1.4. General principles of deductive reasoning

1.4.0. Overview

All the deductive properties and relations of sentences can be seen as special cases of a single relation. We will look at this relation and also see how to study the full range of deductive logic by way of entailment and a couple of auxiliary ideas.

1.4.1. A closer look at entailment

Entailment will be at the heart of our study and we will begin by looking in some detail at a couple ways of formulating its definition.

1.4.2. Division

It will be useful to have a special term for the kind of pattern of truth values that entailment rules out.

1.4.3. Relative exhaustiveness

Although entailment does not encompass all the concepts of deductive logic, there is a similarly defined relation that does.

1.4.4. A general framework

All the deductive properties and relations we will consider can be expressed in terms of relative exhaustiveness and expressed in a way that corresponds directly to definitions of them.

1.4.5. Reduction to entailment

Although relative exhaustiveness provides a way of thinking about deductive properties and relations, entailment is way that they are most naturally established, and we need to consider how this can be done.

1.4.6. Laws for relative exhaustiveness and entailment

The ideas behind the reflexivity and transitivity of implication provide the core of the general principles that hold for the more general relations of relative exhaustiveness and entailment.

1.4.7. Duality

The specific principles concerning ⊤ and ⊥ display a kind of symmetry that we will also find in principles for other logical forms.

Glen Helman 30 Aug 2009

1.4.1. A closer look at entailment

Entailment was introduced in 1.1.6 somewhat informally as a relation between premises and a conclusion that merely extracts information from them and thus brings no risk of new error. Another way of putting the latter point is that a relation of entailment provides a conditional guarantee of the truth of the conclusion: it must be true if the premises are all true.

The discussion of entailment in 1.2.1 developed the resources necessary to give a more formal general definition. In fact it is useful to have in mind two equivalent ways of stating one.

$\Gamma \vDash \varphi$	if and only if	there is no logically possible world in which φ is false while all members of Γ are true
	if and only if	φ is true in every logically possible world in which all members of Γ are true

These are not two different concepts of entailment, for the two statements to the right of `if and only if` say the same thing. Still, they provide different perspectives on the concept. The second—which we will speak of as the *positive form* of the definition—is closely tied to the idea of a conditional guarantee of truth and to the reason why entailment is valuable. The first form —the *negative form*—makes the content of the concept especially clear, and this form of definition will generally be the more useful when we try to prove things concerning entailment. The other deductive properties and relations we have discussed or will go on to discuss can be given analogous pairs of definitions, a negative form ruling out certain patterns of truth values and another form stating a more positive generalization.

The equivalence of the two forms of the definition reflects a feature of all generalizations. When a generalization is false, it is because of a *counterexample*, something that is the sort of thing about which we generalize but that does not have the property we have said that all such things have. A counterexample to the claim that all birds fly is a bird that does not fly. In the positive definition of entailment, the generalization is about all possible worlds in which the premises are all true and such worlds are said to all have the property that the conclusion is true in them. A counterexample to such a generalization is then a world in which the premises are all true but the conclusion is not. The negative form of the definition then affirms the same generalization but by saying that no counterexample exists. As in the case of the generalization use to define entailment, one good way to clarify a generalization is always to ask what sort of counterexample is being ruled out.

It is important to notice how little a claim of entailment says about the actual truth values of the premises and conclusion of an argument. We can distinguish four patterns of truth values that the premises and conclusion could exhibit. Of these, a claim that an argument is valid rules out only the one appearing at the far right of Figure 1.4.1-1.

	Patterns admitted			ruled out
Premises	all **T**	not all **T**	not all **T**	all **T**
Conclusion	**T**	**T**	**F**	**F**

Fig. 1.4.1-1. Patterns of truth values admitted and ruled out by entailment.

So, knowing that an argument is valid tells us about actual truth values only that we do not find the conclusion actually false when the premises are all actually true. The other three patterns all appear in the actual truth values of some valid arguments (though not all are possible for certain valid arguments because other deductive properties and relations of the sentences involved may rule them out).

To see examples of this, consider an argument of the simple sort we will focus on in the next chapter:

It's hot and sunny
It's humid but windy
———
It's hot and humid

This argument is clearly valid since its conclusion merely combines two items of information each of which is extracted from one of the premises. Depending on the state of the weather, the premises may be both true, both false, or one true and the other false; and, in any case where they are not both true the conclusion can be either true or false. In particular, if it's hot and humid but neither sunny nor windy, the conclusion will be true even though both premises are false. This should not be surprising: a false sentence can still contain some true information, so information extracted from a pair of sentences that are not both true might be either true or false.

