show that we cannot accept it given the first, and we add the first to show that the second is related to it in this way.

To complete the derivation, we might exploit the first premise by CR, and this is the only way to proceed using basic rules. Doing this would make the conjunction (A ∧ B) ∧ ¬ C our goal; and, since its components are all resources, it is clear that the gap would close. But, seeing this, we might choose instead to derive that conjunction by Adj.

	$\neg((A \wedge B) \wedge \neg C)$	(5)
	$\neg C$	(4)
	A	(3)
	B	(3)
3 Adj	$A \wedge B$	(4)
4 Adj	$(A \wedge B) \wedge \neg C$	(5)
	●	
5 Nc	$\bot$	2
2 RAA	$\neg B$	1
1 PE	$\neg A \vee \neg B$	

Either way we are completing the *reductio*, in one case under the guidance of the rules and in the other under our own direction.

As noted above, the supposition in PE may be described as *hypothetical*, and this indicates the role it plays, a fourth role on top of those we have seen in Lem and LFR, in RAA and IP, and in PC. In RAA and IP, we make suppositions with the aim of showing that they are false. In Lem and LFR, we make a supposition to investigate the consequences of a claim that we plan to separately show to be true. In PC, we make a pair of suppositions, having already shown that at least one is true. In PE on the other hand, a supposition is made with no expectation of either truth or falsity. It is made instead simply to establish a connection between it and some other claim. As we argue within the scope of the supposition, we are making a *hypothetical argument*, an argument made "under a hypothesis." The conclusion we draw when we discharge the supposition states a connection between the hypothesis and the conclusion of the hypothetical argument. This statement no longer falls under the supposition, and that can be indicated by saying that it is stated *categorically*.

In each of the two forms of PE, shown here as a rule for tree-form proofs,

$$\mathrm{PE}\ \frac{\begin{array}{c}\not\neg\,\not{\pm}\,\not\varphi\\ \vdots\\ \psi\end{array}}{\varphi \vee \psi} \qquad\qquad \mathrm{PE}\ \frac{\begin{array}{c}\not\neg\,\not{\pm}\,\not\psi\\ \vdots\\ \varphi\end{array}}{\varphi \vee \psi}$$

the conclusion after the hypothetical argument says that at least one of two sentences is true. This is to say that, if one is false, the other is true. And this is the connection established between the supposition and conclusion of the hypothetical argument in each form of the rule.

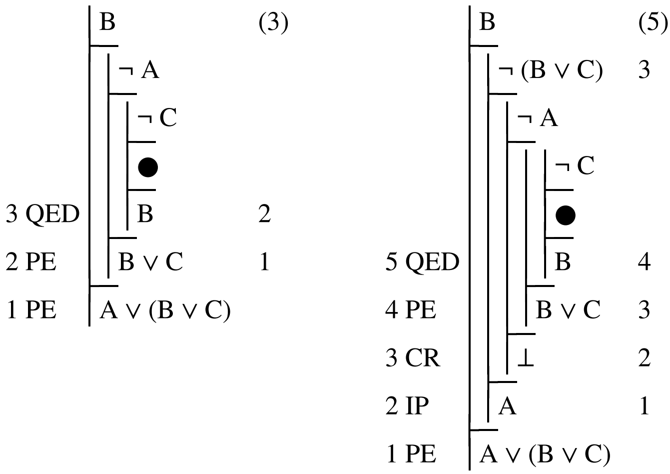
There is some danger of getting tangled in the terminology here, so let's pause and look at it more closely. The terms hypothetical and categorical derive from an ancient classification of sentences into the "categorical," the "disjunctive," and the "hypothetical." Since disjunctions and "hypothetical sentences" (the conditionals to be studied in the next chapter) are ways of hedging claims, the term categorical has acquired the meaning 'unhedged'. Now the disjunctive goal to which we applied this term above certainly hedges each of its components, so it does not state them categorically. But, while sentences along the scope line of the hypothetical argument are stated only "under a hypothesis"—that is, under the supposition of the hypothetical argument—the disjunction following the argument is no longer hedged in this way. That means it is stated categorically with respect to that supposition (though it may still fall in the scope of earlier ones). In short, when the scope line of a hypothetical argument ends, we move from hedged assertion of some claim (in the example above, the assertion of $\neg$ B under the hypothesis A) to unhedged assertion of a claim that incorporates a hedge (i.e., $\neg$ A $\vee$ $\neg$ B in the example).

4.2.3. Further examples

Both disjunction rules are illustrated by the derivation at the right, in which one grouping of a three-part disjunction is shown to entail the other. Choices between the two ways of planning for a goal disjunction were made at stages 2, 3, 5, 6, and 7 in accordance with the rules of thumb given above. Each choice helped to shorten the derivation —though only by a few steps. The derivation is contrived to provide several examples of this rule; we might have instead planned for the initial goal at stage 1 before exploiting the premise rather than planning for it separately in each of three gaps.

	A ∨ (B ∨ C)	1
	A	(4)
	¬ C	
	¬ B	
	●	
4 QED	A	3
3 PE	A ∨ B	2
2 PE	(A ∨ B) ∨ C	1
	B ∨ C	5
	B	(8)
	¬ C	
	¬ A	
	●	
8 QED	B	7
7 PE	A ∨ B	6
6 PE	(A ∨ B) ∨ C	5
	C	(10)
	¬ (A ∨ B)	
	●	
10 QED	C	9
9 PE	(A ∨ B) ∨ C	5
5 PC	(A ∨ B) ∨ C	1
1 PC	(A ∨ B) ∨ C	

The two derivations below illustrate the scale of the difference you can expect a choice between the two forms of PE to make.



Each chooses a different way of planning for the initial goal at stage 1. Notice that in the second, which makes the less efficient choice, we are led back to the goal B ∨ C in a couple of stages.

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4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term dual.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. For example, let us again take X was home and X was out to be contradictories. Then note that to get a sentence contradictory to Ann and Bill were home, we cannot take Ann and Bill were out since both would be false if one of Ann and Bill was home and the other out. To get a contradictory to we need to cover both of those possibilities as well, and Ann or Bill was out will do this. That is, Ann and Bill were home is contradictory to Ann or Bill was out and, similarly, Ann or Bill was home is contradictory to Ann and Bill were out. And this is to say that ¬ Ann and Bill were home ≃ Ann or Bill was out and that ¬ Ann or Bill was home ≃ Ann and Bill were out.

In cases of contradictoriness captured by the $\neg^{\pm}$ notation, these patterns of equivalence are stated in the following principles:

DE MORGAN'S LAWS. *The denial of a conjunction amounts to a disjunction of denials, and the denial of a disjunction amounts to a conjunction of denials.* That is,

$$\neg(\varphi \wedge \psi) \simeq \neg^{\pm}\varphi \vee \neg^{\pm}\psi$$
$$\neg(\varphi \vee \psi) \simeq \neg^{\pm}\varphi \wedge \neg^{\pm}\psi$$

Although these laws are named after Augustus De Morgan (1806-1871), they were known well before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles of conditional exhaustiveness. The table below follows the pattern of the one given for ⊥ and ⊤ in 1.4.7.

	as a premise	as an alternative
Conjunction	$\Gamma, \varphi \wedge \psi \vDash \Delta$ iff $\Gamma, \varphi, \psi \vDash \Delta$	$\Gamma \vDash \varphi \wedge \psi, \Delta$ iff both $\Gamma \vDash \varphi, \Delta$ and $\Gamma \vDash \psi, \Delta$
Disjunction	$\Gamma, \varphi \vee \psi \vDash \Delta$ iff both $\Gamma, \varphi \vDash \Delta$ and $\Gamma, \psi \vDash \Delta$	$\Gamma \vDash \varphi \vee \psi, \Delta$ iff $\Gamma \vDash \varphi, \psi, \Delta$

(Here iff is used as an abbreviation of if and only if.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise much as disjunction behaves as an alternative and disjunction behaves as premise much as conjunction behaves as an alternative.

