1.4. General principles of deductive reasoning

1.4.0. Overview

All the deductive properties and relations of sentences can be seen as special cases of a single relation. We will look at this relation and also see how to study the full range of deductive logic by way of entailment and a couple of auxiliary ideas.

1.4.1. A closer look at entailment
: Entailment will be at the heart of our study and we will begin by looking in some detail at a couple ways of formulating its definition.

1.4.2. Division
: It will be useful to have a special term for the kind of pattern of truth values that entailment rules out.

1.4.3. Conditional exhaustiveness
: Although entailment does not encompass all the concepts of deductive logic, there is a similarly defined relation that does.

1.4.4. A general framework
: All the deductive properties and relations we will consider can be expressed in terms of conditional exhaustiveness and expressed in a way that corresponds directly to definitions of them.

1.4.5. Reduction to entailment
: Although conditional exhaustiveness provides a way of thinking about deductive properties and relations, entailment is way that they are most naturally established, and we need to consider how this can be done.

1.4.6. Laws for entailment
: The ideas behind the reflexivity and transitivity of implication provide the core of the general principles that hold for the more general relations of conditional exhaustiveness and entailment.

1.4.7. Duality
: The specific principles concerning $\top$ and $\bot$ display a kind of symmetry that we will also find in principles for other logical forms.

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1.4.1. A closer look at entailment

Entailment was introduced in 1.1.6 somewhat informally as a relation between premises and a conclusion that merely extracts information from them and thus brings no risk of new error. Another way of putting the latter point is that a relation of entailment provides a conditional guarantee of the truth of the conclusion: it must be true if the premises are all true.

The discussion of entailment in 1.2.1 developed the resources necessary to give a more formal general definition. In fact it is useful to have in mind two equivalent ways of stating one.

$\Gamma \vDash \varphi$	if and only if	there is no logically possible world in which φ is false while all members of Γ are true
	if and only if	φ is true in every logically possible world in which all members of Γ are true

These are not two different concepts of entailment, for the two statements to the right of if and only if say the same thing. Still, they provide different perspectives on the concept. The second—which we will speak of as the *positive form* of the definition—is closely tied to the idea of a conditional guarantee of truth and to the reason why entailment is valuable. The first form—the *negative form*—makes the content of the concept especially clear, and this form of definition will generally be the more useful when we try to prove things concerning entailment. The other deductive properties and relations we have discussed or will go on to discuss can be given analogous pairs of definitions, a negative form ruling out certain patterns of truth values and another form stating a more positive generalization.

The equivalence of the two forms of the definition reflects a feature of all generalizations. When a generalization is false, it is because of a *counterexample*, something that is the sort of thing about which we generalize but that does not have the property we have said that all such things have. A counterexample to the claim that all birds fly is a bird that does not fly. In the positive definition of entailment, the generalization is about all possible worlds in which the premises are all true and such worlds are said to all have the property that the conclusion is true in them. A counterexample to such a generalization is then a world in which the premises are all true but the conclusion is not. The negative form of the definition then affirms the same generalization but by saying that no counterexample exists. As in the case of the generalization use to define entailment, one good way to clarify a generalization is always to ask what sort of counterexample is being ruled out.

It is important to notice how little a claim of entailment says about the actual truth values of the premises and conclusion of an argument. We can distinguish four patterns of truth values that the premises and conclusion could exhibit. Of these, a claim that an argument is valid rules out only the one appearing at the far right of Figure 1.4.1-1.

	Patterns admitted			ruled out
Premises	all **T**	not all **T**	not all **T**	all **T**
Conclusion	**T**	**T**	**F**	**F**

Fig. 1.4.1-1. Patterns of truth values admitted and ruled out by entailment.

So, knowing that an argument is valid tells us about actual truth values only that we do not find the conclusion actually false when the premises are all actually true. The other three patterns all appear in the actual truth values of some valid arguments (though not all are possible for certain valid arguments because other deductive properties and relations of the sentences involved may rule them out).

To see examples of this, consider an argument of the simple sort we will focus on in the next chapter:

It's hot and sunny
It's humid but windy
———————————
It's hot and humid

This argument is clearly valid since its conclusion merely combines two items of information each of which is extracted from one of the premises. Depending on the state of the weather, the premises may be both true, both false, or one true and the other false; and, in any case where they are not both true the conclusion can be either true or false. In particular, if it's hot and humid but neither sunny nor windy, the conclusion will be true even though both premises are false. This should not be surprising: a false sentence can still contain some true information, so information extracted from a pair of sentences that are not both true might be either true or false.

Of course, seeing one of these permitted patterns does not tell us that the argument is valid; no information that is limited to actual truth values can do that because validity concerns all possible worlds, not just the actual one. In particular, having true premises and a true conclusion does not make an argument valid. For example, the following argument is not valid:

Indianapolis is the capital of Indiana
———————————
Springfield is the capital of Illinois

For, although the single premise and the conclusion are both true, there is a logical possibility of the capital of Illinois being different while that of Indiana is as it actually is, so there is a possible world that provides a counterexample to the claim that the argument is valid.

1.4.2. Division

The pattern of truth values for premises and conclusion that is ruled out by entailment (i.e., true premises with a false conclusion) will recur often enough that it will be convenient to have special vocabulary for it. Let us say that a set Γ is *divided* from a set Δ whenever all members of Γ are true and all members of Δ are false. Whatever gives the sentences in Γ and Δ such values will be said to divide these sets. The source of the truth values will differ from context to context though, for the time being, it will be a possible world. When there is something of the appropriate sort that divides a set Γ from a set Δ, we will say that Γ and Δ are *divisible*; otherwise we will say they are *indivisible*.

Notice that these ideas are asymmetric. When one set is divided from another it is the members of the first set that true and the members of the second that are false. You might think of sets being divided vertically, with the first set above the second. In this spatial metaphor, truth is thought of as higher than falsehood; and, although this is only a metaphor, it is a broadly useful one and is consistent with the appearance of Absurdity at the bottom of Figure 1.2.5-2 and Tautology at the top. The asymmetry of division is especially important to remember in the case of the terms divisible and indivisible since this way of expressing the idea could suggest a symmetric relation between the results of a division.