Of course, seeing one of these permitted patterns does not tell us that the argument is valid; no information that is limited to actual truth values can do that because validity concerns all possible worlds, not just the actual one. In particular, having true premises and a true conclusion does not make an argument valid. For example, the following argument is not valid:

Indianapolis is the capital of Indiana
———
Springfield is the capital of Illinois

For, although the single premise and the conclusion are both true, there is a logical possibility of the capital of Illinois being different while that of Indiana is as it actually is, so there is a possible world that provides a counterexample to the claim that the argument is valid.

Glen Helman 30 Aug 2009

1.4.2. Division

The pattern of truth values for premises and conclusion that is ruled out by entailment (i.e., true premises with a false conclusion) will recur often enough that it will be convenient to have special vocabulary for it. Let us say that a set Γ is *divided* from a set Δ whenever all members of Γ are true and all members of Δ are false. Whatever gives the sentences in Γ and Δ such values will be said to divide these sets. The source of the truth values will differ from context to context though, for the time being, it will be a possible world. When there is something of the appropriate sort that divides a set Γ from a set Δ, we will say that Γ and Δ are *divisible*; otherwise we will say they are *indivisible*.

Notice that these ideas are asymmetric. When one set is divided from another it is the members of the first set that true and the members of the second that are false. You might think of sets being divided vertically, with the first set above the second. In this spatial metaphor, truth is thought of as higher than falsehood; and, although this is only a metaphor, it is a broadly useful one and is consistent with the appearance of Absurdity at the bottom of Figure 1.2.5-2 and Tautology at the top. The asymmetry of division is especially important to remember in the case of the terms divisible and indivisible since their grammatical form is often used for symmetric relations.

As with talk of sets of sentences as premises, it is really only the list of members of a set that we care about here, and we speak of sets only because the order of the list and the occurrence of repetitions in it do not matter. In particular, we will not distinguish between a sentence and a set that has only it as a member. So we can restate the negative definition of entailment as follows:

$\Gamma \vDash \varphi$ if and only if there is no possible world that divides Γ from φ.

We will also say that an argument is divided when its premises are divided from its conclusion, so we can say that an argument is valid when no possible world divides it. So to say that a possible world divides an argument is to say that the world is a counterexample to the argument's validity. The divisibility or indivisibility of an argument thus amounts to the existence or non-existence of such a counterexample.

Glen Helman 30 Aug 2009

1.4.3. Relative exhaustiveness

We can use the idea of division to define a relation between sets rather than between a set and a sentence. And it is useful to do this because the relation we define in this way constitutes a single fundamental idea that encompasses all the deductive properties and relations of sentences. We have focused on entailment and will continue to do so, but it doesn't suffice by itself to capture all the ideas of deductive logic. In particular, we needed the idea of absurdity in 1.2.4 to capture the idea of inconsistency, and we have not yet seen how to say, in terms of entailment, when sentences are jointly exhaustive.

The more general relation we will define using division is *relative exhaustiveness*. When it holds between a pair of sets, we will say that one set *renders* the other set *exhaustive*. Our notation for this idea will extend the use of the entailment turnstile to allow a set to appear on the right. The negative and positive forms of its definition are as follows:

$\Gamma \vDash \Delta$	if and only if	there is no possible world in which all members of Δ are false while all members of Γ are true
	if and only if	in each possible world in which all members of Γ are true, at least one member of Δ is true

Or, in terms of division, $\Gamma \vDash \Delta$ if and only if there is no possible world that divides Γ from Δ. Entailment is the special case of this idea where the set Δ consists of a single sentence: to say that φ is entailed by Γ comes to the same thing as saying that φ is rendered exhaustive by Γ.

In cases of relative exhaustiveness that are not cases of entailment, what is rendered exhaustive is either a set with several members or the empty set. In these cases, it does not make sense to speak of a conclusion, for when the set on the right has several members, these sentences need not be valid conclusions from the set that renders them exhaustive. Indeed, a jointly exhaustive pair of sentences will be rendered exhaustive by any set, but often neither member of the pair will be entailed by that set. This is particularly clear in the case of sentences like The glass is full and The glass is not full that are both jointly exhaustive and mutually exlcusive—i.e., that are contradictory. Although the set consisting of such pair is rendered exhaustive by any set, only an inconsistent set could entail both of these sentences.

This means that we need new terminology for sentences on the right of the turnstile when they appear in groups. We will say that such sentences are *alternatives*. The conditional guarantee provided by a claim $\Gamma \vDash \Delta$ of relative exhausitiveness is a guarantee that the alternatives Δ are not all false—i.e., that at least one is true—provided the premises Γ are all true. In particular, when

$\Gamma \vDash \varphi, \psi$, we have a guarantee that, if the members of Γ are all true, then either φ or ψ is true.