Since ⊥ and ⊤ are paired as duals and so are conjunction and disjunction, you might wonder about negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to the two contradictory sentences in order for the results to be contradictory. And negations behavior as a premise is analogous to its behavior as an alternative.

$$\Gamma, \neg\, \varphi \vDash \Delta \text{ iff } \Gamma \vDash \varphi, \Delta$$
$$\Gamma \vDash \neg\, \varphi, \Delta \text{ iff } \Gamma, \varphi \vDash \Delta$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

The term duality points to a certain sort of two-for-one principle. It is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

premise	alternative
⊥	⊤
negation	negation
conjunction	disjunction

So, for example (and to deal only with informal statements of the principles), the principle

> A conjunction as an assumption may be replaced by its components as separate assumptions

(the upper left in the table of principles for conjunction and disjunction above) turns into

> A disjunction as an alternative may be replaced by its components as separate alternatives

(the lower right in that table). And the principle

> A negation as an assumption may be replaced by its immediate component as an alternative

(the first of the principles for negation displayed above) turns into

> A negation as an alternative may be replaced by its immediate component as an assumption

(the second of those principles). We will see more examples of such transformations in the next section but we have already seen some further ones: each of the two forms of De Morgan's laws may be transformed into the other by this association.

Since these transformations treat assumptions and alternatives in a parallel way, not all will apply to entailment, which allows multiple premises but only a single alternative. However, we have also seen that principles for conditional exhaustiveness may be transformed still further into principles of entailment by the law alternatives *via* assumptions.

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4.2.s. Summary

1 A disjunction $\varphi \vee \psi$ is false only when its disjuncts are both false, and it thus says only what both of them say. The law for disjunction as a premise tell us that we can establish a conclusion using such a premise by showing that it is entailed by each of the disjuncts (given our other premises). This way of exploiting a disjunction is known as an proof by cases and it appears in our system of derivations as the rule Proof by Cases (PC) that leads us to divide a gap into two case arguments, each of which takes over the original goal and adds one of the two disjuncts as a supposition.

2 To show that a disjunction is a valid conclusion, we must show that its disjuncts are rendered jointly exhaustive by the premises. We can do this by showing that one of the disjuncts will follow if we add the contradictory of the other to our premises. We use the notation $\neg^{\pm}\varphi$ to indicate the result of either negating or de-negating φ. The law for disjunction as a conclusion then tells us that we can conclude a disjunction if we can conclude one disjunct provided we take the negation or de-negation of the other disjunct as a premise. The rule implementing this idea is Proof of Exhaustion; it enables us to conclude a disjunction from an argument that may be called hypothetical since it bases a disjunct on an assumption (of the negation or de-negation of the other disjunct) that we may not be prepared to assert categorically. It does not matter for the soundness or safety of PE which disjunct figures as the goal of this hypothetical argument and which is negated or de-negated in its supposition.

3 Derivations, especially those that have a disjuction as a goal as well as a premise can often be developed in different ways. Some of these can be significantly longer than others but the choice between forms of PE will usually have only a limited impact on the length.

4 Conjunction and disjunction are opposite in the sense of being dual. One manifestation of this relation is in De Morgan's laws, which tell how to restate the denial of a conjunction or disjunction as an assertion of the other form of compound. Another manifestation is a pattern in laws of conditional exhaustiveness which allows us to interchange conjunctions and disjunctions if at the same time we interchange $\bot$ and $\top$ and also premises and alternatives.

4.2.x. Exercises

1. Use derivations to establish each of the claims of entailment and equivalence shown below. (Remember that claims of equivalence require derivations in both directions.)

a. $A \wedge B \vDash A \vee B$

b. $A \wedge B \vDash B \vee C$

c. $A \vee B, \neg A \vDash B$

d. $A \vee (A \wedge B) \vDash A$

e. $A \vee B, \neg (A \wedge C), \neg (B \wedge C) \vDash \neg C$

f. $A \wedge (B \vee C) \vDash (A \wedge B) \vee C$

g. $A \vee B, C \vDash (A \wedge C) \vee (B \wedge C)$

h. $A \vee B, \neg A \vee C \vDash B \vee C$

i. $A \simeq (A \wedge B) \vee (A \wedge \neg B)$

2. Use derivations to establish each of the claims of equivalence below.

a. $A \vee A \simeq A$

b. $A \vee B \simeq B \vee A$

c. $A \vee (B \vee C) \simeq (A \vee B) \vee C$

d. $A \vee (B \wedge \neg B) \simeq A$

e. $\neg (A \vee B) \simeq \neg A \wedge \neg B$

f. $\neg (A \wedge B) \simeq \neg A \vee \neg B$

3. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap.

a. $A \vee B, A \vDash \neg B$

b. $A \vee (B \wedge C) \simeq (A \vee B) \wedge C$

c. $\neg (A \vee B) \simeq \neg A \vee \neg B$

For more exercises, use the exercise machine.

4.2.xa. Exercise answers

1. **a.**

```
           │A ∧ B      1
           ├─
1 Ext      │A
1 Ext      │B           (3)
           │ │¬ A
           │ ├─
           │ │●
           │ ├─
3 QED      │ │B         2
           ├─
2 PE       │A ∨ B
```

b.

```
           │A ∧ B      1
           ├─
1 Ext      │A
1 Ext      │B           (3)
           │ │¬ C
           │ ├─
           │ │●
           │ ├─
3 QED      │ │B         2
           ├─
2 PE       │B ∨ C
```

c.

```
           │A ∨ B        1
           │¬ A          (3)
           ├─
           │ │A          (3)
           │ ├─
           │ │ │¬ B
           │ │ ├─
           │ │ │●
           │ │ ├─
3 Nc       │ │ │⊥        2
           │ ├─
2 IP       │ │B          1
           │
           │ │B          (4)
           │ ├─
           │ │●
           │ ├─
4 QED      │ │B          1
           ├─
1 PC       │B
```

d.

```
          │ A ∨ (A ∧ B)    1
          ├─
          │ │ A            (2)
          │ ├─
          │ │ ●
          │ ├─
2 QED     │ │ A            1
          │
          │ │ A ∧ B        3
          │ ├─
3 Ext     │ │ A            (4)
3 Ext     │ │ B
          │ │ ●
          │ ├─
4 QED     │ │ A            1
          ├─
1 PC      │ A
```