As with talk of sets of sentences as premises, it is really only the list of members of a set that we care about here, and we speak of sets only because the order of the list and the occurrence of repetitions in it do not matter. In particular, we will not distinguish between a sentence and a set that has only it as a member. So we can restate the negative definition of entailment as follows:

$$\Gamma \vDash \varphi \text{ if and only if there is no possible world that divides } \Gamma \text{ from } \varphi.$$

We will also say that an argument is divided when its premises are divided from its conclusion, so we can say that an argument is valid when no possible world divides it. So to say that a possible world divides an argument is to say that the world is a counterexample to the argument's valdity. The divisibility or indivisibility of an argument thus amounts to the existence or non-existence of such a counterexample.

It can help when thinking about cases of division where one or both of the sets Γ and Δ is empty to restate the requirement all members of Γ are true as no member of Γ is false and restate the requirement for Δ analogously. That is, the most generally useful form of definition of division is this:

Γ is divided from Δ	if and only if	no member of Γ is false and no member of Δ is true

Notice that the requirement this places on a set is automatically satisfied when that set is the empty set ∅. That means that we can say:

$$\Gamma \text{ is divided from } \varnothing \text{ if and only if no member of } \Gamma \text{ is false}$$

$$\varnothing \text{ is divided from } \Delta \text{ if and only if no member of } \Delta \text{ is is true}$$

Either way, we can see in particular that the empty is bound to be divided from itself. This consequence is no more than a curiosity, but it serves to emphasize that we are using the term divides in a rather special sense.

1.4.3. Conditional exhaustiveness

We can use the idea of division to generalize entailment to a relation between sets. And it is useful to do this because the more general relation encompasses all the deductive properties and relations of sentences. Although we have focused on entailment and will continue to do so, it doesn't suffice by itself to capture all the ideas of deductive logic. In particular, we need the idea of the absurdity $\bot$ to describe inconsistency in terms of entailment, and we have not yet seen how to say, in terms of entailment, when sentences are jointly exhaustive. But the more general relation can serve to define both of these ideas.

This new relation associated with joint exhaustiveness in much the way entailment is associated with tautologousness. Actually, it is associated in this way with a more general idea of exhautiveness that concerns any number of sentences, not merely two. Just as a pair of sentences are jointly exhaustive when we can be sure that, no matter what, at least one of the two is true, we will say that a set Δ of any size is *exhaustive* when we can be sure that at least one of its members is true. We will speak of these members as *alternatives*, so a set of alternatives is exhaustive when we can be sure that always at least one of these alternatives is true.

For example, the alternatives The glass is full, The glass is empty, and The glass is partly full form a set that is exhuastive in this sense. You might notice that it happens that any two of these alternatives are mutually exclusive, but that is an accident of this example. Replacing the first two alternatives with The glass is at least 90% full and The glass is no more than 10% full would not damage exhaustiveness since the new alternatives are true in even more possibilities, and neither of them excludes the claim that the glass is partly full. For another, more artificial, example, consider The book is not red, The book is not green, and The book is not blue. It is possible for all three of these alternatives to be true, so certainly no two of them are mutually exclusive; and if one is false the other two are true, so we are bound to have at least two of them true and the three are certainly an exhaustive set of alternatives.

We will use the notation $\vDash \Delta$ for this general idea of exhaustiveness and define it more formally (in a negative and positive form, respectively) as follows:

$\vDash \Delta$	if and only if	there is no possible world in which all members of Δ are false
	if and only if	in each possible world, at least one member of Δ is true

The notation for exhaustiveness provides notation for tautologousness; for, if φ is the sole member of Δ, a guarantee that at least one alternative from Δ is true is a guarantee that φ is true. So we can write $\vDash \varphi$ to say that φ is a tautology —i.e., that $\varphi \simeq \top$. The extended use of the entailment turnstile also provides us with a new notation for the idea of joint exhaustiveness: $\varphi \triangledown \psi$ if and only if $\vDash \varphi, \psi$.

Now let us return to the project of generalizing entailment. While tautologousness is an unconditional guarantee of truth, entailment guarantees the truth of its conclusion only given the truth of a set of assumptions. Entailment is thus a guarantee of truth for a single sentence only given the conditions set out in the assumptions, and we can think about an analogous conditional guarantee that a set is exhaustive. Saying that Δ is exhaustive unconditionally tells us that ranges of possibilities left open by its alternatives taken together cover all possibilities whatsoever. We can say that a set Δ is *exhaustive given* a set Γ when the ranges of possibilities left open by the alternatives in Δ taken together cover all possibilities in which every assumption in Γ is true. When this is so we have a guarantee that in any possible world in which all assumptions in Γ are true at least one alternative in Δ is true. For example, while the two alternatives The glass is full and The glass is empty are not jointly exhaustive, they are exhaustive given the assumption The glass is not partly full since it rules out all possibilities where they are both false.

Our notation for conditional exhaustiveness will again use the entailment turnstile, writing $\Gamma \vDash \Delta$ with the set of assumptions on the left and the set of alternatives on the right. It will help in reading this notation to have vocabulary that makes Γ the subject, so we will say that Γ *renders* Δ *exhaustive* when Δ is exhaustive given Γ. The negative and positive forms of the definition of this idea are as follows:

$\Gamma \vDash \Delta$	if and only if	there is no possible world in which all members of Δ are false while all members of Γ are true
	if and only if	in each possible world in which all members of Γ are true, at least one member of Δ is true

And, as promised, this idea can be stated very directly in terms of division:

$\Gamma \vDash \Delta$ if and only if there is no possible world that divides Γ from Δ. Entailment is the special case where the set Δ consists of a single sentence, for to say that φ is entailed by Γ comes to the same thing as saying that φ is rendered exhaustive by Γ. Either way we are claiming that there is no possible world that divides Γ from φ.

