8.3. Numerical quantification

8.3.0. Overview

Claims of exemplification speak of quantity in a very limited way; but, when combined with identity, the existential quantifier can be used to express quite a variety of clearly numerical claims.

8.3.1. Else

The key device we will use appears in the English word else, which can be used to claim the existence of a new example.

8.3.2. Numerical quantifier phrases

The phrase something else provides a way to claim the existence of ever more new examples, allowing to express phrases of the form at least *n* symbolically; and a variety of specific numerical claims can be captured by considering truth-functional compounds employing these phrases.

8.3.3. Exactly *n*

A simpler analysis of exactly *n* is possible by using a device that is captured in English by the phrase nothing else.

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8.3.1. Else

Consider the sentence Ed signed up and someone else did, too. To analyze it as a conjunction, we need to fill out the second clause, not only by replacing did by the phrase signed up but also by making explicit an implicit reference to Ed. The full analysis would proceed as follows:

Ed signed up and someone else did, too

Ed signed up ∧ someone other than Ed signed up

Ed signed up ∧ someone other than Ed is such that (he or she signed up)

Se ∧ (∃x: x is a person other than Ed) x signed up

Se ∧ (∃x: x is a person ∧ x is other than Ed) Sx

Se ∧ (∃x: Px ∧ ¬ x is Ed) Sx

Se ∧ (∃x: Px ∧ ¬ x = e) Sx

Se ∧ ∃x ((Px ∧ ¬ x = e) ∧ Sx)

P: [_ is a person]; S: _ signed up; e: Ed

That is, the function of the word else here is to restrict an existential claim by requiring that the example it claims to exist be different from a previous reference; in short, else serves to indicate a new example. The restriction of existential claims so that they claim the existence of new examples can be found not only with the word else but also, though less obviously, in a variety of quantifier phrases we have not yet attempted to analyze.

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8.3.2. Numerical quantifier phrases

So far the only numerical claims we have seen have been ones asserting or denying that a class is empty. We will now move on to a much wider group, considering claims of the three sorts

At least *n* Cs are such that … they …

At most *n* Cs are such that … they …

Exactly *n* Cs are such that … they …,

where *n* may be any positive integer.

To see how to approach these quantificational claims, let us first consider existential claims regarding pairs. We looked at generalizations about pairs in 7.4.1, giving special attention to the example Not every employer and employee get along. This is the denial of generalization, so it can be understood to claim the existence of a counterexample, and we can restate it as follows to make this more explicit:

Some employer and employee do not get along.

Now we can analyze this sentence using two existential quantifiers, restricting the second by relation to the first. We would get this:

Some employer and employee do not get along

Something is such that (it and some employee of it do not get along)

∃x x and some employee of x do not get along

∃x some employee of x is such that (x and he or she do not get along)

∃x (∃y: x employs y) x and y do not get along

∃x (∃y: Exy) ¬ x and y get along

∃x (∃y: Exy) ¬ Gxy

∃x ∃y (Exy ∧ ¬ Gxy)

E: [_ employs _]; G: _ and _ get along

The English sentence claims the existence of a pair of examples whose members are related in a certain way (as employer and employee). As with generalizations, the limitations of our notation have forced us to treat the two quantifiers asymmetrically in the symbolic form.

Now consider the sentence At least 2 things are on the agenda. It can be understood to claim the existence of a pair of examples whose members are specified to be non-identical. Following the pattern we have just used to analyze restricted existential claims concerning pairs, we can express this idea as follows:

At least 2 things are on the agenda

Something is such that (it and something else are on the agenda)

∃x x and something else are on the agenda

∃x something other than x is such that (x and it are on the agenda)

∃x (∃y: y is other than x) x and y are on the agenda

∃x (∃y: ¬ y is x) (x is on the agenda ∧ y is on the agenda)

∃x (∃y: ¬ y = x) (Nxa ∧ Nya)

∃x ∃y (¬ y = x ∧ (Nxa ∧ Nya))

N: [_ is on _]; a: the agenda

The quantifier phrase something else has been analyzed here before and in order to separate the vocabulary found in the quantifier phrase from that found in the quantified predicate of the original English sentence. We might have analyzed the conjunction before the second quantifier by way of an intermediate form like this:

∃x x is on the agenda and so is something else

We would have ended up with the form ∃x (Nxa ∧ (∃y: ¬ y = x) Nya), which is equivalent to the form above by a confinement equivalence discussed in 8.1.4.