e.

```
          │ A ∨ B              2
          │ ¬ (A ∧ C)          3
          │ ¬ (B ∧ C)          7
          ├─
          │ │ C                (6),(10)
          │ ├─
          │ │ │ A              (5)
          │ │ ├─
          │ │ │ │ │ ●
          │ │ │ │ ├─
5 QED     │ │ │ │ │ A          4
          │ │ │ │
          │ │ │ │ │ ●
          │ │ │ │ ├─
6 QED     │ │ │ │ │ C          4
          │ │ │ ├─
4 Cnj     │ │ │ │ A ∧ C        3
          │ │ ├─
3 CR      │ │ │ ⊥              2
          │ │
          │ │ │ B              (9)
          │ │ ├─
          │ │ │ │ │ ●
          │ │ │ │ ├─
9 QED     │ │ │ │ │ B          8
          │ │ │ │
          │ │ │ │ │ ●
          │ │ │ │ ├─
10 QED    │ │ │ │ │ C          8
          │ │ │ ├─
8 Cnj     │ │ │ │ B ∧ C        7
          │ │ ├─
7 CR      │ │ │ ⊥              2
          │ ├─
2 PC      │ │ ⊥                1
          ├─
1 RAA     │ ¬ C
```

f.

```
          │ A ∧ (B ∨ C)              1
          ├─
1 Ext     │ A                        (5)
1 Ext     │ B ∨ C                    2
          │
          │ │ B                      (6)
          │ ├─
          │ │ │ ¬ C
          │ │ ├─
          │ │ │ │ ●
          │ │ │ ├─
5 QED     │ │ │ │ A                  4
          │ │ │
          │ │ │ │ ●
          │ │ │ ├─
6 QED     │ │ │ │ B                  4
          │ │ ├─
4 Cnj     │ │ │ A ∧ B                3
          │ ├─
3 PE      │ │ (A ∧ B) ∨ C            2
          │
          │ │ C                      (8)
          │ ├─
          │ │ │ ¬ (A ∧ B)
          │ │ ├─
          │ │ │ ●
          │ │ ├─
8 QED     │ │ │ C                    7
          │ ├─
7 PE      │ │ (A ∧ B) ∨ C            2
          ├─
2 PC      │ (A ∧ B) ∨ C
```

g.

```
          │ A ∨ B                          1
          │ C                              (5),(9)
          ├──
          │ │ A                            (4)
          │ ├──
          │ │ │ ¬ (B ∧ C)
          │ │ ├──
          │ │ │ │ ●
          │ │ │ ├──
4 QED     │ │ │ │ A                        3
          │ │ │
          │ │ │ │ ●
          │ │ │ ├──
5 QED     │ │ │ │ C                        3
          │ │ ├──
3 Cnj     │ │ │ A ∧ C                      2
          │ ├──
2 PE      │ │ (A ∧ C) ∨ (B ∧ C)            1
          │
          │ │ B                            (8)
          │ ├──
          │ │ │ ¬ (A ∧ C)
          │ │ ├──
          │ │ │ │ ●
          │ │ │ ├──
8 QED     │ │ │ │ B                        7
          │ │ │
          │ │ │ │ ●
          │ │ │ ├──
9 QED     │ │ │ │ C                        7
          │ │ ├──
7 Cnj     │ │ │ B ∧ C                      6
          │ ├──
6 PE      │ │ (A ∧ C) ∨ (B ∧ C)            1
          ├──
1 PC      │ (A ∧ C) ∨ (B ∧ C)
```

h.

```
          │ A ∨ B                 1
          │ ¬ A ∨ C               2
          ├──
          │ │ A                   (5)
          │ ├──
          │ │ │ ¬ A               (5)
          │ │ ├──
          │ │ │ │ ¬ B
          │ │ │ ├──
          │ │ │ │ │ ¬ C
          │ │ │ │ ├──
          │ │ │ │ │ ●
          │ │ │ │ ├──
5 Nc      │ │ │ │ │ ⊥             4
          │ │ │ ├──
4 IP      │ │ │ │ C               3
          │ │ ├──
3 PE      │ │ │ B ∨ C             2
          │ │
          │ │ │ C                 (7)
          │ │ ├──
          │ │ │ │ ¬ B
          │ │ │ ├──
          │ │ │ │ ●
          │ │ │ ├──
7 QED     │ │ │ │ C               6
          │ │ ├──
6 PE      │ │ │ B ∨ C             2
          │ ├──
2 PC      │ │ B ∨ C               1
          │
          │ │ B                   (9)
          │ ├──
          │ │ │ ¬ C
          │ │ ├──
          │ │ │ ●
          │ │ ├──
9 QED     │ │ │ B                 8
          │ ├──
8 PE      │ │ B ∨ C               1
          ├──
1 PC      │ B ∨ C
```

i.

	A	(3),(7)
	¬ (A ∧ B)	5
	●	
3 QED	A	2
	B	(8)
	●	
7 QED	A	6
	●	
8 QED	B	6
6 Cnj	A ∧ B	5
5 CR	⊥	4
4 RAA	¬ B	2
2 Cnj	A ∧ ¬ B	1
1 PE	(A ∧ B) ∨ (A ∧ ¬ B)	

	(A ∧ B) ∨ (A ∧ ¬ B)	1
	A ∧ B	2
2 Ext	A	(3)
2 Ext	B	
	●	
3 QED	A	1
	A ∧ ¬ B	4
4 Ext	A	(5)
4 Ext	¬ B	
	●	
5 QED	A	1
1 PC	A	

2. a.

	A ∨ A	1
	A	(2)
	●	
2 QED	A	1
	A	(3)
	●	
3 QED	A	1
1 PC	A	

	A	(2)
	¬ A	
	●	
2 QED	A	1
1 PE	A ∨ A	

b.

```
        | A ∨ B        1
        |----
        | | A          (3)
        | |----
        | | | ¬ B
        | | |----
        | | | ●
        | | |----
3 QED   | | | A        2
        | |----
2 PE    | | B ∨ A      1
        |
        | | B
        | |----
        | | | ¬ A      (5)
        | | |----
        | | | ●
        | | |----
5 QED   | | | B        4
        | |----
4 PE    | | B ∨ A      1
        |----
1 PC    | B ∨ A
```

```
        | B ∨ A          2
        |----
        | | ¬ A          (5)
        | |----
        | | | B          (3)
        | | |----
        | | | ●
        | | |----
3 QED   | | | B          2
        | |
        | | | A          (5)
        | | |----
        | | | | ¬ B
        | | | |----
        | | | | ●
        | | | |----
5 Nc    | | | | ⊥      4
        | | |----
4 IP    | | | B        2
        | |----
2 PC    | | B          1
        |----
1 PE    | A ∨ B
```

c.

```
        | (A ∨ B) ∨ C          3
        |----
        | | ¬ A                (6)
        | |----
        | | | ¬ B              (8)
        | | |----
        | | | | A ∨ B          4
        | | | |----
        | | | | | A            (6)
        | | | | |----
        | | | | | | ¬ C
        | | | | | |----
        | | | | | | ●
        | | | | | |----
6 Nc    | | | | | | ⊥        5
        | | | | |----
5 IP    | | | | | C          4
        | | | |
        | | | | | B          (8)
        | | | | |----
        | | | | | | ¬ C
        | | | | | |----
        | | | | | | ●
        | | | | | |----
8 Nc    | | | | | | ⊥        7
        | | | | |----
7 IP    | | | | | C          4
        | | | |----
4 PC    | | | | C            3
        | | |
        | | | | C            (9)
        | | | |----
        | | | | ●
        | | | |----
9 QED   | | | | C            3
        | | |----
3 PC    | | | C              2
        | |----
2 PE    | | B ∨ C            1
        |----
1 PE    | A ∨ (B ∨ C)
```