In cases of conditional exhaustiveness that are not cases of entailment, what is rendered exhaustive is either a set of several alternatives or the empty set. In these cases, it does not make sense to speak of a conclusion, for when the set on the right has several members, these sentences need not be valid conclusions from the set that renders them exhaustive. Indeed, a jointly exhaustive pair of alternatives will be rendered exhaustive by any set, but often neither member of the pair will be entailed by that set. This is particularly clear in the case of sentences like The glass is full and The glass is not full that are both jointly exhaustive and mutually exlcusive—i.e., that are contradictory. Although the set consisting of such pair is rendered exhaustive by any set, only an inconsistent set could entail both of these alternatives. So the term conclusion will be reserved for cases where there is a single alternative.

1.4.4. A general framework

It was noted in the last section that conditional exhaustiveness does not merely generalize entailment and unconditional exhaustiveness but encompasses all deductive properties and relations. It is not surprising that does so if these properties and relations are understood to all consist in guarantees that certain parterns of truth values appear in no possible world. For any claim there is no world where certain sentences Γ are true and other sentences Δ are false is a claim that $\Gamma \vDash \Delta$. Of course, a given deductive property or relation may rule out a number of different patterns—i.e., rule out a number of different ways of distributing truth values among the sentences it applies to—but this just means that a deductive property or relation may consist of a number of different claims of conditional exhaustiveness. In the case of the properties and relations we will consider, only equivalence and contradictoriness involve more than one claim of conditional exhaustiveness.

The table below summarizes the deductive properties and relations that involve only one claim of conditional exhaustiveness along with the vocabulary we have used for various special cases. The ideas discussed in the last subsection appear in the three columns at the right. Moving down one of these columns, we move from an unconditional guarantee of truth somewhere in a set of alternatives to a conditional guarantee that is hedged with one or more assumptions. Moving left to right in a one of the rows, we move from a guarantee of truth that focuses on a single alternative, a definite conclusion, to one that applies to a set of two or more alternatives.

assumptions \ *alternatives*	*none*	*one*	*two*	*any no.*
none		$\vDash \psi$ *tautologous*	$\vDash \psi, \psi'$ (or $\psi \triangledown \psi'$) *jointly exhaustive*	$\vDash \Delta$ *exhaustive*
one	$\varphi \vDash$ *absurd*	$\varphi \vDash \psi$ *implies*	$\varphi \vDash \psi, \psi'$	$\varphi \vDash \Delta$
two	$\varphi, \varphi' \vDash$ (or $\varphi \vartriangle \varphi'$) *mutually exclusive*	$\varphi, \varphi' \vDash \psi$	$\varphi, \varphi' \vDash \psi, \psi'$	$\varphi, \varphi' \vDash \Delta$
any no.	$\Gamma \vDash$ *inconsistent*	$\Gamma \vDash \psi$ *entails*	$\Gamma \vDash \psi, \psi'$	$\Gamma \vDash \Delta$ *renders exhaustive*

The column to the left of these three covers the cases where the set of alternatives is empty. There can be no unconditional guarantee of this sort, so there is no entry in the first row. The entry would not be a property or relation but instead the false statement $\varnothing \vDash \varnothing$ (which asserts an unconditional guarantee that some member of the empty set is true).

Since there are no alternatives in question, the ideas in the first column are really properties of sets of assumptions (just as those in the first row are properties of sets of alternatives). Absurdity, mutual exlucsiveness, and inconsistency are negative properties, each of which guarantees that a certain group of assumptions cannot all be true. They do this indirectly by making these assumptions conditions of a guarantee of something that is bound to be false—i.e., that the empty set of alternatives exhausts all possibilities. That is, they use the same device as a sentence like If that's a good book, then I'm the King of France which denies something by stating it as a sufficient condition for an absurd claim.

So, in each of these columns, movement down from one row to the next is a matter of making a guarantee of truth conditional on further assumptions. It is possible to think of movement to the right within each row in a somewhat analogous way: adding alternatives modifies a guarantee by adding exceptions. To claim that It is raining and It isn't raining are jointly exhaustive is not to guarantee the truth of either sentence, but such a claim does assert the existence of a guarantee that for each sentence that it is true apart from cases where the other is true. Similarly, a claim of entailment is a guarantee that the premises of an argument are not all true unless the conclusion is, so it can be seen to differ from a claim of inconsistency by adding an exception.

Terms like except and unless carry implicatures that can interfere with understanding this idea. It is important to understand them as you would in a guarantee. A guarantee that a product will function for three years unless it has been abused merely makes the guarantee conditional on the absence of abuse. It does not "guarantee" in addition that the product will not function if it has been abused although the statement The product will function unless it has been abused might suggest this under other circumstances.

The ideas of division and conditional exhaustiveness also provide ways of extending to any set the idea of logical independence introduced in 1.2.6 in the case of a pair of sentences. First, let us look at this general idea of logical independence directly. We will say that a set Γ of sentences is *logically independent* when every way of assigning a truth value to each member of Γ is exhibited in at least one possible world. This is the same as saying that for every part of the set (counting both the empty set and the whole set Γ as parts of Γ) it is possible to divide that part from the rest of the set. When the set has two members, this is the same as the earlier idea. When the set $\{\varphi\}$ containing a single sentence φ is logically independent in this sense, the sentence φ is said to be *logically contingent* because there is at least one possible world in which it is true and at least one where it is false, so its truth or falsity is not settled by logic.

Conditional exhaustiveness provides an alternative way of describing this idea. When the sentences in a set are *not* independent, not every way of dividing them into a set of true sentences and a set of false sentences is logically possible. And when some way of dividing them is not possible, the set contains at least one pair of non-overlapping subsets Γ and Δ such that $\Gamma \vDash \Delta$. So the members of a set are logically independent when the relation of conditional exhaustiveness never holds between non-overlapping subsets. (It always holds between sets that overlap because there is no way of dividing such sets.)

