

2.4. Using lemmas

2.4.0. Overview

Although our system of derivations as it stands is both sound and complete, we will add rules that reflect the use of lemmas, both because of the importance of lemmas in ordinary explicit deductive reasoning and because the sorts of organization and simplification they provide in that context are of value for our work, too.

2.4.1. The dangers of lemmas

Although the use of lemmas is valuable in general, not all individual lemmas are valuable: the uncontrolled use of lemmas can lead us into blind alleys or delay the progress of a derivation.

2.4.2. Lemmas for *reductio* arguments

A lemma that is entailed by our goal is safe (though not necessarily progressive); this means that any lemma is safe when the goal is \perp .

2.4.3. Attachment rules

Lemmas are certainly safe when we know we can prove them. We will use such lemmas to add to the available resources. The sentences added in this way may be more complex than those already present, so this use of lemmas can interfere with decisiveness.

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2.4.1. The dangers of lemmas

A fully general rule for introducing lemmas was cited in 2.3.4 as an example of an unsafe rule because the resources of a gap might not entail the lemma even when the proximate argument of the gap is valid. Such a rule would also prevent a system from being decisive because it would always be possible to develop a gap further by introducing a lemma. However, as was noted earlier, more limited rules for introducing lemmas can be safe, and we will see later that they can also be progressive. In this section we will look at the problems posed by lemmas more closely before going on, in 2.4.2 and 2.4.3, to consider a couple of special cases where these problems do not arise.

The law for lemmas of 1.4.6 can be stated as follows:

$$\Gamma \vDash \varphi \text{ if both } \Gamma \vDash \psi \text{ and } \Gamma, \psi \vDash \varphi$$

Let us recall why this is true: any possible world that is a counterexample to the first entailment will be a counterexample to one of the two on the right—the first of them if it makes ψ false and the second if it makes it true. So if both entailments on the right hold (that is, neither has a counterexample), then the one on the left will hold, too.

But this principle is stated only as an **if** claim, and the corresponding **only if** statement is not always true. When $\Gamma \vDash \varphi$, we know that $\Gamma, \psi \vDash \varphi$ by the monotonicity of \vDash ; but, since φ and ψ need have no connection with one another, knowing that $\Gamma \vDash \varphi$ would by itself give us no reason to suppose that $\Gamma \vDash \psi$. Of course, in a case where we know that $\varphi \vDash \psi$, we would know $\Gamma \vDash \psi$ because of the chain law, and there are other cases where we would know $\Gamma \vDash \psi$ because of special connections between Γ and ψ . We will use lemmas in special cases like these; but, before turning to them, let's look at what a fully general rule for lemmas would be like.

If used in tree-form proofs a rule for lemmas would take the following form:

$$\text{Lem} \frac{\psi \quad \varphi}{\varphi}$$

This rule divides the proof of φ into two branches, and represents a division of the task of proving φ into two components. The first is to prove the lemma ψ , and the second is to prove φ using the lemma ψ in addition to the already available assumptions and conclusions derived from them. The occurrence of ψ above φ at the right is intended to indicate that ψ is an assumption from which conclusions can be drawn in this branch, and the slash through it indicates that

is not an assumption of the proof after the use of Lem. Such an assumption is said to be *discharged*, and we will use the same term when an assumption is no longer available in a derivation.

The assumption ψ may occur at the tips of several branches on the right, and it is legitimate to discharge some or all of them. The effect of Lem is understood to be the same as replacing the discharged assumptions ψ in the right branch with conclusions proved in the way shown by the left branch, so the effect of the rule is to erase the discharged assumptions as assumptions.

The value of using Lem rather than proving ψ on the right to begin with lies in allowing us to consider the two components of the argument separately and to save some work in cases where the assumption ψ appears more than once in the tree on the right. Both of these functions lie behind the use of lemmas in mathematical proofs. In the next section, we will see a simple example of the second sort of use.