Glen Helman 30 Aug 2009

1.4.4. A general framework

It is not surprising that relative exhaustiveness should encompass deductive properties and relations if these properties and relations are understood to all consist in guarantees that certain parterns of truth values appear in no possible world. For to say that there is no world where certain sentences Γ are true and other sentences Δ are false is to say that $\Gamma \vDash \Delta$. Of course, a given deductive property or relation may rule out a number of different patterns—i.e., rule out a number of different ways of distributing truth values among the sentences it applies to—but this just means that a deductive property or relation may consist of a number of different claims of relative exhaustiveness.

In the case of the properties and relations we will consider, no more than two claims of relative exhaustiveness are ever required, as can be seen in the following table. (When nothing appears to the left or the right of the turnstile, the set on that side is the empty set.)

Concept	*described in terms of relative exhaustiveness*
Γ entails φ	$\Gamma \vDash \varphi$
φ is a tautology	$\vDash \varphi$
φ and ψ are equivalent	both $\varphi \vDash \psi$ and $\psi \vDash \varphi$
Γ excludes φ	$\Gamma, \varphi \vDash$
Γ is inconsistent	$\Gamma \vDash$
φ and ψ are mutually exclusive	$\varphi, \psi \vDash$
φ is absurd	$\varphi \vDash$
Γ is exhaustive	$\vDash \Gamma$
φ and ψ are jointly exhaustive	$\vDash \varphi, \psi$
φ and ψ are contradictory	both $\varphi, \psi \vDash$ and $\vDash \varphi, \psi$

This list adds only one concept to those already discussed, a generalization of the idea of a pair of jointly exhaustivess sentences to the exhaustiveness of a set, and this is a good example of how these descriptions work. A direct definition of this new idea can be read off its description in terms of relative exhaustiveness in the following way. To say that $\vDash \Gamma$ is to say that there is no possible world that divides the empty set and Γ. That is, there is no possible world that makes every member of the empty set true and every member of Γ false. But, since the empty set has no members, there is no way for any possible world to fail to make all its members true because there is nothing to serve as an exception. This means that the property of making all members of the empty set true adds nothing to the description of the sort of world ruled out by the claim that $\vDash \Gamma$, and this claim can be stated more simply by saying that there is no possible world that makes all members of Γ false. To state the definition in positive form, a set Γ is exhaustive when, in every possible world, at least one member of Γ is true. That is, if we take the sets of possible worlds left open by the various members of Γ and put them all together, they will all exhaust all possibilities. In the same way, the definition of each of the properties and relations in the table above can be read off the right side of the table by applying the definition of relative exhaustiveness to the case or cases indicated.

The ideas of division and relative exhaustiveness also provide ways of extending to a set the idea of logical independence introduced in 1.2.6 to speak of the absence of any deductive property or relation in a pair of sentences. First, let us look at this general idea of logical independence directly. We will say that a set Γ of sentences is *logically independent* when every way of assigning a truth value to each member of Γ is exhibited in at least one possible world. This is the same as saying that for every part of the set (counting both the empty set and the whole set Γ as parts of Γ) it is possible to divide that part from the rest of the set. When the set has two members, this is the same as the earlier idea. When the set {φ} containing a single sentence φ is logically independent in this sense, the sentence φ is said to be *logically contingent* because there is at least one possible world in which it is true and at least one where it is false, so its truth or falsity is not settled by logic.

Relative exhaustiveness provides an alternative way of describing this idea. When the sentences in a set are *not* independent, not every way of dividing them into a set of true sentences and a set of false sentences is logically possible. And when that is so, the set contains at least one pair of non-overlapping subsets Γ and Δ such that $\Gamma \vDash \Delta$. So the members of a set are logically independent when the relation of relative exhaustiveness never holds between non-overlapping subsets. (It always holds between sets that overlap because there is no way of dividing such sets.)

When a set is logically independent, each member is contingent and any two of its members are logically independent, but the contingency of members and the independence of pairs does not by itself imply that the set as a whole is logically independent. For example, assume that the sentences X is fast, X is strong, X has skill, and X has stamina form an independent set. Then the sentences

X is fast and strong	X has skill and stamina	X is fast and has stamina

are each contingent, and any two of them can be seen to be independent. However, the first two taken together entail the third, so these three more complex sentences do not form an independent set.

Glen Helman 30 Aug 2009

1.4.5. Reduction to entailment

Relative exhaustiveness relaxes the restriction to a single conclusion found in entailment to allow several alternatives or none at all. To express the ideas captured by relative exhaustiveness in terms of entailment, we need to add ways of capturing each of these added cases.