When a set is logically independent, each member is contingent and any two of its members are logically independent, but the contingency of members and the independence of pairs does not by itself imply that the set as a whole is logically independent. For example, assume that the sentences X is fast, X is strong, X has skill, and X has stamina form an independent set. Then the sentences

X is fast and strong	X has skill and stamina	X is fast and has stamina

are each contingent, and any two of them can be seen to be independent. However, the first two taken together entail the third, so these three more complex sentences do not form an independent set.

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1.4.5. Reduction to entailment

Conditional exhaustiveness relaxes the restriction to a single conclusion found in entailment to include cases where there are several alternatives or none at all. To express the ideas captured by conditional exhaustiveness in terms of entailment, we need to add ways of capturing each of these added cases.

When a claim of conditional exhaustiveness offers no alternatives, it asserts the inconsistency of the assumptions; and that comes to the same thing as entailing the specific absurdity $\bot$. That is, we can state the following:

INCONSISTENCY *VIA* ABSURDITY. $\Gamma \vDash$ (i.e., $\Gamma \vDash \varnothing$) if and only if $\Gamma \vDash \bot$.

This law holds because rendering exhaustive the empty set and entailing $\bot$ both offer conditional guarantees of a truth that cannot exist, so each has the effect of ruling out the possibility of meeting the conditions of the guarantee.

To express the idea of rendering exhaustive multiple alternatives using entailment we need help from the concept of contradictoriness. Contradictoriness comes in here because having an exception in a guarantee comes to the same thing as having its contradictory as a condition. For example, the guarantee The product will funciton for three years unless it is abused is equivalent to The product will function for three years if it hasn't been abused, and the guarantee The product will function for three years if it is serviced regularly is equivalent to The product will function for three years unless it is not serviced regularly. To make this intuitive point more formally, note first that when sentences are contradictory, they always have opposite truth values. So making one true comes to the same thing as making the other false, and contradictory sentences play opposite roles when sets are being divided. More specifically, if φ and $\overline{\varphi}$ are contradictory sentences, then

$$\Gamma \text{ is divided from } (\Delta \text{ together with } \varphi)$$
$$\textit{if and only if}$$
$$(\Gamma \text{ together with } \overline{\varphi}) \text{ is divided from } \Delta$$

because each of these divisions requires that φ be made false and $\overline{\varphi}$ be made true. Since a claim of conditional exhaustiveness asserts that a division is not possible, having a sentence as an alternative comes to the same thing as having a sentence contradictory to it as an assumption; that is,

$$\text{if } \varphi \text{ and } \overline{\varphi} \text{ are contradictory, then } \Gamma \vDash \varphi, \Delta \text{ if and only if } \Gamma, \overline{\varphi} \vDash \Delta$$

If we apply this idea repeatedly (perhaps infinitely many times), we can move any set of alternatives to the left of the turnstile, and that is the basis of the following law:

ALTERNATIVES *VIA* ASSUMPTIONS. Let $\overline{\Delta}$ be the result of replacing each member of Δ by a sentence contradictory to it. Then $\Gamma \vDash \Delta, \Sigma$ if and only if $\Gamma, \overline{\Delta} \vDash \Sigma$.

In short, we can remove alternatives if we put sentences contradictory to them among the assumptions.

The laws we have seen give us two approaches to restating claims of conditional exhaustiveness as entailments. A claim with no alternatives—i.e., a claim of inconsistency—can be turned into an entailment by adding $\bot$ as the conclusion. And we may replace any alternatives by assumptions contradictory to them to reduce multiple alternatives to a single conclusion. The two may be combined by replacing all alternatives by contradictory assumptions and then adding $\bot$ as conclusion. The following table uses these two approaches to state all the deductive properties we have considered in terms of the general ideas of entailment and contradictoriness and of the specific absurdity $\bot$:

Concept	*in terms of entailment and other ideas*
φ is a tautology	$\vDash \varphi$
Γ entails φ	$\Gamma \vDash \varphi$
φ is absurd—i.e., $\varphi \vDash$	$\varphi \vDash \bot$
φ and ψ are mutually exclusive—i.e., $\varphi \vartriangle \psi$ (or $\varphi, \psi \vDash$)	$\varphi, \psi \vDash \bot$
Γ excludes φ—i.e., $\Gamma, \varphi \vDash$	$\Gamma, \varphi \vDash \bot$
Γ is inconsistent—i.e., $\Gamma \vDash$	$\Gamma \vDash \bot$
φ and ψ are jointly exhaustive —i.e., $\varphi \triangledown \psi$ (or $\vDash \varphi, \psi$)	$\overline{\varphi} \vDash \psi$ (or $\overline{\psi} \vDash \varphi$, or $\overline{\varphi}, \overline{\psi} \vDash \bot$)
Γ is exhaustive—i.e., $\vDash \Gamma$	$\overline{\Gamma} \vDash \bot$
φ and ψ are equivalent—i.e., $\varphi \simeq \psi$	both $\varphi \vDash \psi$ and $\psi \vDash \varphi$
φ and ψ are contradictory—i.e., φ ⧖ ψ (or both $\varphi, \psi \vDash$ and $\vDash \varphi, \psi$)	both $\varphi, \psi \vDash \bot$ and $\overline{\varphi} \vDash \psi$ (or $\overline{\psi} \vDash \varphi$, or $\overline{\varphi}, \overline{\psi} \vDash \bot$)

Here $\overline{\varphi}$ is any sentence contradictory to φ, and $\overline{\Gamma}$ is the result of replacing each member of Γ by a sentence contradictory to it

There are alternative ways of stating each of these ideas in terms of entailment. Any time $\bot$ appears as the conclusion and there is at least one assumption, $\bot$ could be replaced as the conclusion by a sentence contradictory to an assumption, which is then dropped. That is, $\Gamma, \varphi \vDash \bot$ if and only if $\Gamma \vDash \overline{\varphi}$. And whenever $\bot$ is not the conclusion, it could be made the conclusion if the a sentence contradictory to the previous conclusion is added to the assumptions —i.e., $\Gamma \vDash \varphi$ if and only if $\Gamma, \overline{\varphi} \vDash \bot$. Also, we may replace an assumption and the conclusion both by putting a sentence contradictory to each on the other side of the turnstile—i.e., Γ, φ ⊨ ψ if and only if Γ, $\overline{\psi}$ ⊨ $\overline{\varphi}$.