The appearance of a corresponding step in a sequent proof is shown below. Notice that the assumption ψ is dropped between the second of the premise sequents and the conclusion sequent.

$$\text{law for lemmas} \frac{\Gamma \vDash \psi \quad \Gamma, \psi \vDash \varphi}{\Gamma \vDash \varphi}$$

Sequent proofs with such a rule would work in only one direction. The law for lemmas does justify the conclusion sequent if we are able to establish the premise sequents, but we cannot go in the other direction. That is, since the conclusion sequent may hold even if the first of the premise sequents does not hold, we cannot investigate the requirements for an entailment to hold by moving up from the root.

In the notation of derivations, we use scope lines to mark the scope of added assumptions, which are marked off from other resources along a scope line by the sort of horizontal line we use to indicate the premises of the ultimate argument. In the diagram below, notice that the proximate arguments of the gaps before and after the rule is applied follow the pattern shown in the sequent proof step above.

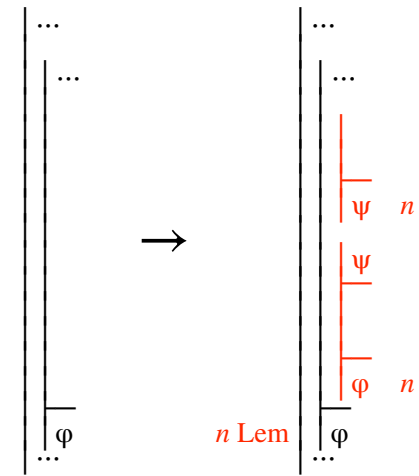


Fig. 2.4.1-1. Developing a derivation by introducing a lemma at stage n (a rule that will be part of our systems of derivations only in more restricted forms).

The assumption ψ is available only to the right of its scope line. After that scope line ends, it is said to have been discharged. The part of the proof in which this assumption is available corresponds to the right side of the tree in the tree-form and sequent proofs we have been looking at.

The effect of this rule on an argument tree is the following pattern of branching:

$$\frac{\Gamma / \psi \quad \Gamma, \psi / \varphi}{\Gamma / \varphi}$$

Again notice that ψ is added to the premises in the right-hand child, and it is no longer among the assumptions when we move from this child back down the tree to its parent. (Nothing rules out the possibility that ψ already appears in Γ ; this would be equivalent to having a tree-form proof in which the use of Lem discharges only some of the occurrences of the assumption ψ .)

The second of the two new gaps in a derivation developed using Lem should be, if anything, easier to close than the original gap because it has a further resource. This increased ease is the point of introducing a lemma. The price we pay for this is the need to close the first new gap also. If the lemma is properly chosen, that may also be easier than closing the original gap; but, because this rule is unsafe, we cannot be sure in general that the first new gap can be closed at all even if the original one could be closed eventually in other ways. Because of this, when lemmas are introduced in ordinary deductive reasoning we must be prepared to backtrack, to abandon the attempt to work by way of the lemma

and look for another approach to the proof. The notation of derivations is not designed to incorporate backtracking, so we will use lemmas only in cases where we can be sure there will be no need to do that.

Indeed, we will not incorporate the rule Lem in the general form given here into our system of derivations. Instead, we will employ more specific rules that are based on the idea behind it. Even in cases where we can sure backtracking is not necessary, the introduction of lemmas could interfere with decisiveness if there were enough safe lemmas to keep introducing them forever. So our restrictions on the use of alternatives to Lem will be more severe than would be required merely to insure their safety.

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2.4.2. Lemmas for *reductio* arguments

We have seen that one case where a lemma is bound to be safe is when it is entailed by the goal we seek. That is, we can state following principle:

If $\varphi \vDash \psi$, then $\Gamma \vDash \varphi$ if and only if both $\Gamma \vDash \psi$ and $\Gamma, \psi \vDash \varphi$

which tells us that, when $\varphi \vDash \psi$, it is not only sound but also safe to introduce a lemma ψ in a derivation whose goal is φ . It is the **only if** part of this principle—and, more specifically, the part that says $\Gamma \vDash \varphi$ only if $\Gamma \vDash \psi$ —that requires the assumption that $\varphi \vDash \psi$.