This is the second of the two derivations needed; the first appears in 4.2.3. In that one, disjunctive resources are exploited before disjunctive goals are planned for while the derivation at the left here illustrates the opposite approach.

d.

```
        │ A ∨ (B ∧ ¬ B)      1
        ├─
        │ │ A                (2)
        │ ├─
        │ │ ●
        │ ├─
2 QED   │ │ A                1
        │ │ B ∧ ¬ B          3
        │ ├─
3 Ext   │ │ B                (5)
3 Ext   │ │ ¬ B              (5)
        │ │ │ ¬ A
        │ │ ├─
        │ │ │ ●
        │ │ ├─
5 Nc    │ │ │ ⊥              4
        │ ├─
4 IP    │ │ A                1
        ├─
1 PC    │ A
```

```
        │ A                  (2)
        ├─
        │ │ ¬ (B ∧ ¬ B)
        │ ├─
        │ │ ●
        │ ├─
2 QED   │ │ A                1
        ├─
1 PE    │ A ∨ (B ∧ ¬ B)
```

e.

```
        │ ¬ (A ∨ B)           3,7
        ├─
        │ │ │ A               (5)
        │ │ ├─
        │ │ │ │ │ ¬ B
        │ │ │ │ ├─
        │ │ │ │ │ ●
        │ │ │ │ ├─
5 QED   │ │ │ │ │ A           4
        │ │ │ ├─
4 PE    │ │ │ │ A ∨ B         3
        │ │ ├─
3 CR    │ │ │ ⊥               2
        │ ├─
2 RAA   │ │ ¬ A               1
        │ │ │ B               (9)
        │ │ ├─
        │ │ │ │ │ ¬ A
        │ │ │ │ ├─
        │ │ │ │ │ ●
        │ │ │ │ ├─
9 QED   │ │ │ │ │ B           8
        │ │ │ ├─
8 PE    │ │ │ │ A ∨ B         7
        │ │ ├─
7 CR    │ │ │ ⊥               6
        │ ├─
6 RAA   │ │ ¬ B               1
        ├─
1 Cnj   │ ¬ A ∧ ¬ B
```

```
        │ ¬ A ∧ ¬ B            1
        ├─
1 Ext   │ ¬ A                  (4)
1 Ext   │ ¬ B                  (5)
        │ │ A ∨ B              3
        │ ├─
        │ │ │ A                (4)
        │ │ ├─
        │ │ │ ●
        │ │ ├─
4 Nc    │ │ │ ⊥                3
        │ │ │ B                (5)
        │ │ ├─
        │ │ │ ●
        │ │ ├─
        │ │ │ ⊥                3
        │ ├─
3 PC    │ │ ⊥                  2
        ├─
2 RAA   │ ¬ (A ∨ B)
```

f.

	$\neg (A \wedge B)$	3
	A	(5)
	B	(6)
	●	
5 QED	A	4
	●	
6 QED	B	4
4 Cnj	$A \wedge B$	3
3 CR	$\bot$	2
2 RAA	$\neg B$	1
1 PE	$\neg A \vee \neg B$	

	$\neg A \vee \neg B$	3
	$A \wedge B$	2
2 Ext	A	(4)
2 Ext	B	(5)
	$\neg A$	(4)
	●	
4 Nc	$\bot$	3
	$\neg B$	(5)
	●	
5 Nc	$\bot$	3
3 PC	$\bot$	1
1 RAA	$\neg (A \wedge B)$	

3. **a.**

	$A \vee B$	2
	A	
	B	
	A	
	○	$A, B \nvDash \bot$
	$\bot$	2
	B	
	○	$A, B \nvDash \bot$
	$\bot$	2
2 PC	$\bot$	1
1 RAA	$\neg B$	

A	B	$A \vee B$,	A /	$\neg B$
T	T	Ⓣ	Ⓣ	Ⓕ

b.

	A ∨ (B ∧ C)	3,8
	│ │ ¬ A	(5)
	│ │ │ A	(5)
	│ │ │ │ ¬ B	
	│ │ │ │ ●	
5 Nc	│ │ │ │ ⊥	4
4 IP	│ │ │ B	3
	│ │ │ B ∧ C	
6 Ext	│ │ │ B	7
6 Ext	│ │ │ C	
	│ │ │ ●	
7 QED	│ │ │ B	3
3 PC	│ │ B	2
2 PE	│ A ∨ B	1
	│ │ A	
	│ │ │ ¬ C	
	│ │ │ ○	A, ¬ C ⊭ ⊥
	│ │ │ ⊥	9
9 IP	│ │ C	8
	│ │ B ∧ C	10
10 Ext	│ │ B	
10 Ext	│ │ C	11
	│ │ ●	
11 QED	│ │ C	8
8 PC	│ C	1
1 Cnj	(A ∨ B) ∧ C	

Since entailment fails in one direction, equivalence must fail, so a second derivation for entailment in the other direction need not be pursued; but that entailment does hold, as is shown below.

	(A ∨ B) ∧ C	1
1 Ext	A ∨ B	2
1 Ext	C	(8)
	│ A	(4)
	│ │ ¬ (B ∧ C)	
	│ │ ●	
4 QED	│ │ A	3
3 PE	│ A ∨ (B ∧ C)	2
	│ B	(7)
	│ │ ¬ A	
	│ │ │ ●	
7 QED	│ │ │ B	6
	│ │ │ ●	
8 QED	│ │ │ C	6
6 Cnj	│ │ B ∧ C	5
5 PE	│ A ∨ (B ∧ C)	2
2 PC	A ∨ (B ∧ C)	

Each of the following divides the one open gap:

A	B	C	A ∨ (B ∧ C) / (A ∨ B) ∧ C			
T	T	F	Ⓣ	F	T	Ⓕ
T	F	F	Ⓣ	F	T	Ⓕ

c.

	¬ (A ∨ B)	3
	│ A	(5)
	│ │ B	
	│ │ │ ¬ B	
	│ │ │ ●	
5 QED	│ │ │ A	4
4 PE	│ │ A ∨ B	3
3 CR	│ ⊥	2
2 RAA	¬ B	1
1 PE	¬ A ∨ ¬ B	

	¬ A ∨ ¬ B	2
	│ A ∨ B	3,5
	│ │ ¬ A	(4)
	│ │ │ A	(4)
	│ │ │ ●	
4 Nc	│ │ │ ⊥	3
	│ │ │ B	
	│ │ │ ○	¬ A, B ⊭ ⊥
	│ │ │ ⊥	3
3 PC	│ │ ⊥	2
	│ │ ¬ B	(6)
	│ │ │ A	
	│ │ │ ○	A, ¬ B ⊭ ⊥
	│ │ │ ⊥	5
	│ │ │ B	(6)
	│ │ │ ●	
6 Nc	│ │ │ ⊥	5
5 PC	│ │ ⊥	2
2 PC	│ ⊥	1
1 RAA	¬ (A ∨ B)	

The following divide the first and second open gap, respectively:

A	B	¬ A	∨	¬ B	/	¬	(A ∨ B)
F	T	T	Ⓣ	F		Ⓕ	T
T	F	F	Ⓣ	T		Ⓕ	T

Glen Helman 03 Aug 2010

4.3. Detachment: eliminating alternatives

4.3.0. Overview

Since disjunctions (and negated conjunctions) make weak claims, the most general forms of reasoning about them are not simple; but there are simple patterns of argument involving them that work in special cases.