It may seem pointless to define the relation of contradictoriness in terms of entailment, as is done in the last row of this table, since we need to use the idea of contradictoriness in order to do this. But the definition does mean that, once we know a single sentence contradictory to a given sentence, we can say what other sentences are contradictory to it using only the ideas of entailment and absurdity.

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1.4.6. Laws for entailment

Most of the laws of deductive reasoning we will study will be generalizations about specific logical forms that will be introduced chapter by chapter, but some very general laws can be stated at this point. We have already seen some of these. We have just seen the laws tying inconsistency to Absurdity alternatives to assumptions. And the principles of reflexivity and transitivity for implication discussed in 1.2.3 can be generalized to provide basic laws for entailment and conditional exhaustiveness. We will look first at the case of entailment.

Two basic laws suffice to capture the basic properties of entailment considered in its own right:

LAW FOR PREMISES. *Any set of assumptions entails each of its members.* That is, Γ, φ ⊨ φ (for any sentence φ and any set Γ).

CHAIN LAW. *A set of assumptions entails anything entailed by things it entails.* That is, if Γ ⊨ φ for each assumption φ in Δ and Δ ⊨ ψ, then Γ ⊨ ψ (for any sentence ψ and any sets Γ and Δ).

Taken together, these laws tell us that the relation which holds between sets Γ and Δ when Γ entails all members of Δ is both reflexive and transitive. For the law for premises tells us that any set entail every member of itself. And, if Γ entails every member of Δ and Δ entails every member of the Σ, then Γ also entails every member of Σ by the chain law. Although this reflexive and transitive relation is, like conditional exhaustiveness, a relation between sets of sentences, they are different relations, and we will see later that conditional exhaustiveness is neither reflexive nor transitive.

These two principles have as a consequence two further principles the addition and subtraction of assumptions that will play an important role in our study of entailment:

MONOTONICITY. *Adding assumptions never undermines entailment.* That is, if Γ ⊨ φ, then Γ, Δ ⊨ φ (for any sets Γ and Δ and any sentence φ).

LAW FOR LEMMAS. *Any assumption that is entailed by other assumptions may be dropped without undermining entailment.* That is, if Γ, φ ⊨ ψ and Γ ⊨ φ, then Γ ⊨ ψ (for any sentence φ and set Γ).

Each of these principles is based on both the law for premises and the chain law. In the case of the first, the law for premises tells us that Γ together with Δ entails every member of Γ alone, so Γ, Δ ⊨ φ if Γ ⊨ φ by the chain law. The assumption of the second that Γ ⊨ φ combines with the law for premises to tell us that Γ entails every member of the result of adding the further assumption

Φ, and the chain law then tells us that Γ entails anything Ψ entailed by this enlarged set of assumptions.

The term lemma can be used for a conclusion that is drawn not because it is of interest in its own right but because it helps us to draw further conclusions. The second law tells us that if we add to our premises Γ a lemma φ that we can conclude from them, anything ψ we can conclude using the enlarged set of premises can be concluded from the original set Γ.

The idea behind the law of monotonicity is that adding assumptions can only make it harder to find a possible world that divides the assumptions from the conclusion, so, if no possible world will divide Γ from φ, we can be sure that no world will divide from φ the larger set of assumptions we get by adding some further assumptions Δ. The term monotonic is applied to trends that never change direction. More specifically, it is applied to a quantity that does not both increase and decrease in response to changes in another quantity. In this case, it reflects the fact that adding assumptions will never lead to a decrease in the sets of alternatives rendered exhaustive by them and adding alternatives will never lead to a decrease in the sets of assumptions rendering them exhaustive.

It is a distinguishing characteristic of deductive reasoning that a principle of monotonicity holds. For, when reasoning is not risk free, additional data can show that a initially well-supported conclusion is false and do so without undermining the original premises on which the conclusion was based. If such further data were added to the original premises, the result would no longer support the conclusion. This means that risky inference is, in general, *non-monotonic* in the sense that additions to the premises can reduce the set of conclusions that are justified. This is true of inductive generalization and of inference to the best explanation of available data, but the term non-monotonic is most often applied to another sort of non-deductive inference, an inference in which features of typical or normal cases are applied when there is no evidence to the contrary. One standard example is the argument from the premise Tweety is a bird to the conclusion Tweety flies. This conclusion is reasonable when the premise exhausts our knowledge of Tweety; but the inference is not free of risk, and the conclusion would no longer be reasonable if we were to add the premise that Tweety is a penguin.

The law for premises and the chain can be shown to give a complete account of the general laws of entailment in the sense that any relation between sets of sentences and sentences that obeys them is an entailment relation for some set of possibile worlds and assignment of truth values to sentences in each world. But this is not to say that they provide a complete general account of deductive properties and relations, because our definitions of the may of these in terms of entailment also used the ideas of contradiction and the absurdity $\bot$. The laws providing for inconsistency *via* absurdity and alternatives *via* assumptions govern these ideas but they were stated for conditional exhaustiveness rather than entailment. Although laws for inconsistency and contradictoriness might be stated in terms of entailment, doing so now would pointlessly anticipate later topics, so we will let the two laws we began with suffice.