In order to apply the idea of this principle in derivations, we can look for convenient ways of insuring that $\varphi \vDash \psi$. The obvious valid arguments of this form among those we have identified so far are EFQ and the two forms of Ext. Although EFQ will prove to be the more important, Ext is a better source of examples at the moment and we will consider it first. Here is a derivation which uses the rule Lem to introduce a lemma that is the result of applying left extraction to the final goal.

	$A \wedge B$	1
1 Ext	A	(5),(9)
1 Ext	B	(4)
	<div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;">●</div> </div> </div>	
4 QED	B	3
	<div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;">●</div> </div>	
5 QED	A	3
3 Cnj	$B \wedge A$	2
	$B \wedge A$	(7),(10)
	<div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;">●</div> </div>	
7 QED	$B \wedge A$	6
	<div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;">●</div> </div>	
9 QED	A	8
	<div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;"> <div style="border-left: 1px solid black; padding-left: 5px; margin-left: 5px;">●</div> </div>	
10 QED	$B \wedge A$	8
8 Cnj	$A \wedge (B \wedge A)$	6
6 Cnj	$(B \wedge A) \wedge (A \wedge (B \wedge A))$	2
2 Lem	$(B \wedge A) \wedge (A \wedge (B \wedge A))$	

Here the rule Lem is applied at stage 2 with the left component of the goal as the lemma. This yields a slight shortening of the derivation since we are able to use the lemma to conclude $B \wedge A$ by QED at stages 7 and 9 rather than

repeating the proof used at stages 3-5 twice.

The basic idea here—isolating a component of an argument to avoid repeating it—is an important one. However, the actual simplification in this case is limited, and we would have few opportunities to use lemmas whose safety was assured by Ext. So we will not build this use of lemmas into our system of derivations, it will serve us only as an initial example.

The pattern *Ex Falso Quodlibet* provides the basis for a much more important use of lemmas. An argument whose conclusion is \perp is often called a *reductio argument*; *reductio* here is short for the Latin phrase *reductio ad absurdum* ('reduction to absurdity'). We will often need to use a lemma to complete such an argument and, since EFQ tells us that \perp entails any sentence, we know that any lemma we choose is safe. We will call the rule implementing this idea *Lemma for Reductio* or LFR:

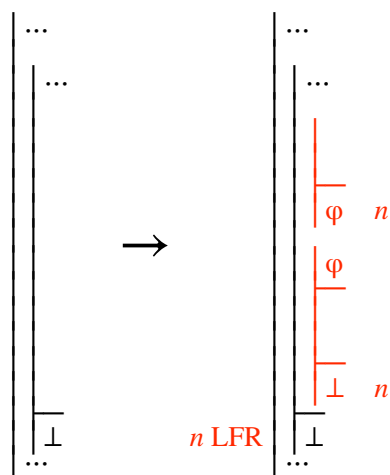


Fig. 2.4.2-1. Developing a derivation by introducing a lemma for a *reductio* at stage n .

The principle associated with this rule is the following:

$\Gamma \vDash \perp$ if and only if both $\Gamma \vDash \varphi$ and $\Gamma, \varphi \vDash \perp$

Although this follows from the principle stated at the beginning of the section and the validity of EFQ, it is instructive to look at its justification more directly. The *if* part again is just an instance of the law for lemmas. The *only if* part tells us that any interpretation dividing one of the child gaps will also divide the parent. But this must be so because any interpretation dividing a child will make the active resources of the parent true (since they all remain active in each of the children) and every interpretation makes \perp false.

Unfortunately, we are not yet in a position to illustrate this rule because we

have no non-trivial examples of formally valid *reductio* arguments. A *reductio* is formally valid only if its premises constitute a formally inconsistent set (that is, one whose members cannot be all true on any extensional interpretation) and the only formally inconsistent sets available with our current analyses of sentences contain \perp either as a member or as a component of one. Such a set can be shown to entail \perp with use of nothing but Ext and QED, so introducing LFR would merely complicate that argument.