We have already seen a way to use entailment to say what it is to render exhaustive the empty set of alternatives. In 1.2.4, we characterized inconsistency in terms of entailment and absurdity by what was called the Basic Law for Inconsistency. If we restate that law by expressing inconsistency in terms of relative exhaustiveness, it says

$$\Gamma \vDash \text{ if and only if } \Gamma \vDash \bot$$

so a set renders exhaustive an empty set of alternatives if and only if it entails the absurdity ⊥. Rendering exhaustive an empty set and entailing ⊥ are both *conditional* guarantees of something that cannot happen, so each has the effect of ruling out the possibility of meeting the conditions of the guarantee (i.e., of having all members of Γ true).

To express the idea of rendering exhaustive multiple alternatives using entailment we need help from the concept of contradictoriness. When sentences φ and ψ are contradictory (i.e., when $\varphi \bowtie \psi$), they always have opposite truth values. so making one true comes to the same thing as making the other false. But the difference between having a sentence as an assumption and having it as an alternative lies in the truth value assigned to it in the pattern that is being ruled out by the claim of relative exhaustiveness. This means that having a sentence as an alternative comes to the same thing as having a sentence contradictory to it as an assumption; that is,

$$\text{if } \varphi \bowtie \varphi', \text{ then } \Gamma \vDash \varphi, \Delta \text{ if and only if } \Gamma, \varphi' \vDash \Delta$$

If we apply this idea repeatedly (perhaps infinitely many times), we can move any set of alternatives to the left of the turnstile.

> Basic law for relative exhaustiveness. Suppose Δ′ is the result of replacing each member of Δ by a sentence contradictory to it. Then $\Gamma \vDash \Delta, \Sigma$ if and only if $\Gamma, \Delta' \vDash \Sigma$.

That is, we can remove alternatives if we put sentences contradictory to them among the assumptions. This gives us two ways of restating claims of relative exhaustiveness as entailments: (i) we may replace all but one alternative by assumptions contradictory to them or (ii) we may replace all alternatives by asssumptions contradictory to them and replace the resulting empty set of alternatives by ⊥.

The following table summarizes the application of these ideas to state all the deductive properties we have considered using entailment, absurdity, and contradictoriness:

Concept	*in terms of entailment and other ideas*
Γ entails φ	$\Gamma \vDash \varphi$
φ is a tautology	$\vDash \varphi$
φ and ψ are equivalent	both $\varphi \vDash \psi$ and $\psi \vDash \varphi$
Γ excludes φ (i.e., $\Gamma, \varphi \vDash$)	$\Gamma, \varphi \vDash \bot$
Γ is inconsistent (i.e., $\Gamma \vDash$)	$\Gamma \vDash \bot$
φ and ψ are mutually exclusive (i.e., $\varphi, \psi \vDash$)	$\varphi, \psi \vDash \bot$
φ is absurd (i.e., $\varphi \vDash$)	$\varphi \vDash \bot$
Γ is exhaustive (i.e., $\vDash \Gamma$)	$\Gamma' \vDash \bot$
φ and ψ are jointly exhaustive (i.e., $\vDash \varphi, \psi$)	$\varphi', \psi' \vDash \bot$ (or $\varphi' \vDash \psi$ or $\psi' \vDash \varphi$)
φ and ψ are contradictory (i.e., both $\varphi, \psi \vDash$ and $\vDash \varphi, \psi$)	both $\varphi, \psi \vDash \bot$ and $\varphi', \psi' \vDash \bot$

Here φ′ is any sentence contradictory to φ, and Γ′ is the result of replacing each member of Γ by a sentence contradictory to it

Either of the two further ways of stating exhaustiveness shown in the next-to-last row could be used instead of the second entailment in the last row. And, when a non-empty set Γ is said to be exhaustive, we could leave one member behind as a conclusion rather than adding ⊥; that is, the idea that $\vDash \Gamma, \varphi$ could be restated not only as $\Gamma', \varphi' \vDash \bot$ but also as $\Gamma' \vDash \varphi$. That is, a set containing a sentence φ is exhaustive if we have a guarantee that φ is true when the others members of the set are false. (It may seem pointless to define the relation of contradictoriness in terms of entailment when we need to use the relation in order to do this, but the definition does mean that, once we know a single sentence contradictory to a given sentence, we say what other sentences are contradictory to it using only the ideas of entailment and absurdity.)

1.4.6. Laws for relative exhaustiveness and entailment

Most of the laws of deductive reasoning we will study will be generalizations about specific logical forms that will be introduced chapter by chapter, but some very general laws can be stated at this point. We have already seen a couple of these, the principles of reflexivity and transitivity for implication. And, in fact, more general forms of these principles are at the heart of the basic laws for entailment and relative exhaustiveness.