Let us look briefly at conditional exhaustiveness. As noted earlier, it is neither reflexive nor transitive. Although $\Gamma \vDash \Gamma$ whenever Γ has at least one member, we have already seen that $\varnothing \nvDash \varnothing$. And if conditional exhaustiveness were transitive every sentence φ would imply every other sentence ψ since $\varphi \vDash \varphi, \psi$ and $\varphi, \psi \vDash \psi$. In spite of this, we can state laws for relative exhaustiveness that are somewhat analogous to the basic laws for entailment. First two basic laws:

REPETITION. *A set of assumptions renders exhaustive any set of alternatives that it overlaps.* That is, $\Gamma, \varphi \vDash \varphi, \Delta$ (for any sentence φ and any sets Γ and Δ).

CHAIN LAW. *If a set of sentences each of which is a sufficient exception to a claim of exhaustiveness itself renders exhaustive a set of sentences each of which is a sufficient additional assumption for the claim, the claim holds without exceptions or additional assumptions.* Suppose (i) $\Gamma \vDash \varphi, \Delta$ for each φ in Σ, (ii) $\Gamma, \psi \vDash \Delta$ for each ψ in Θ, and (iii) $\Sigma \vDash \Theta$. Then $\Gamma \vDash \Delta$ (for any sentences φ and ψ and any sets Γ, Δ, Σ, and Θ).

Although the first of these is similar to the law for premises, it is given a different name because this law is as much about alternatives as about assumptions. The metaphor of a chain does not apply very directly to the second law, but this law does play a role for conditional exhaustiveness that is analogous to the chain law for entailment. Its verbal statement is more complex than the other laws, and it may not be clear how to fit it with what follows. The idea is that condition (i) tells us that the claim of conditional exhaustiveness of Δ given Γ holds when we add to Δ any member φ of Σ as a further alternative (i.e., as an exception to the claim). Condition (iii) guarantees the exhaustiveness of Θ given Σ, and condition (ii) tells us that the exhaustiveness of Δ holds given Γ together with any member ψ of Θ as a further assumption. The law then holds because, if each member of Γ is true, then by (i) we must have at least one member of Δ true unless each member of Σ is true; and, if the latter is the case, by (iii) we must have at least one member of Θ true and, by (ii), this is enough to insure that at least one member of Δ is true as we wished.

As with entailment, we will consider two laws that follow from this basic pair.

MONOTONICITY. *Adding assumptions or alternatives never undermines conditional exhaustiveness.* That is, if $\Gamma \vDash \Delta$, then $\Gamma, \Sigma \vDash \Delta, \Theta$ (for any sets Γ, Δ, Σ, and Θ);

CUT. *An alternative may be dropped if adding it as an assumption is enough to render the remaining alternatives exhaustive.* That is, if $\Gamma, \varphi \vDash \Delta$ and $\Gamma \vDash \varphi, \Delta$, then $\Gamma \vDash \Delta$ (for any sentence φ and any sets Γ and Δ).

The second is relatively close in form to the law for lemmas but it given a given a different name, as was the repetition law, because assumptions and alternatives play parallel roles in it. The significance of the term cut lies simply in its effect of dropping the sentence φ. The idea behind that is that, given the truth of all members of Γ, at least one of the alternatives Δ to be true in a case where φ is true because $\Gamma, \varphi \vDash \Delta$ and in a case where φ false because $\Gamma \vDash \varphi, \Delta$ and φ cannot be the alternative that is true.

One of the reasons for considering conditional exhaustiveness is that a law providing alternatives *via* assumptions follows from the basic laws. This law takes the following form:

ALTERNATIVES *VIA* ASSUMPTIONS. If both $\varphi, \psi \vDash$ and $\vDash \varphi, \psi$ (i.e., φ and ψ are contradictory), then $\Gamma \vDash \varphi, \Delta$ if and only if $\Gamma, \psi \vDash \Delta$.

To see why this follows, suppose that φ and ψ are contradictory and $\Gamma \vDash \varphi, \Delta$. We can apply the chain law with $\Gamma, \psi \vDash \Delta$ as the claim we wish to establish and $\varphi, \psi \vDash$ (i.e., $\varphi, \psi \vDash \varnothing$) as the claim cited in condition (iii). Because the Θ mentioned in the law is the empty set $\varnothing$ in this case, there is nothing to show for (ii) since there is no member of Θ for which it might fail; and (i) says merely that $\Gamma, \psi \vDash \varphi, \Delta$, which holds by monotonicty (since we have assumed that $\Gamma \vDash \varphi, \Delta$), and $\Gamma, \psi \vDash \psi, \Delta$, which holds by repetition. We can use $\vDash \varphi, \psi$ in a similar way to show $\Gamma \vDash \varphi, \Delta$ when we suppose $\Gamma, \psi \vDash \Delta$.

We cannot expect to get the law providing for inconsistency *via* Absurdity without some principle stating the logical properties of $\bot$ (something we will consider in the next subsection), but we can say that $\Gamma \vDash$ (i.e., Γ is inconsistent) if $\Gamma \vDash \varphi$ and $\varphi \vDash$ (i.e., φ is absurd). (The argument applies the chain law in a way similar too, but simpler than, the one we just saw.) In the other direction, knowing that Γ is inconsistent does not enable us to conclude that it entails some inconsistent sentence because we don't yet have a law telling us that there are any inconsistent sentences. But we can say that if Γ is inconsistent, it entails any inconsistent sentence there is because an inconsistent set entails any sentence whatsoever: we know that if $\Gamma \vDash$ (i.e., $\Gamma \vDash \varnothing$) then $\Gamma \vDash \varphi$, for any sentence φ, by monotonicity. This gives us the following law pointing the way to, if not providing, inconsistency *via* absurdity:

INCONSISTENCY *VIA* ABSURDITY. If $\varphi \vDash$, then $\Gamma \vDash$ if and only if $\Gamma \vDash \varphi$.