In the next chapter, the rule LFR will serve us as a temporary expedient, but we will eventually introduce other special rules that are designed to cover the case where LFR would be most useful, and LFR itself will be ignored. One reason is that the free use of LFR would undermine decisiveness since the form of the rule places no constraints on the number of different lemmas that might be introduced. Something like a limitation to sentences that already appear as components of active resources and goals would be sufficient to insure decisiveness and would still permit the more important uses of the rule, but we will not attempt to formulate the sort of restriction that would enable us to prove decisiveness for a system with LFR. It is simply not important enough to bother. Apart from its role as a temporary expedient, it will serve us mainly as a way of displaying the connection between the special rules to be introduced later and the idea of a lemma.

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2.4.3. Attachment rules

The second sort of case in which use of a lemma is bound to safe is one in which it is clearly entailed by the available resources. A principle justifying that use would take the following form:

If $\Gamma \vDash \psi$, then $\Gamma \vDash \phi$ if and only if $\Gamma, \psi \vDash \phi$

If we were to drop the **only if**, this would be just another way of stating the law for lemmas, and we know that $\Gamma \vDash \phi$ only if $\Gamma, \psi \vDash \phi$ by monotonicity.

There might seem to be little point in regarding this as a case of a lemma at all. If a statement is entailed by the resources, why not just have a rule that allows us to add it to our active resources? Indeed, as far as soundness and safety are concerned, this would come to the same thing. But this second use of lemmas is motivated (as well as constrained) by considerations of decisiveness. That is, our system is decisive in part because active resources are added only to reduce the complexity of proximate arguments, and it may be useful to reach our goal by way of a lemma that is entailed by our resources but is more complex than they are. Adding such a sentence as an active resource would open the door to going around in circles in which we add complexity only to simplify and then add complexity again, and so on.

When discussing the soundness of QED in 2.3.4, we saw that it would be legitimate for a rule to close a gap when its goal is not among its active resources—or even among the active resources of its ancestor gaps—if the goal was entailed by available resources. We will not use such a sweeping rule but we will introduce a few rules in special cases that add to the available resources of a gap without changing either its active resources or its goal.

An example is the following way of developing a gap, which we will call *Adjunction*:

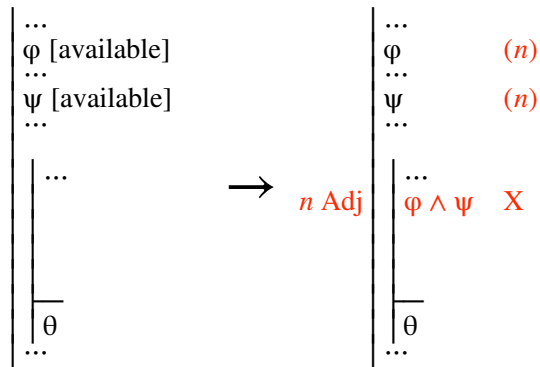


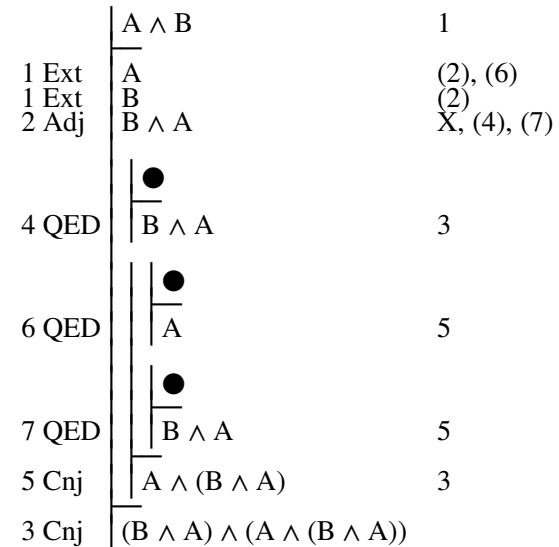
Fig. 2.4.3-1. Developing a derivation by applying Adj at stage n .