4.3.1. Detachment rules

If we add to a disjunction the information that one of its disjuncts is false, we can conclude the other disjunct; and a related principle applies to negated conjunctions.

4.3.2. More attachment rules

A disjunction is entailed by each of its disjuncts; and, while this does not provide a safe way of planning to reach a goal, it is a useful way of adding to the inactive resources. Again, a similar principle applies to negated conjunctions.

4.3.1. Detachment rules

When we exploit a disjunction using a proof by cases, we divide the parent gap into two children. Something like this is essential in any rule that allows us to exploit a disjunction by way of reasoning about its disjuncts, for the truth of a disjunction does not settle the truth values of its disjuncts. However, if we add to the disjunction information about the truth value of one disjunct, it can be possible to conclude something about the other one.

In particular, if we know both that a disjunction is true and that one of its disjuncts is false, we can conclude that the other disjunct is true. This idea appears in a pattern of argument, which has been recognized long enough to have acquired a Latin name: *modus tollendo ponens*

$$\text{MTP}\ \frac{\varphi \vee \psi \qquad \neg^{\pm}\varphi}{\psi} \qquad \text{MTP}\ \frac{\varphi \vee \psi \qquad \neg^{\pm}\psi}{\varphi}$$

The name refers to what the second premise and conclusion say about the two disjuncts. It can be translated, very roughly, as way, by taking, of putting. That is, the argument enables you to put forth one component as the conclusion if you take away the other component by asserting a premise that negates or de-negates it.

The use of this idea in derivations will be based on a somewhat stronger pair of principles for which we will also use the name *modus tollendo ponens*.

$$\Gamma, \varphi \vee \psi, \neg^{\pm}\varphi \vDash \chi \text{ if and only if } \Gamma, \psi, \neg^{\pm}\varphi \vDash \chi$$
$$\Gamma, \varphi \vee \psi, \neg^{\pm}\psi \vDash \chi \text{ if and only if } \Gamma, \varphi, \neg^{\pm}\psi \vDash \chi$$

Taken together, these say that in the presence of a sentence negating or de-negating one component of a disjunction, having the disjunction as a premise comes to the same thing as having its other component as a premise. The if parts of the principles are tied to the validity of the arguments MTP while the only if parts are tied to the fact that a disjunction is entailed by each of its components. More fundamentally, both rest on the fact that, if we make one component of disjunction false, we make the disjunction true if and only if we make the remaining component true.

The *modus tollendo ponens* principles describe grounds under which we can drop a disjunction from our active resources (and replace it by one of its disjuncts), so they justify a rule *Modus Tollendo Ponens* (MTP) that provides an added way of exploiting a disjunction.

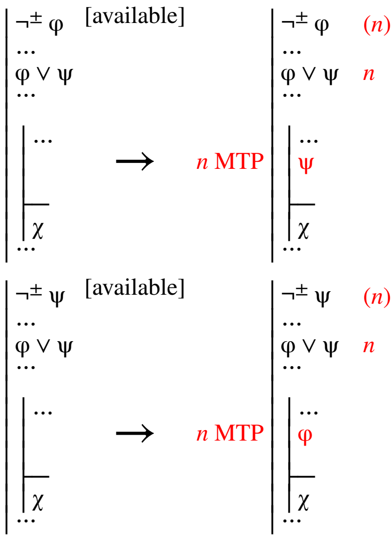

Fig. 4.3.1-1. Developing a derivation at stage *n* by exploiting a disjunction when a sentence negating or de-negating one component is also an active resource.

Notice that the negated or de-negated component is not exploited, so the stage number to its right is enclosed in parentheses. And, since we are not exploiting this resource, there is no need for it to be active: as is the case with the resources required by adjunction rules or rules for closing gaps, it is enough that this resource be available. On the other hand, the disjunction itself is exploited, so it must be active and the stage number added at its right is not parenthesized.

This is only the first of a number of rules that will enable us to exploit weak compounds in the presence of information about a component. We will label as *detachment rules* these rules, and we will use the same name for certain other rules that enable us to exploit resources when we have further information. The resource that is exploited by such a rule will be called the *main resource* while the resource that must be available but is not exploited will be called the *auxiliary resource*. In the case of MTP, the disjunction is the main resource and the sentence negating or de-negating one of its disjuncts is the auxiliary resource.

The second detachment rule we will add concerns the not-both form. De Morgan's laws tell us that the form $\neg\,(\varphi \wedge \psi)$ is equivalent to the disjunction $\neg^{\pm}\,\varphi \vee \neg^{\pm}\,\psi$, so we should expect some appropriate modification of *modus tollendo ponens* to be valid. The proper form is this:

$$\text{MPT}\ \frac{\neg(\varphi \wedge \psi) \qquad \varphi}{\neg^{\pm}\,\psi} \qquad \text{MPT}\ \frac{\neg(\varphi \wedge \psi) \qquad \psi}{\neg^{\pm}\,\varphi}$$

These arguments are called *modus ponendo tollens*: they are a way of, by putting, taking. That is, if we know that φ and ψ are not both true, adding the information that one of them is true (i.e., putting it forth), enables us to conclude that the other is not true (i.e., we can take it away). The corresponding principles, also called *modus ponendo tollens*, are these:

$$\Gamma, \neg(\varphi \wedge \psi), \varphi \vDash \chi \text{ if and only if } \Gamma, \neg^{\pm}\psi, \varphi \vDash \chi$$
$$\Gamma, \neg(\varphi \wedge \psi), \psi \vDash \chi \text{ if and only if } \Gamma, \neg^{\pm}\varphi, \psi \vDash \chi$$

They are based on the *modus ponendo tollens* arguments and also on the fact that a not-both form $\neg(\varphi \wedge \psi)$ is entailed by a sentence negating or de-negating either φ or ψ. That is, in the presence of a premise asserting φ or ψ, the not-both $\neg(\varphi \wedge \psi)$ can be replaced by a sentence that negates or de-negates the other component.

The rule *Modus Ponendo Tollens* (MPT) is this:

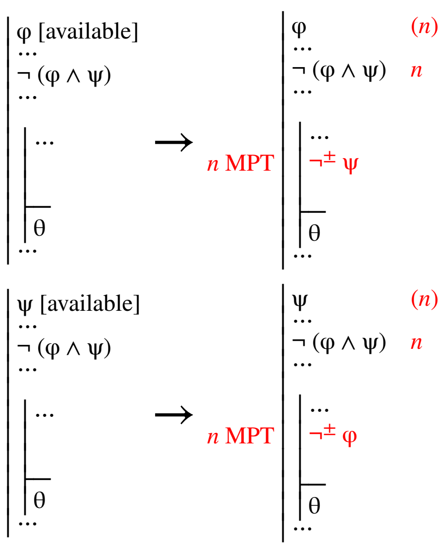

Fig. 4.3.1-2. Developing a derivation at stage *n* by exploiting a negated conjunction when a conjunct is also an active resource.

As with MTP, one resource, the main resource, is exploited (and should be active) while the other, auxiliary resource, is not exploited and need only be available.