This basic idea can be extended to any quantifier phrase of the form at least *n* Cs. For example, at least 3 Cs can be understood to claim the existence of an example, an example different from the first, and an example different from the first two. Let us apply this idea to a case where the restrictions of non-identity are added to other specifications:

At least 3 birds are in the tree

(∃x: x is a bird) x and at least 2 other birds are in the tree

(∃x: Bx) (∃y: y is a bird other than x) x and y and another bird are in the tree

(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: z is a bird other than x and y) x and y and z are in the tree

(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: Bz ∧ (¬ z = x ∧ ¬ z = y)) (x is in the tree ∧ y is in the tree ∧ z is in the tree)

(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: Bz ∧ (¬ z = x ∧ ¬ z = y)) (Nxt ∧ Nyt ∧ Nzt)

B: [_ is a bird]; N: _ is in _ ; t: the tree

This can be restated in a number of different ways by using unrestricted quantifiers and applying confinement principles. The following may help in thinking about the net result of the three quantifier phrases above:

$$\exists x\ \exists y\ \exists z\ ((\neg\, y = x \wedge \neg\, z = x \wedge \neg\, z = y) \wedge (Bx \wedge By \wedge Bz) \wedge (Nxt \wedge Nyt \wedge Nzt))$$

That is, we assert the existence of a triple with three properties: (i) no two of its members are the same, (ii) each member is a bird, and (iii) each member is in the tree. The sentence Heinz produces at least 57 varieties could be handled (in principle if not in practice) by extending the same ideas to assert the existence of a series of 57 things no two of which are the same and each of which is both a variety and produced by Heinz. If you are mathematically minded, you might try calculating the number of denied equations you would need in that case.

In the other direction, if the scopes of quantifier phrases are confined to parts of the sentence in which they bind variables, we would have instead

$$(\exists x\!: Bx)\ (Nxt \wedge (\exists y\!: By \wedge \neg\, y = x)\ (Nyt \wedge (\exists z\!: Bz \wedge (\neg\, z = x \wedge \neg\, z = y))\ Nzt))$$

which might be expressed in English as Some bird is such that it is in the tree and some bird other than it is such that it, too, is in the tree and some bird different from both of the them is in the tree also.

As a general pattern for At least *n* things are such that … they …, we might use either of the following:

$$\exists x_1\ (\exists x_2\!: \neg\, x_2{=}x_1)\ \ldots\ (\exists x_n\!: \neg\, x_n{=}x_1 \wedge \neg\, x_n{=}x_2 \wedge \ldots \wedge \neg\, x_n{=}x_{n\text{-}1})\ (\theta x_1 \wedge \theta x_2 \wedge \ldots \wedge \theta x_n)$$

$$\exists x_1\ \exists x_2\ \ldots\ \exists x_n\ ((\neg\, x_2{=}x_1) \wedge \ldots \wedge (\neg\, x_n{=}x_1 \wedge \neg\, x_n{=}x_2 \wedge \ldots \wedge \neg\, x_n{=}x_{n-1}) \wedge (\theta x_1 \wedge \theta x_2 \wedge \ldots \wedge \theta x_n))$$

where $\theta\tau$ abbreviates … τ …. These logical forms differ in whether the denied equations appear as restrictions on quantifiers or as conjuncts of the formula to which the quantifiers are applied. In either case, the list of denied equations should include $\neg\, x_i = x_j$ for each $i > j$ where $i, j \leq n$. At least *n* Cs are such that … they … can be captured by adding the formulas x_i is a C, for each $i \leq n$, either as restrictions on the relevant quantifiers or as further conjuncts of the quantified formula.

The corresponding pattern with the quantifiers confined would be:

$$\exists x_1\ (\theta x_1 \wedge (\exists x_2\!: \neg\, x_2{=}x_1)\ (\theta x_2 \wedge \ldots\ (\exists x_n\!: \neg\, x_n{=}x_1 \wedge \neg\, x_n{=}x_2 \wedge \ldots \wedge \neg\, x_n{=}x_{n\text{-}1})\ \theta x_n \ldots))$$

This says roughly, Something is such that …it… and so is something else … and so is something else. In spite of appearances, this English sentence is not a conjunction because each use of else refers implicitly to all of the previous uses of something and cannot be separated from them in an independent component.

We are also now in a position to analyze the other two sorts of numerical quantifier phrases mentioned earlier, for claims made using them can be restated as truth-functional compounds of claims made using at least *n*.

At most *n* Cs are such that … they …

may be paraphrased as

¬ at least n+1 Cs are such that … they …

and

Exactly *n* Cs are such that … they …

may be paraphrased as

At least *n* Cs are such that … they …
∧ at most *n* Cs are such that … they …

For example, to claim that there was at most one winner is to deny that there were at least two, and to claim that there was exactly one is to say both there was at least one and that there was at most one—i.e., it is to say that there was at least one and deny that there were at least two.

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8.3.3. Exactly *n*

It is also possible to give a somewhat simpler symbolic representations of the quantifier phrase exactly *n* Cs than we get by way of truth-functional compounds of at least-*m* forms. Here are a couple of approaches for the case of exactly 1:

I forgot just one thing
Something is such that (I forgot it and nothing else)
∃x I forgot x and nothing else
∃x (I forgot x ∧ I forgot nothing other than x)
∃x (Fix ∧ nothing other than x is such that (I forgot it))
∃x (Fix ∧ (∀y: y is other than x) ¬ I forgot y)

$$\exists x\, (Fix \land (\forall y\colon \neg\, y = x)\ \neg\, Fiy)$$
$$\exists x\, (Fix \land \forall y\, (\neg\, y = x \rightarrow \neg\, Fiy))$$