The added conjunction functions as a lemma, so this rule represents a way of using lemmas. However, it has a number of special features, both by comparison with a rule like LFR and by comparison with other rules we have seen.

The lemma $\phi \wedge \psi$ does not lie to the right of a new scope line, as it does in the second gap introduced by LFR, for two reasons. First, we have not branched the gap so the added resource is available throughout the gap. And, second, we do not need to mark this new resource off as an added assumption because it is entailed by those already present.

Notice also that we treat this rule not as a way to plan for our goal but simply as a way to add resources. However, it does not exploit resources in order to add others and the X to the right of $\phi \wedge \psi$ is intended to indicate that this resource need not be exploited further. One way to think about this is to suppose that $\phi \wedge \psi$ has been introduced as something already exploited. That is, although it need not have once been an active resource that has since been exploited (and it would already be part of the available resources if it had been), it has a status similar to such resources.

One example of the use of Adj is provided by the [example in 2.4.2](#) (though we are thinking of the lemma differently now: there we thought of it as something entailed by the goal while here we think of it as something entailed by the resources).



With two more uses of Cnj, we would not have needed Adj; and, with two more uses of Adj, we would not have needed Cnj. Still, it is this sort of mixed use of the two rules that brings us closest to typical patterns of explicit deductive argument.

This example also exhibits the sort of foresight or insight that is required to Adj and similar rules. At stage 2, after simplifying our resources as far as possible, we look ahead to the analysis of the goal (before we have built that analysis into our derivation at stages 3 and 5), and we see that we will need to establish $B \wedge A$ twice to reach it. Noticing that we already have the makings of this sentence among our resources, we then assemble it using Adj to have it available for later use.

Adjunction is one example of a group of rules we will refer to as *attachment rules*. Any such rule R will exhibit the following general pattern.

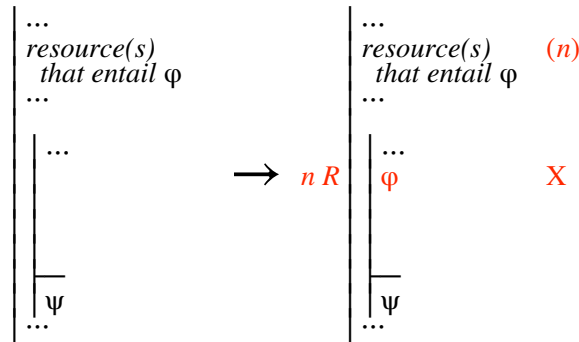


Fig. 2.4.3-2. Developing a derivation by applying an attachment rule R at stage n .

Since the lemma ϕ is not an active resource, the proximate argument of child gap is the same as the parent's proximate argument. This means that safety and soundness (even strictness) hold as they would for a gap that is completely unchanged. A rule like this must be considered when arguing for the soundness of rules like QED that use merely available resources, but the required argument was already considered in 2.3.4: an interpretation that divides a gap and all its ancestors will already make true not only all the available resources but also any sentence ϕ that is entailed by them, so we do nothing to change the situation by adding such a sentence ϕ to the available resources.

Of course, a rule like Adj does raise questions about decisiveness since the lemma it introduces is more complex than the premises it is based on. This increased complexity will be typical of attachment rules and is the reason for their name. We will not state the sort restriction on the use of attachment rules that would enable us to prove decisiveness; and, for practical purposes, the most valuable constraint on their use is simple good sense. But, as a rule of thumb, it is natural to limit the use of such rules to cases where the lemma is a component of a goal or active resource since such cases will represent the principal grounds for using these rules in any case. But it is important to remember that a sentence is a component of itself, and one common use of

these rules will be to introduce the goal itself as an available resource in order to apply QED.