Both entailment and relative exhaustiveness differ from implication by covering cases of multiple assumptions or no assumptions at all, and relative exhaustiveness similarly flexible with regard to alternatives. This difference is mainly reflected in form taken by the principles analogous to reflexivity and transitivity, and we will turn to them shortly. In addition to those principles we need only say that adding assumptions or alternatives never undermines a claim of entailment or relative exhaustiveness. This idea can be stated formally as follows, where Σ and Θ are the added assumptions and alternatives, respectively:

> MONOTONICITY. If $\Gamma \vDash \Delta$, then $\Gamma, \Sigma \vDash \Delta, \Theta$ (for any sets Γ, Δ, Σ, and Θ);
> *For entailment*. If $\Gamma \vDash \varphi$, then $\Gamma, \Sigma \vDash \varphi$ (for any sets Γ and Σ and any sentence φ).

Although the principle for entailment is stated separately, it is just the special case of the first where the set Δ has φ as its only member and the set Θ is empty.

The idea behind monotonicity is that adding assumptions or alternatives can only make it harder to find a possible world that divides one group from the other, so, if no possible world will divide Γ from Δ, we can be sure that no world will divide the (possibly) larger set of assumptions we get by adding the set Σ from the (possibly) larger set of alternatives we get by adding the set Θ. (The groups are only possibly larger because either set of additions might be empty.)

The term monotonic is applied to trends that never change direction. More specifically, it is applied to a quantity that does not both increase and decrease in response to changes in another quantity. In this case, it reflects the fact that adding assumptions will never lead to a decrease in the sets of alternatives rendered exhaustive by them and adding alternatives will never lead to a decrease in the sets of assumptions rendering them exhaustive.

It is a distinguishing characteristic of deductive reasoning that such a principle holds. For, when reasoning is not risk free, additional data can show that a initially well-supported conclusion is false and do so without undermining the original premises on which the conclusion was based. If such further data were added to the original premises, the result would no longer support the conclusion. This means that risky inference is, in general, *non-monotonic* in the sense that additions to the premises can reduce the set of conclusions that are justified. This is true of inductive generalization and of inference to the best explanation of available data, but the term non-monotonic is most often applied to another sort of non-deductive inference, an inference in which features of typical or normal cases are applied when there is no evidence to the contrary. One standard example is the argument from the premise Tweety is a bird to the conclusion Tweety flies. This conclusion is reasonable when the premise exhausts our knowledge of Tweety; but the inference is not free of risk, and the conclusion would no longer be reasonable if we were to add the premise that Tweety is a penguin.

Now let us turn to principles analogous to the principles of reflexivity and transitivity for implication. Whenever we have an implication $\varphi \vDash \psi$, monotonicity tells us that we also have $\varphi, \Sigma \vDash \psi, \Theta$ for any assumptions Σ and alternatives Θ. Applying this idea to implications given by reflexivity (and adjusting the notation), we have the following principle:

REPETITION. $\Gamma, \varphi \vDash \varphi, \Delta$ (for any sentence φ and any sets Γ and Δ).

LAW FOR PREMISES. $\Gamma, \varphi \vDash \varphi$ (for any sentence φ and any set Γ).

The principle for entailment here has been given a different name to reflect its use in proofs, but it is again a special case of the principle for relative exhaustiveness. The idea in both cases is just the same as that behind reflexivity: since there is no danger of a sentence φ being false given that it is true, there is no way to divide a set of assumptions containing φ from a set of alternatives in which φ also appears.

Before looking at the analogues to transitivity for relative exhaustiveness and entailment, let us look a little more closely at transitivity itself. Recall, that it says that if $\varphi \vDash \psi$ and $\psi \vDash \chi$, then $\varphi \vDash \chi$. This comes to the same thing as saying that if $\varphi \nvDash \chi$ then either $\varphi \nvDash \psi$ or $\psi \nvDash \chi$. And we can justify it in this latter form by noting, first, that if $\varphi \nvDash \chi$, there is a possible world that divides φ from χ and this world must also make ψ either true or false. If the world makes ψ false, it will divide φ from it, showing that $\varphi \nvDash \psi$, and, if it makes the ψ true, it divide it from χ, showing that $\psi \nvDash \chi$. So, when $\varphi \nvDash \chi$, we are bound to have either $\varphi \nvDash \psi$ or $\psi \nvDash \chi$; and if $\varphi \vDash \psi$ and $\psi \vDash \chi$ (so we have neither $\varphi \nvDash \psi$ nor $\psi \nvDash \chi$), we must have $\varphi \vDash \chi$.

When we consider division of sets rather than just sentences, the same idea gives us the following principles:

CUT. If $\Gamma, \varphi \vDash \Delta$ and $\Gamma \vDash \varphi, \Delta$, then $\Gamma \vDash \Delta$ (for any sentence φ and any sets Γ and Δ).