As an example of these new rules, here is an alternative version of the derivation at the end of 4.2.1:

```
         │¬ (P ∧ ¬ G)   2
         │¬ (C ∧ ¬ G)   4
         │P ∨ C         3
         ├─
         ││¬ G          (2),(5)
         │├─
2 MPT    ││¬ P          (3)
3 MTP    ││C            (4)
4 MPT    ││G            (5)
         ││●
         │├─
5 Nc     ││⊥            1
         ├─
1 IP     │G
```

This is far from the only way of using the new rules to complete the derivation. To choose only the most minor variation of the derivation above, notice that in the second use of MPT either G or ¬ ¬ G could be concluded (since both can be described as $\neg^{\pm}$ ¬ G). And either could be used along with ¬ G to conclude ⊥ by Nc.

One oddity of the argument above is that the supposition ¬ G (Sam didn't grant the proposal's significance) enables us to conclude first that ¬ P (Sam didn't praise the proposal), then C (Sam condemned the proposal), and finally G itself. An argument by which a claim is shown to follow from its own denial is traditionally called a *consequentia mirabilis* (an amazing consequence) and has been a standard form of philosophical argumentation since antiquity. (For example, a common way of arguing against a skeptic who denies the existence of knowledge is to try to show that this claim, that there is no knowledge, in fact implies that there is knowledge, which leads to the conclusion that knowledge must exist. Any reply to this argument must question the moves by which one is supposed to get from the claim that there is no knowledge to the consequence that there is knowledge because, if this transition is valid, an indirect proof will show that knowledge does exist.)

4.3.2. Attachment rules

The principles that lie behind the rules MTP and MPT were based in part on the fact that the weak compounds $\varphi \vee \psi$ and $\neg (\varphi \wedge \psi)$ are entailed by certain information about their components. We will refer to the principles asserting these entailments as *weakening principles*:

$$\varphi \vDash \varphi \vee \psi$$
$$\psi \vDash \varphi \vee \psi$$
$$\neg^{\pm} \varphi \vDash \neg (\varphi \wedge \psi)$$
$$\neg^{\pm} \psi \vDash \neg (\varphi \wedge \psi)$$

They provide the basis for further attachment rules (i.e., ones in addition to Adj). These rules allow us to enter the conclusions of the weakening principles as inactive resources when their premises are already available.

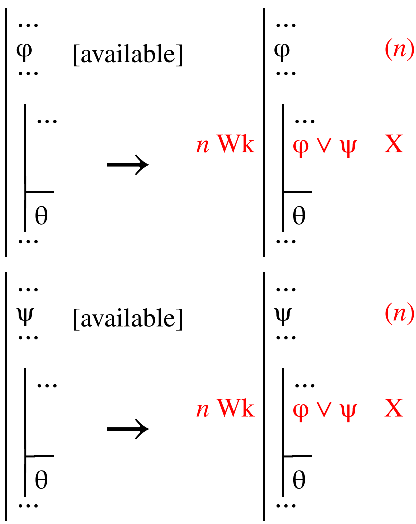

Fig. 4.3.2-1. Developing a derivation at stage *n* by adding an inactive disjunction that weakens one of the available resources.

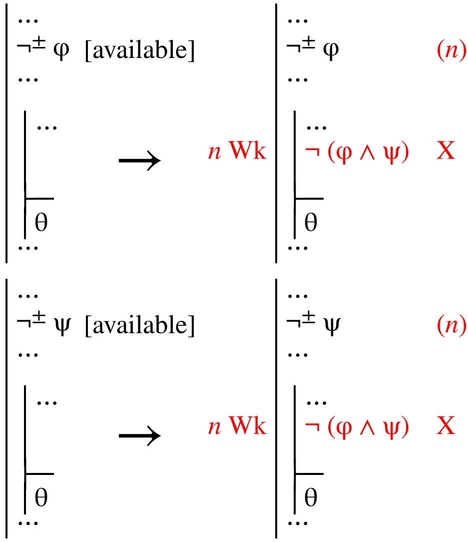

Fig. 4.3.2-2. Developing a derivation at stage *n* by adding an inactive negated conjunction that weakens one of the available resources.

These rules can be used, as we have used Adj, to provide material for closing gaps. But, since the detachment rules MTP and MPT can use inactive resources, attachment rules can provide material for them, too. For example, below are two approaches to the same argument. The argument is designed as an illustration but can be given the English interpretation that appears between them:

	(A ∧ B) ∨ (C ∧ D)	6
	¬ ((B ∧ E) ∧ (F ∨ G))	3
	E ∧ F	1
1 Ext	E	(4)
1 Ext	F	(2)
2 Wk	F ∨ G	X,(3)
3 MPT	¬ (B ∧ E)	4
4 MPT	¬ B	(5)
5 Wk	¬ (A ∧ B)	X,(6)
6 MTP	C ∧ D	7
7 Ext	C	(8)
7 Ext	D	
	●	
8 QED	C	

Assume we know in general that either Ann and Bill were both at the party or Carol and Dave were both there. And assume also that it is not the case that both Bill and Ed were there along with either Fred or Gail. Then, assuming we know in particular that Ed and Fred were both there, we can conclude that Carol was, too.

	$(A \wedge B) \vee (C \wedge D)$	4
	$\neg ((B \wedge E) \wedge (F \vee G))$	7
	$E \wedge F$	1
1 Ext	E	(6)
1 Ext	F	(8)
	│ $\neg C$	
3 Wk	│ $\neg (C \wedge D)$	X,(4)
4 MTP	│ $A \wedge B$	5
5 Ext	│ A	
5 Ext	│ B	(6)
6 Adj	│ $B \wedge E$	X,(7)
7 MPT	│ $\neg (F \vee G)$	(9)
8 Wk	│ $F \vee G$	X,(9)
	│ ●	
9 Nc	│ $\bot$	
		2
2 IP	C	

Both derivations begin by exploiting the third premise, but they exploit the other two premises in a different order. The first derivation produces a direct proof of the conclusion C while the second reaches C by an indirect proof showing that ¬ C is incompatible with the premises.

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4.3.s. Summary

1 While a disjunction does not settle the truth values of its disjuncts, it says enough about them that adding the information that one is false will tell us that the other is true. This principle is known traditionally as *modus tollendo ponens*. Since each disjunct entails the disjunction, if we know that one disjunct is false, then the disjunction and the other disjunct add the same information. This idea is implemented in a further rule for exploiting disjunctions, also known as *Modus Tollendo Ponens* (MTP). The not-both form $\neg (\varphi \wedge \psi)$ is analogous to disjunction and analogous principles apply. Specifically, a principle *modus ponendo tollens* tells us that $\neg (\varphi \wedge \psi)$ together with the assertion of one of φ and ψ entails the denial of the other. And, since the denial of either φ or ψ entails $\neg (\varphi \wedge \psi)$, we can have a rule *Modus Ponendo Tollens* (MPT) for exploiting not-both forms. The rules MTP and MPT are examples of detachment rules. The resource exploited in each is its main resource and the additional resource that must be available is the auxiliary resource.

2 We will refer to as weakening the principle that disjunctions and not-both forms are entailed by assertions of components (in the case of disjunctions) or their denials (in the case of the not-both form). This principle provides the basis for two further attachment rules, both called Weakening (Wk), that license the addition of inactive resources. Since the second resource we must have in order to apply a detachment rule need only be available, attachment rules can be used to prepare for the use of detachment rules as well to prepare for the use of rules that close gaps.