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2.4.s. Summary

- 1 Using a lemma is one way of dividing up the work of a proof. We might use lemmas in derivations by dividing a gap into two gaps, one with the lemma as a goal and the other with it as a further assumption to use in reaching the original goal. A rule Lemma (Lem) that does this is not safe in general, and we will use only special instances of it.
- 2 A lemma is always safe when it is entailed by the current goal. We can use this idea in reductio arguments, arguments whose goal is \perp . Since \perp entails any sentence, the rule that introduces lemmas in such circumstances, Lemma for Reductio (LFR), will be safe (though some restriction on its use is needed to insure it is progressive).
- 3 A lemma is also safe if we know we can establish it. We will use this sort of lemma only in attachment rules, rules that add the lemma as an available but inactive resource. The first example of this sort of rule is Adjunction (Adj) which adds a conjunction when both conjuncts are already available. Although attachment rules can help us to close gaps sooner, care is needed in their use if they are to be progressive.

The derivation rules we have so far are summarized in the table below. The names of the rules are links to the point in the text where they were initially described; look there to see the actual form taken by the rule.

<i>Rules for developing gaps</i>		<i>Rules for closing gaps</i>		<i>Basic system</i>	
<i>for resources</i>	<i>for goals</i>	<i>when to close</i>	<i>rule</i>		
conjunction $\varphi \wedge \psi$	<u>Ext</u>	<u>Cnj</u>	the goal is also a resource		<u>QED</u>
			T is the goal		<u>ENV</u>
			\perp is a resource	<u>EFQ</u>	
			<i>Attachment rule</i>	<i>Added rules</i> <i>(optional)</i>	
			<i>added resource</i>	<i>rule</i>	
			$\varphi \wedge \psi$	<u>Adj</u>	
			<i>Rule for lemmas</i>		
			<i>prerequisite</i>	<i>rule</i>	
			the goal is \perp	<u>LFR</u>	

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2.4.x. Exercise questions

Use the basic system of derivations along with the attachment rule Adj to establish the following. These repeat entailments from earlier exercises and examples (specifically, **b** and **d** of exercise 2.2.x.2 and exercises **2** and **4** of 2.3.x). They will work best as exercises in the use of Adj if you avoid using Cnj.

1. $A \vDash A \wedge A$
2. $A \wedge B, B \wedge C, C \wedge D \vDash A \wedge D$
3. $A \wedge B \vDash A \wedge (B \wedge A)$
4. $A, B \wedge C, D \vDash (C \wedge (B \wedge A)) \wedge B$

The exercise machine doesn't incorporate attachment rules, so, while it can generate exercises where Adj would be useful, that rule won't be used in any answers it produces.

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2.4.xa. Exercise answers

The answers below avoid the use of Cnj in order to maximize the use of the rule Adj. In some cases, a mixed use of the two would have produced a more natural argument.

1.	A	(1)
1 Adj	A \wedge A	X,(2)
	●	
2 QED	A \wedge A	
2.	A \wedge B	1
	B \wedge C	2
	B \wedge D	3
	A	(4)
1 Ext	B	
1 Ext	B	
2 Ext	B	
2 Ext	C	
3 Ext	B	
3 Ext	D	(4)
4 Adj	A \wedge D	X,(5)
	●	
5 QED	A \wedge D	
3.	A \wedge B	1
1 Ext	A	(2),(3)
1 Ext	B	(2)
2 Adj	B \wedge A	X,(3)
3 Adj	A \wedge (B \wedge A)	X,(4)
	●	
4 QED	A \wedge (B \wedge A)	
4.	A	(2)
	B \wedge C	1
	D	
	B	(2),(4)
1 Ext	C	(3)
1 Ext	B \wedge A	X,(3)
2 Adj	C \wedge (B \wedge A)	X,(4)
3 Adj	(C \wedge (B \wedge A)) \wedge B	X,(5)
4 Adj	●	
5 QED	(C \wedge (B \wedge A)) \wedge B	