1.4.7. Duality

In the context of conditional exhaustiveness all that need be said about the logical properties of Tautology ⊤ and Absurdity ⊥ is that Tautology is a tautology (i.e., $\vDash \top$) and that Absurdity is absurd (i.e., $\bot \vDash$). The first of these makes sense for entailment and, together with the basic laws of entailment provides with the sort of laws we will go on to consider shortly. However, it is the latter laws that we will focus on since they state the role of ⊤ in entailment. And, in the case of ⊥, saying merely that it is absurd tells us nothing from the point of view of entailment since that is to say only that $\bot \vDash \bot$.

Tautology ⊤ is entailed by any set of premises (the empty set included) because it cannot go beyond the information contained in any set of sentences; and, for the same reason, the presence of ⊤ among the premises of an argument contributes nothing to the argument's validity. These two ideas can be expressed more formally in the following laws.

LAW FOR ⊤ AS A CONCLUSION. $\Gamma \vDash \top$ (for any set Γ).

LAW FOR ⊤ AS A PREMISE. $\Gamma, \top \vDash \varphi$ if and only if $\Gamma \vDash \varphi$ (for any set Γ and sentence φ).

Although they are stated for ⊤, these laws will hold for all tautologies since they hold simply in virtue of the proposition expressed by ⊤.

These laws are different in character from the ones consider in the last subsection because they concern the logical properties of a specific sort of sentence rather than the general principles governing logical relations. They are also a first sample of a common pattern in the laws of deductive reasoning that we will consider. Entailment is so central to deductive reasoning that an account of the role of a kind of sentence in entailment as a conclusion and as a premise will usually tell us all we need to know about it.

A simple law describes the role of absurdities as premises. We state it for the specific absurdity ⊥.

LAW FOR ⊥ AS A PREMISE. $\Gamma, \bot \vDash \varphi$ (for any set Γ and sentence φ).

An argument with an absurdity among its premises is valid by default. Since its premises cannot all be true, there is no risk of *new* error no matter what the conclusion is. There is no law restating the significance of having ⊥ as a conclusion because that is simplest way we have of using entailment to say that a set of assumptions is inconsistent.

Although entailment will be our focus, it is enlightening to consider analogues for conditional exhaustiveness of the laws just stated. In particular, we can state a law for ⊥ as an alternative in the context of conditional exhaustiveness, and all the properties of ⊤ and ⊥ take a particularly symmetric form when stated in terms of that relation.

	as a premise	*as an alternative*
Tautology	if $\Gamma, \top \vDash \Delta$, then $\Gamma \vDash \Delta$	$\Gamma \vDash \top, \Delta$
Absurdity	$\Gamma, \bot \vDash \Delta$	if $\Gamma \vDash \bot, \Delta$, then $\Gamma \vDash \Delta$

That is, while ⊤ contributes nothing as a premise and may be dropped, it is enough for a claim of conditional exhaustiveness to hold that it be an alternative (no matter how small the set Γ of premises or the set Δ of other alternatives). And while it is enough to have ⊥ as a premise (no matter how small the set of alternatives is), it contributes nothing as an alternative and may be dropped.

Notice that the converses of the principles at the upper left and lower right hold by monotonicity because they are just the addition of a premise in one case and an alternative in the other. If we take the if and only if principle that results from adding the converse to the lower right and consider a case where Δ is empty, we get

$$\Gamma \vDash \bot \text{ if and only if } \Gamma \vDash$$

This is the principle for conditional exhaustiveness that lies behind the law providing inconsistency *via* Absurdity of 1.4.5 (and it follows from the law promising inconsistency *via* absurdity that was stated at the end of the last section once we have stated that $\bot \vDash$). The moral is that our use of ⊥ as a conclusion to define inconsistency in terms of entailment really involves the same idea as the principle for ⊥ as an alternative that may be stated for conditional exhaustiveness.

The symmetry exhibited by the set of principles in the table above might be traced to the symmetry of conditional exhaustiveness: since ⊤ and ⊥ are contradictory, having one as an assumption comes to the same thing as having the other as an alternative according to the law of 1.4.5 providing alternatives *via* assumptions. However, there is a more general idea behind this symmetry that will apply also to cases where sentences are not contradictory.

The essential difference between the lower left and upper right in the table above lies in interchanging ⊥ and ⊤ and, at the same time, interchanging premises and alternatives. And the same is true of the upper left and lower right. That is, if we apply this transition to the lower left, we get

$$\Delta \vDash \Gamma, \top$$

and that differs from the upper right only in the order of the alternatives and the exchange of Δ for Γ. And neither of these differences is important.

Alternatives function only as a set, so the order in which they are listed does not matter. And, since each of Γ and Δ could be any set, exchanging them does not alter the content of the principle. Either way, we say that it is enough to have ⊤ as an alternative no matter what premises and what further alternatives we have. The possibility of the sort of transformation used to get from the lower left to the upper right can be expressed by saying that ⊤ and ⊥ on the one hand and premise (or assumption) and alternative on the other constitute pairs of *dual* terms. We will run into other pairs of terms later that fit into the same sort of duality.

Glen Helman 03 Aug 2010

1.4.s. Summary

1 Entailment may be defined in two equivalent ways, negatively as the relation that holds when the conclusion is false in no possible world in which all the premises are true or positively as the relation which holds when the conclusion is true in all such worlds. The negative form has the advantage of focusing attention on the sort of possible world that serves as a counterexample to a claim of entailment. The positive form characterizes a relation of entailment as a conditional guarantee of the truth of the conclusion, a guarantee conditional on the truth of the premises.

2 The requirements for a world to serve as a counterexample to entailment suggest the general idea of dividing a pair of sets by making all members of the first true and all members of the second false. A world will be said to divide an argument when it divides the premises from the conclusion.

3 The idea of division enables us to define a relation of conditional exhaustiveness between sets: one set renders another exhaustive when there is no possible world that divides the two sets. We will extend the notation for entailment to express this relation between sets Γ and Δ as Γ ⊨ Δ. Entailment is the special case of this where Δ has only one member. When Δ has more than one member, its members will be referred to as alternatives because a relation of conditional exhaustive provides a conditional guarantee only that at least one member of the second set it true.