LAW FOR LEMMAS. If $\Gamma, \varphi \vDash \psi$ and $\Gamma \vDash \varphi$, then $\Gamma \vDash \psi$ (for any sentence φ and set Γ);

Any world that divides a set Γ from a set Δ must assign some truth value to the sentence φ; and, depending on the value it assigns, it will either divide the set Γ together with φ from Δ or it will divide Γ by itself from φ together with the set Δ. If we take the special case where the set Δ is a single sentence ψ, we get something like the second principle—in particular, if $\Gamma, \varphi \vDash \psi$ and $\Gamma \vDash \varphi, \psi$, then $\Gamma \vDash \psi$—and the second principle follows by monotonicity since knowing that $\Gamma \vDash \varphi$ is enough to tell us that $\Gamma \vDash \varphi, \psi$.

The name cut simply reflects the appearance of the principle for relative exhaustiveness: we drop reference to φ in moving from the left side to the right. The name of the principle for entailment again reflects a use of the principle in proofs. The term lemma can be used for a conclusion that is drawn not because it is of interest in its own right but because it helps us to draw further conclusions. This law tells us that if we add to our premises Γ a lemma φ that we can conclude from them, anything ψ we can conclude using the enlarged set of premises can be concluded from the original set Γ. Or, to put it in a way that emphasizes its relation to cut, we can drop from a set of premises any sentence that is entailed by the remaining premises.

Although the cut law and the law for lemmas are based on the same idea as the principle of transitivity, they do not themselves assert transitivity for the more general relations they concern (though transitivity for implication follows from them using monotonicity). We should not expect relative exhaustiveness to obey any principle like transitivity because the significance of sets on the left and right of the turnstile is so different: while $\Gamma \vDash \Delta$ tells us that the truth of all members of Γ guarantees the truth of *at least one* member of Δ, something like $\Delta \vDash \Sigma$ would only guarantee the truth of at least one member of Σ if we were assured of the truth of *all* members of Δ. And a true transitivity principle would not even make sense for entailment because it is relation between different sorts of things (sets and single sentences) so individual cases of entailment usually cannot be linked in a chain.

However, we can state a principle that amounts to the transitivity of a relation closely tied to entailment.

CHAIN LAW. If $\Gamma \vDash \varphi$ for each assumption φ in Δ and $\Delta \vDash \psi$, then $\Gamma \vDash \psi$ (for any sentence ψ and any sets Γ and Δ).

This principle follows from the same line of reasoning as the cut law and the law for lemmas: if a possible world divides Γ from ψ, it must also either make all members of Δ true, in which case it divides $\Delta \vDash \psi$, or make some member

Δ false, in which case it divides Γ from that member of Δ and Γ would not entail every member of Δ.

The name of this principle reflects its role as a principle of transitivity for a relation that holds between sets Γ and Δ when Γ entails every member of Δ. Let us call this *set entailment*. The chain law tells us that, if Γ entails each member of Δ and Δ entails each member of Σ, then Γ entails each member of Σ. Set entailment is also reflexive since the law for premises tells us that any set Γ entails each of its members. Since, in fact, the chain law can be combined with the law for premises to yield both the law for lemmas and the monotonicity of entailment, principles stating the reflexivity and transitivity of set entailment might be thought of as the basic principles of entailment.

Glen Helman 30 Aug 2009

1.4.7. Duality

⊤ is entailed by any set of premises because it cannot go beyond the information contained in any set of sentences; and, for the same reason, the presence of ⊤ among the premises of an argument contributes nothing to the argument's validity. These two ideas can be expressed more formally in the following laws.

LAW FOR ⊤ AS A CONCLUSION. Γ ⊨ ⊤ (for any set Γ).

LAW FOR ⊤ AS A PREMISE. Γ,⊤ ⊨ φ if and only if Γ ⊨ φ (for any set Γ and sentence φ).

Although they are stated for ⊤, these laws will hold for all tautologies since they hold simply in virtue of the proposition expressed by ⊤.

These laws are different in character from the ones we have been considering since they concern the logical properties of a specific sort of sentence rather than the general principles governing logical relations. They are also a first sample of a common pattern in the laws of deductive reasoning that we will consider. Entailment is so central to deductive reasoning that an account of the role of a kind of sentence in entailment as a conclusion and as a premise will usually tell us all we need to know about it.

A simple law describes the role of absurdities as premises. We state it for the specific absurdity ⊥.

LAW FOR ⊥ AS A PREMISE. Γ, ⊥ ⊨ φ (for any set Γ and sentence φ).

An argument with an absurdity among its premises is valid by default. Since its premises cannot all be true, there is no risk of *new* error no matter what the conclusion is.