We now have examples of all the types of rules we will employ in this course:

Basic system

Rules for developing gaps		
	for resources	*for goals*
atomic sentence		IP
negation ¬ φ	CR (if φ is not atomic and the goal is ⊥)	RAA
conjunction φ ∧ ψ	Ext	Cnj
disjunction φ ∨ ψ	PC	PE

Rules for closing gaps	
when to close	*rule*
the goal is also a resource	QED
sentences φ and ¬ φ are resources & the goal is ⊥	Nc
⊤ is the goal	ENV
⊥ is a resource	EFQ

Added rules (optional)

Detachment rules (optional)		
main resource	*auxiliary resource*	*rule*
φ ∨ ψ	$\neg^{\pm}$ φ or $\neg^{\pm}$ ψ	MTP
¬ (φ ∧ ψ)	φ or ψ	MPT

Attachment rules	
added resource	*rule*
φ ∧ ψ	Adj
φ ∨ ψ	Wk
¬ (φ ∧ ψ)	Wk

Rule for lemmas	
prerequisite	*rule*
the goal is ⊥	LFR

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4.3.x. Exercises

Redo the exercises of 4.2.x, looking for opportunities to use the new rules. (Each of the answers in 4.2.xa has as least one alternative using the new rules; and, in most cases, the alternative is much shorter than the one given there.)

Since the exercise machine incorporates detachment rules but not attachment rules, it can be used to produce only some of the alternative derivations that are possible using the rules of this section.

4.3.xa. Exercise answers

1. **a.**

```
         | A ∧ B     1
         |---
1 Ext    | A         (2)
1 Ext    | B
2 Wk     | A ∨ B     X,(3)
         | ●
         |---
3 QED    | A ∨ B
```

b.

```
         | A ∧ B     1
         |---
1 Ext    | A
1 Ext    | B         (2)
2 Wk     | B ∨ C     X,(3)
         | ●
         |---
3 QED    | B ∨ C
```

c.

```
         | A ∨ B     1
         | ¬ A       (1)
         |---
1 MTP    | B         (2)
         | ●
         |---
2 QED    | B
```

d. Although the following is a possible approach, the derivation in 4.2.xa is probably more natural:

```
         | A ∨ (A ∧ B)   2
         |---
         | | ¬ A         (2),(4)
         | |---
2 MTP    | | A ∧ B       3
3 Ext    | | A           (4)
3 Ext    | | B
         | | ●
         | |---
4 Nc     | | ⊥           1
         |---
1 IP     | A
```

e.

```
         | A ∨ B         3
         | ¬ (A ∧ C)     2
         | ¬ (B ∧ C)     4
         |---
         | | C           (2),(5)
         | |---
2 MPT    | | ¬ A         (3)
3 MTP    | | B           (4)
4 MPT    | | ¬ C         (5)
         | | ●
         | |---
5 Nc     | | ⊥           1
         |---
1 RAA    | ¬ C
```

f.

```
         | A ∧ (B ∨ C)   1
         |---
1 Ext    | A             (4)
1 Ext    | B ∨ C         3
         |
         | | ¬ C         (3)
         | |---
3 MTP    | | B           (4)
4 Adj    | | A ∧ B       X,(5)
         | | ●
         | |---
5 QED    | | A ∧ B       2
         |---
2 PE     | (A ∧ B) ∨ C
```

or

```
         | A ∧ (B ∨ C)       1
         |---
1 Ext    | A                 (3)
1 Ext    | B ∨ C             2
         |
         | | B               (3)
         | |---
3 Adj    | | A ∧ B           X,(4)
4 Wk     | | (A ∧ B) ∨ C     X,(5)
         | | ●
         | |---
5 QED    | | (A ∧ B) ∨ C     2
         |
         | | C               (6)
         | |---
6 Wk     | | (A ∧ B) ∨ C     X,(7)
         | | ●
         | |---
7 QED    | | (A ∧ B) ∨ C     2
         |---
2 PC     | (A ∧ B) ∨ C
```

g.

```
            | A ∨ B                  1
            | C                      (2),(5)
            |----
            | | A                    (2)
            | |----
2 Adj       | | A ∧ C                X,(3)
3 Wk        | | (A ∧ C) ∨ (B ∧ C)    X,(4)
            | | ●
            | |----
4 QED       | | (A ∧ C) ∨ (B ∧ C)    1
            |
            | | B                    (5)
            | |----
5 Adj       | | B ∧ C                X,(6)
6 Wk        | | (A ∧ C) ∨ (B ∧ C)    X,(7)
            | | ●
            | |----
7 QED       | | (A ∧ C) ∨ (B ∧ C)    1
            |----
1 PC        | (A ∧ C) ∨ (B ∧ C)
```

or

```
            | A ∨ B                  1
            | C                      (2),(4)
            |----
            | | ¬ (A ∧ C)            2
            | |----
2 MPT       | | ¬ A                  (3)
3 MTP       | | B                    (4)
4 Adj       | | B ∧ C                X,(5)
            | | ●
            | |----
5 QED       | | B ∧ C                1
            |----
1 PE        | (A ∧ C) ∨ (B ∧ C)
```

h.

```
            | A ∨ B        1
            | ¬ A ∨ C      2
            |----
            | | A          (2)
            | |----
2 MTP       | | C          (3)
3 Wk        | | B ∨ C      X,(4)
            | | ●
            | |----
4 QED       | | B ∨ C      1
            |
            | | B          (5)
            | |----
5 Wk        | | B ∨ C      X,(6)
            | | ●
            | |----
6 QED       | | B ∨ C      1
            |----
1 PC        | B ∨ C
```

or

```
            | A ∨ B        2
            | ¬ A ∨ C      3
            |----
            | | ¬ B        (2)
            | |----
2 MTP       | | A          (3)
3 MTP       | | C          (4)
            | | ●
            | |----
4 QED       | | C          1
            |----
1 PE        | B ∨ C
```

i.

```
            | A                      (2),(3)
            |----
            | | ¬ (A ∧ B)            2
            | |----
2 MPT       | | ¬ B                  (3)
3 Adj       | | A ∧ ¬ B              X,(4)
            | | ●
            | |----
4 QED       | | A ∧ ¬ B              1
            |----
1 PE        | (A ∧ B) ∨ (A ∧ ¬ B)
```

```
            | (A ∧ B) ∨ (A ∧ ¬ B)    3
            |----
            | | ¬ A                  (2),(5)
            | |----
2 Wk        | | ¬ (A ∧ B)            X,(3)
3 MTP       | | A ∧ ¬ B              4
4 Ext       | | A                    (5)
4 Ext       | | ¬ B
            | | ●
            | |----
5 Nc        | | ⊥                    1
            |----
1 IP        | A
```

Although the derivation above for the second entailment is possible, the derivation for it in 4.2.xa is probably more natural

2. **a.**

	│ A ∨ A	2
	│ │ ¬ A	(2),(3)
2 MTP	│ │ A ●	(3)
3 Nc	│ │ ⊥	1
1 IP	│ A	

	│ A	(1)
1 Wk	│ A ∨ A ●	X,(2)
2 QED	│ A ∨ A	

Another somewhat artificial approach.

b.