4 Since a set of alternatives can have more than one member or be empty, conditional exhaustiveness encompasses all the deductive properties and relations we have considered (as well as an extension of the idea of joint exhaustiveness to any set of sentences). The way a property or relation is expressed using conditional exhaustiveness is tied directly to the negative form of the definition of the property or relation. When no relation of conditional exhaustiveness holds no matter how a set is divided into two parts, all patterns of truth values for its members are possible and the set is logically independent. A single sentence that forms a logically independent set is logically contingent.

5 Definitions in terms of conditional exhaustiveness can be converted into definitions in terms of entailment by replacing empty sets of alternatives with ⊥ and reducing the size of multiple sets of alternatives by replacing members by adding assumptions that are contradictory to them (using the basic law for conditional exhaustiveness).

6 Conditional exhaustiveness and entailment satisfy a principle of

monotonicity. The term monotonic reflects the fact that conditional exhaustiveness or entailment will never stop holding because of additions to the set of assumptions or set of alternatives. This principle is significant in distinguishing entailment from other forms of good inference, whose riskiness means that they are non-monotonic because adding information telling us that the risk does not pay off will undermine their quality. Both conditional exhaustiveness and entailment also satisfy analogues to the principles of reflexivity and transitivity for implication. In the case of reflexivity, these laws are repetition for conditional exhaustiveness and the law for premises for entailment. For transitivity, they are cut for conditional exhaustiveness and the law for lemmas for entailment. The latter licenses the use of lemmas, valid conclusions that are of interest only as premises in further arguments. A more general law, called the chain law, together with a law for premises, yields all laws of entailment, and these two principles amount to principles of reflexivity and transitivity for a relation of set entailment that holds when one set entails each member of another.

7 The laws describing the behavior of ⊤ and ⊥ in the context of conditional exhaustiveness exhibit a kind of symmetry that we will see in other laws later. The sentences ⊤ and ⊥ are dual as are the terms premise and alternative (or the left and right of an turnstile) in the sense that replacing each such term in a law by the one dual to it will produce another law.

1.4.x. Exercise questions

1. Any claim that a deductive relation holds can be stated as one or more claims that one set of sentences cannot be divided from another. (i) Restate each of the following claims in that way, and (ii) explicitly describe the sort of possibility that would divide the sets in question and is thus ruled out by claiming that the deductive relation holds. Nonsense words have been used to help you think to think how a possibility would be described without worrying whether that possibility could really occur.

For example, the claim that the sentences The widget plonked and The widget plinked are equivalent can be restated by saying that (i) the set consisting of the first sentence cannot be divided from the set consisting of the second sentence and vice versa. That is, (ii) it rules out any possibility in which the widget plonked but did not plink and any possibility in which the widget plinked but did not plonk.

- **a.** The gizmo is a widget and The gizmo is a gadget are mutually exclusive
- **b.** The gizmo is a widget and The gizmo is a gadget are jointly exhaustive
- **c.** The widget plinked is a tautology
- **d.** The widget plonked is absurd
- **e.** The widget was a gadget renders exhaustive the alternatives The widget plinked and The widget plonked
- **f.** The widget was a gizmo, The widget plinked, and The widget plonked are inconsistent

2. The basic law for conditional exhaustiveness can be used not only to replace alternatives by assumptions but also to replace assumptions by alternatives. For example, the claim that The widget is blue entails The widget is colored can be restated to say (i) The widget is blue and The widget is not colored are inconsistent, (ii) The widget is not blue and The widget is colored form an exhuastive set, or (iii) The widget is not colored entails The widget is not blue.

In the following, you will be asked to restate some statements of deductive relations by replacing alternatives with assumptions or assumptions with alternatives. You may add or remove ordinary negation to state the contradictories of sentences.

a. Restate the following as a claim of entailment: The gadget is red and The gadget is green are mutually exclusive

b. Restate the following as a claim of entailment: Someone is in the auditorium and There are empty seats in the auditorium are jointly exhaustive

c. Restate the following as a claim of absurdity: A widget is a widget is a tautology

d. Restate the following as a claim of tautologousness: A widget is a gadget is absurd

e. Restate the following as a claim of inconsistency: The widget is a gadget or gizmo and The widget is not a gadget entail The widget is a gizmo

f. Restate the following so that each assumption is replaced by an alternative and each alternative by an assumption: The widget has advanced and The widget has plonked render exhaustive the alternatives The widget has finished the task and The widget has broken

1.4.xa. Exercise answers

1. a. (i) The set consisting of The gizmo is a widget and The gizmo is a gadget cannot be divided from the empty set; that is, (ii) there is no possibility of the gizmo being both a widget and a gadget.

b. (i) The empty set cannot be divided from the set consisting of The gizmo is a widget and The gizmo is a gadget; that is, (ii) there is no possibility of the gizmo being neither a widget nor a gadget

c. (i) The empty set cannot be divided from the set consisting of only The widget plinked; that is, (ii) there is no possibility that the widget did not plink

d. (i) The set consisting of only The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget plonked

e. (i) The set consisting of only The widget was a gadget cannot be divided from the set consisting of The widget plinked and The widget plonked; that is, (ii) there is no possbility that the widget was a gadget while not either plinking or plonking.

f. (i) The set consisting of The widget was a gizmo, The widget plinked, and The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget was a gizmo and both plinked and plonked

2. a. The gadget is red entails The gadget is not green (*or:* The gadget is green entails The gadget is not red)

b. The auditorium is empty entails There are empty seats in the auditorium (*or:* There are no empty seats in the auditorium entails The auditorium is not empty)

c. A widget is a not widget is absurd

d. A widget is a not gadget is a tautology

e. The widget is a gadget or gizmo, The widget is not a gadget, and The widget is not a gizmo are inconsistent

f. The widget has not finished the task and The widget has not broken render exhaustive The widget has not advanced and The widget has not plonked