There is no law restating the significance of having ⊥ as a conclusion; that is simplest way we have of using entailment to say that a set of assumptions is inconsistent. But we can state a law for ⊥ as an alternative in the context of relative exhaustiveness, and all the properties of ⊤ and ⊥ take a particularly symmetric form when stated in terms of that relation.

	as a premise	*as an alternative*
Tautology	if Γ, ⊤ ⊨ Δ, then Γ ⊨ Δ	Γ ⊨ ⊤, Δ
Absurdity	Γ, ⊥ ⊨ Δ	if Γ ⊨ ⊥, Δ, then Γ ⊨ Δ

That is, while ⊤ contributes nothing as a premise and may be dropped, it is enough for relative exhaustiveness to have it as alternative (no matter how small the set Γ of premises or the set Δ of other alternatives). And while it is enough to have ⊥ as a premise (no matter how small the set of alternatives is), it contributes nothing as an alternative and may be dropped.

Notice that the converses of the principles at the upper left and lower right hold by monotonicity because they are just the addition of a premise in one case and an alternative in the other. If we take the if and only if principle that results from addition the converse to the lower right and consider a case where Δ is empty, we get

$$\Gamma \vDash \bot \text{ if and only if } \Gamma \vDash$$

This is the principle we saw in 1.4.5 that describes inconsistency in terms of entailment. That is, our use of $\bot$ as a conclusion to define inconsistency in terms of entailment really involves the same idea as the principle for $\bot$ as an alternative.

The symmetry exhibited by the set of principles in the table above might be traced to the symmetry of relative exhaustiveness: since $\top$ and $\bot$ are contradictory, having one as an assumption comes to the same thing as having the other as an alternative according to the basic law of relative exhaustiveness discussed in 1.4.5. However, there is a more general idea behind this symmetry that will apply also to cases where sentences are not contradictory.

The essential difference between the lower left and upper right in the table lies in interchanging Absurdity and Tautology and, at the same time, interchanging premises and alternatives. And the same is true of the upper left and lower right. This is, if we apply this transition to the lower left, we get

$$\Delta \vDash \Gamma, \top$$

and that differs from the upper right only in the order of the alternatives and the exchange of Δ for Γ. Neither of these differences are essential. Alternatives function only as a set, so the order in which they are listed does not matter. And, since each of Γ and Δ could be any set, exchanging them does not alter the content of the principle. Either way, we say that it is enough to have $\top$ as an alternative no matter what premises and what further alternatives we have. The possibility of this sort of transformation can be expressed by saying that $\top$ and $\bot$ on the one hand and premise (or assumption) and alternative on the other constitute pairs of *dual* terms. We will run into other pairs of terms later that fit into the same sort of duality.

Glen Helman 30 Aug 2009

1.4.s. Summary

1 Entailment may be defined in two equivalent ways, negatively as the relation that holds when the conclusion is false in no possible world in which all the premises are true or positively as the relation which holds when the conclusion is true in all such worlds. The negative form has the advantage of focusing attention on the sort of possible world that serves as a counterexample to a claim of entailment. The positive form characterizes a relation of entailment as a conditional guarantee of the truth of the conclusion, a guarantee conditional on the truth of the premises.

2 The requirements for a world to serve as a counterexample to entailment suggest the general idea of dividing a pair of sets by making all members of the first true and all members of the second false. A world will be said to divide an argument when it divides the premises from the conclusion.

3 The idea of division enables us to define a relation of relative exhaustiveness between sets: one set renders another exhaustive when there is no possible world that divides the two sets. We will extend the notation for entailment to express this relation between sets Γ and Δ as $\Gamma \vDash \Delta$. Entailment is the special case of this where Δ has only one member. When Δ has more than one member, its members will be referred to as alternatives because a relation of relative exhaustive provides a conditional guarantee only that at least one member of the second set it true.

4 Since a set of alternatives can have more than one member or be empty, relative exhaustiveness encompasses all the deductive properties and relations we have considered (as well as an extension of the idea of joint exhaustiveness to any set of sentences). The way a property or relation is expressed using relative exhaustiveness is tied directly to the negative form of the definition of the property or relation. When no relation of relative exhaustiveness holds no matter how a set is divided into two parts, all patterns of truth values for its members are possible and the set is logically independent. A single sentence that forms a logically independent set is logically contingent.

5 Definitions in terms of relative exhaustiveness can be converted into definitions in terms of entailment by replacing empty sets of alternatives with $\bot$ and reducing the size of multiple sets of alternatives by replacing members by adding assumptions that are contradictory to them (using the basic law for relative exhaustiveness).