	│ A ∨ B	1
	│ │ A	(2)
2 Wk	│ │ B ∨ A ●	X,(3)
3 QED	│ │ B ∨ A	1
	│ │ B	(4)
4 Wk	│ │ B ∨ A ●	X,(5)
5 QED	│ │ B ∨ A	1
1 PC	│ B ∨ A	

	│ B ∨ A	2
	│ │ ¬ A	(2)
2 MTP	│ │ B ●	(3)
3 QED	│ │ B	1
1 PE	│ A ∨ B	

As was the case with the derivations in 4.2.xa, each of the above approaches could have been used for both entailments.

c.

	│ (A ∨ B) ∨ C	3
	│ │ ¬ A	(4)
	│ │ │ ¬ C	(3)
3 MTP	│ │ │ A ∨ B	4
4 MTP	│ │ │ B ●	(5)
5 QED	│ │ │ B	2
2 PE	│ │ B ∨ C	1
1 PE	│ A ∨ (B ∨ C)	

The derivation at the right can be compared to the one in 4.2.3

	│ A ∨ (B ∨ C)	1
	│ │ A	(2)
2 Wk	│ │ A ∨ B	X,(3)
3 Wk	│ │ (A ∨ B) ∨ C ●	X,(4)
4 QED	│ │ (A ∨ B) ∨ C	1
	│ │ B ∨ C	5
	│ │ │ B	(6)
6 Wk	│ │ │ A ∨ B	X,(7)
7 Wk	│ │ │ (A ∨ B) ∨ C ●	X,(8)
8 QED	│ │ │ (A ∨ B) ∨ C	5
	│ │ │ C	(9)
9 Wk	│ │ │ (A ∨ B) ∨ C ●	(10)
10 QED	│ │ │ (A ∨ B) ∨ C	5
5 PC	│ │ (A ∨ B) ∨ C	1
1 PC	│ (A ∨ B) ∨ C	

d.

```
          | A ∨ (B ∧ ¬ B)   2
          |---
          | | ¬ A           (2)
          | |---
2 MTP     | | B ∧ ¬ B       3
3 Ext     | | B             (4)
3 Ext     | | ¬ B           (4)
          | | ●
          | |---
4 Nc      | | ⊥             4
          |---
1 IP      | A
```

```
          | A               (1)
          |---
1 Wk      | A ∨ (B ∧ ¬ B)   X,(2)
          | ●
          |---
2 QED     | A ∨ (B ∧ ¬ B)
```

e.

```
          | ¬ (A ∨ B)       (4),(7)
          |---
          | | | A           (3)
          | | |---
3 Wk      | | | A ∨ B       X,(4)
          | | | ●
          | | |---
4 Nc      | | | ⊥           2
          | |---
2 RAA     | | ¬ A           1
          | | | B           (6)
          | | |---
6 Wk      | | | A ∨ B       X,(7)
          | | | ●
          | | |---
7 Nc      | | | ⊥           5
          | |---
5 RAA     | | ¬ B           1
          |---
1 Cnj     | ¬ A ∧ ¬ B
```

```
          | ¬ A ∧ ¬ B       1
          |---
1 Ext     | ¬ A             (3)
1 Ext     | ¬ B             (4)
          | | A ∨ B         3
          | |---
3 MTP     | | B             (4)
          | | ●
          | |---
4 Nc      | | ⊥             2
          |---
2 RAA     | ¬ (A ∨ B)
```

f.

```
          | ¬ (A ∧ B)       2
          |---
          | | A             (2)
          | |---
2 MPT     | | ¬ B           (3)
          | | ●
          | |---
3 QED     | | ¬ B           1
          |---
1 PE      | ¬ A ∨ ¬ B
```

```
          | ¬ A ∨ ¬ B       3
          |---
          | | A ∧ B         2
          | |---
2 Ext     | | A             (3)
2 Ext     | | B             (4)
3 MTP     | | ¬ B           (4)
          | | ●
          | |---
4 Nc      | | ⊥             1
          |---
1 RAA     | ¬ (A ∧ B)
```

3. **a.** This derivation is unchanged from 4.2.xa

```
         │A ∨ B    2
         │A
         ├─
         │ │B
         │ ├─
         │ │ │A
         │ │ ├─
         │ │ │○     A, B ⊭ ⊥
         │ │ ├─
         │ │ │⊥     2
         │ │
         │ │ │B
         │ │ ├─
         │ │ │○     A, B ⊭ ⊥
         │ │ ├─
         │ │ │⊥     2
         │ ├─
2 PC     │ │⊥       1
         ├─
1 RAA    │¬ B
```

A	B	A ∨ B , A / ¬ B		
T	T	Ⓣ	Ⓣ	Ⓕ

b.

```
             │A ∨ (B ∧ C)     3,8
             ├─
             │ │ │¬ A         (3)
             │ │ ├─
3 MTP        │ │ │B ∧ C       4
4 Ext        │ │ │B           (5)
4 Ext        │ │ │C
             │ │ │●
             │ │ ├─
5 QED        │ │ │B           2
             │ ├─
2 PE         │ │A ∨ B         1
             │
             │ │ │¬ C         (7)
             │ │ ├─
7 Wk         │ │ │¬ (B ∧ C)   X,(8)
8 MTP        │ │ │A
             │ │ │○           A, ¬ C ⊭ ⊥
             │ │ ├─
             │ │ │⊥           9
             │ ├─
6 IP         │ │C             1
             ├─
1 Cnj        │(A ∨ B) ∧ C
```

```
             │(A ∨ B) ∧ C     1
             ├─
1 Ext        │A ∨ B           3
1 Ext        │C               (4)
             │ │¬ A           (3)
             │ ├─
3 MTP        │ │B             (4)
4 Adj        │ │B ∧ C         X,(5)
             │ │●
             │ ├─
5 QED        │ │B ∧ C         2
             ├─
2 PE         │A ∨ (B ∧ C)
```

Each of the following divides the one open gap:

A	B	C	A ∨ (B ∧ C) / (A ∨ B) ∧ C			
T	T	F	Ⓣ	F	T	Ⓕ
T	F	F	Ⓣ	F	T	Ⓕ

Although the use of Wk and MTP shortens the whole first derivation, it actually delays the dead end, which would have been reached after stage 7 if the first premise had been exploited by PC in the second gap. As in 4.2.xa, the second derivation is unnecessary once a dead-end gap is found in the first.

c.

	│ ¬ (A ∨ B)	(4)
	│ │ A	(3)
	│ │ │ B	
3 Wk	│ │ │ A ∨ B	X,(4)
	│ │ │ ●	
4 Nc	│ │ │ ⊥	2
2 RAA	│ │ ¬ B	1
1 PE	│ ¬ A ∨ ¬ B	

The following divide the first and second open gap, respectively:

A	B	¬ A	∨	¬ B	/ ¬	(A ∨ B)
F	T	T	Ⓣ	F	Ⓕ	T
T	F	F	Ⓣ	T	Ⓕ	T

	│ ¬ A ∨ ¬ B	2
	│ │ A ∨ B	3,4
	│ │ │ ¬ A	(3)
3 MTP	│ │ │ B	
	│ │ │ ○	¬ A, B ⊭ ⊥
	│ │ │ ⊥	2
	│ │ │ ¬ B	(4)
4 MTP	│ │ │ A	
	│ │ │ ○	A, ¬ B ⊭ ⊥
	│ │ │ ⊥	2
2 PC	│ │ ⊥	1
1 RAA	│ ¬ (A ∨ B)	