6 Relative exhaustiveness and entailment satisfy a principle of monotonicity.

The term monotonic reflects the fact that relative exhaustiveness or entailment will never stop holding because of additions to the set of assumptions or set of alternatives. This principle is significant in distinguishing entailment from other forms of good inference, whose riskiness means that they are non-monotonic because adding information telling us that the risk does not pay off will undermine their quality. Both relative exhaustiveness and entailment also satisfy analogues to the principles of reflexivity and transitivity for implication. In the case of reflexivity, these laws are repetition for relative exhaustiveness and the law for premises for entailment. For transitivity, they are cut for relative exhaustiveness and the law for lemmas for entailment. The latter licenses the use of lemmas, valid conclusions that are of interest only as premises in further arguments. A more general law, called the chain law, together with a law for premises, yields all laws of entailment, and these two principles amount to principles of reflexivity and transitivity for a relation of set entailment that holds when one set entails each member of another.

7 The laws describing the behavior of ⊤ and ⊥ in the context of relative exhaustiveness exhibit a kind of symmetry that we will see in other laws later. The sentences ⊤ and ⊥ are dual as are the terms premise and alternative (or the left and right of an turnstile) in the sense that replacing each such term in a law by the one dual to it will produce another law.

Glen Helman 30 Aug 2009

1.4.x. Exercise questions

1. Any claim that a deductive relation holds can be stated as one or more claims that one set of sentences cannot be divided from another. (i) Restate each of the following claims in that way, and (ii) explicitly describe the sort of possibility that would divide the sets in question and is thus ruled out by claiming that the deductive relation holds. Nonsense words have been used to help you think to think how a possibility would be described without worrying whether that possibility could really occur.

For example, the claim that the sentences The widget plonked and The widget plinked are equivalent can be restated by saying that (i) the set consisting of the first sentence cannot be divided from the set consisting of the second sentence and vice versa. That is, (ii) it rules out any possibility in which the widget plonked but did not plink and any possibility in which the widget plinked but did not plonk.

- **a.** The gizmo is a widget and The gizmo is a gadget are mutually exclusive
- **b.** The gizmo is a widget and The gizmo is a gadget are jointly exhaustive
- **c.** The widget plinked is a tautology
- **d.** The widget plonked is absurd
- **e.** The widget was a gadget renders exhaustive the alternatives The widget plinked and The widget plonked
- **f.** The widget was a gizmo, The widget plinked, and The widget plonked are inconsistent

2. The basic law for relative exhaustiveness can be used not only to replace alternatives by assumptions but also to replace assumptions by alternatives. For example, the claim that The widget is blue entails The widget is colored can be restated to say (i) The widget is blue and The widget is not colored are inconsistent, (ii) The widget is not blue and The widget is colored form an exhuastive set, or (iii) The widget is not colored entails The widget is not blue.

In the following, you will be asked to restate some statements of deductive relations by replacing alternatives with assumptions or assumptions with alternatives. You may add or remove ordinary negation to state the contradictories of sentences.

a. Restate the following as a claim of entailment: The gadget is red and The gadget is green are mutually exclusive

b. Restate the following as a claim of entailment: Someone is in the auditorium and There are empty seats in the auditorium are jointly exhaustive

c. Restate the following as a claim of absurdity: A widget is a widget is a tautology

d. Restate the following as a claim of tautologousness: A widget is a gadget is absurd

e. Restate the following as a claim of inconsistency: The widget is a gadget or gizmo and The widget is not a gadget entail The widget is a gizmo

f. Restate the following so that each assumption is replaced by an alternative and each alternative by an assumption: The widget has advanced and The widget has plonked render exhaustive the alternatives The widget has finished the task and The widget has broken

Glen Helman 30 Aug 2009

1.4.xa. Exercise answers

1. a. (i) The set consisting of The gizmo is a widget and The gizmo is a gadget cannot be divided from the empty set; that is, (ii) there is no possibility of the gizmo being both a widget and a gadget.

b. (i) The empty set cannot be divided from the set consisting of The gizmo is a widget and The gizmo is a gadget; that is, (ii) there is no possibility of the gizmo being neither a widget nor a gadget

c. (i) The empty set cannot be divided from the set consisting of only The widget plinked; that is, (ii) there is no possibility that the widget did not plink

d. (i) The set consisting of only The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget plonked

e. (i) The set consisting of only The widget was a gadget cannot be divided from the set consisting of The widget plinked and The widget plonked; that is, (ii) there is no possbility that the widget was a gadget while not either plinking or plonking.

f. (i) The set consisting of The widget was a gizmo, The widget plinked, and The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget was a gizmo and both plinked and plonked

2. a. The gadget is red entails The gadget is not green (*or:* The gadget is green entails The gadget is not red)

b. The auditorium is empty entails There are empty seats in the auditorium (*or:* There are no empty seats in the auditorium entails The auditorium is not empty)

c. A widget is a not widget is absurd

d. A widget is a not gadget is a tautology

e. The widget is a gadget or gizmo, The widget is not a gadget, and The widget is not a gizmo are inconsistent

f. The widget has not finished the task and The widget has not broken render exhaustive The widget has not advanced and The widget has not plonked

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