I forgot just one thing
Something is such that (I forgot it and it was all I forgot)
∃x I forgot x and x was all I forgot
∃x (I forgot x ∧ x was all I forgot)
∃x (Fix ∧ everything I forgot is such that (x was it))
∃x (Fix ∧ (∀y: I forgot y) x was y)

$$\exists x\, (Fix \land (\forall y\colon Fiy)\ x = y)$$
$$\exists x\, (Fix \land \forall y\, (Fiy \rightarrow x = y))$$

F: [_ forgot _]; i: me

And, in general, Exactly one thing is such that (… it …) can be analyzed as any of the following (where θx abbreviates … x …):

$\exists x\, (\theta x \land (\forall y\colon \neg\, y = x)\ \neg\, \theta y)$	$\exists x\, (\theta x \land \forall y\, (\neg\, y = x \rightarrow \neg\, \theta y))$
$\exists x\, (\theta x \land (\forall y\colon \theta y)\ x = y)$	$\exists x\, (\theta x \land \forall y\, (\theta y \rightarrow x = y))$

The forms in columns are equivalent by the symmetry of identity and the following equivalences:

$$(\forall x\colon \rho x)\ \theta x \Leftrightarrow (\forall x\colon \neg^{\pm}\, \theta x)\ \neg^{\pm}\, \rho x$$
$$\varphi \rightarrow \psi \Leftrightarrow \neg^{\pm}\, \psi \rightarrow \neg^{\pm}\, \varphi$$

The first of these is traditionally called *contraposition* and that name is sometimes used for the second also. The first licenses the restatement of Only dogs barked by Everything that barked was a dog. The second would apply to the same pair of sentences when they are represented using unrestricted quantifiers and also to the restatement of The match burned only if oxygen was present by If the match burned, then oxygen was present.

The initial unrestricted quantifier in the above analyses of exactly 1 thing can also be replaced by a restricted quantifier. The following analysis of a slightly more complex example uses this sort of variation on the second pattern above:

I forgot just one number
Some number I forgot is such that (it was all the numbers I forgot)
(∃x: x is a number I forgot) x was all the numbers I forgot
(∃x: x is a number ∧ I forgot x) every number I forgot is such that (x was it)
(∃x: x is a number ∧ I forgot x) (∀y: y is a number I forgot) x was y
(∃x: Nx ∧ Fix) (∀y: y is a number ∧ I forgot y) x was y

$$(\exists x\colon Nx \land Fix)\ (\forall y\colon Ny \land Fiy)\ x = y$$

And, in general, Exactly 1 C is such that (… it …) can be analyzed as

(∃x: x is a C ∧ … x …) (∀y: y is a C ∧ … y …) x = y

The analogous variation on the first pattern would be

(∃x: x is a C ∧ … x …) (∀y: y is a C ∧ ¬ y = x) ¬ … y …

In the case of, I forgot just one number, this pattern would amount to saying Some number that I forgot is such that I forgot no other number.

The sentence There is exactly 1 C can be understood as Exactly 1 C is such that (it is) and the dummy predicate [_ is] can be dropped to yield the analysis

(∃x: x is a C) (∀y: y is a C) x = y

which can be understood to say Some C is such that (it is all the Cs there are).

This sort of pattern will be important for the analysis of definite descriptions in 8.4.2, but the first approach (i.e., by way of *nothing else*) is probably the more natural way of extending the analysis to claims of exactly *n* for numbers $n > 1$—as in the following example:

Exactly 2 things are in the room

2 things are such that (they are in the room but and nothing else is)

∃x (∃y: ¬ y = x) x and y are in the room but and nothing else is

∃x (∃y: ¬ y = x) ((x is in the room ∧ y is in the room) ∧ nothing other than x and y is in the room)

∃x (∃y: ¬ y = x) ((Nxr ∧ Nyr) ∧ (∀z: z is other than x and y) ¬ z is in the room)

∃x (∃y: ¬ y = x) ((Nxr ∧ Nyr) ∧ (∀z: z is other than x ∧ z is other than y) ¬ Nzr)

$$\exists x\ (\exists y: \neg\, y = x)\ ((Nxr \wedge Nyr) \wedge (\forall z: \neg\, z = x \wedge \neg\, z = y)\ \neg\, Nzr)$$

N: [_ is in _]; r: the room

The general forms for exactly 2 things are such that (… they …) and exactly 2 Cs are such that (… they …) along these lines are the following (using θ for $[\dots x \dots]_x$ and ρ for [_ is a C]):

$$\exists x\ (\exists y: \neg\, y = x)\ ((\theta x \wedge \theta y) \wedge (\forall z: \neg\, z = x \wedge \neg\, z = y)\ \neg\, \theta z)$$

$$(\exists x: \rho x)\ (\exists y: \rho y \wedge \neg\, y = x)\ ((\theta x \wedge \theta y) \wedge (\forall z: \rho z \wedge \neg\, z = x \wedge \neg\, z = y)\ \neg\, \theta z)$$

Notice that the restricting predicate ρ is added to each of the three quantifiers in the second. In particular, Exactly 2 boxes are in the room means 2 boxes are such that (they are in the room and no other boxes are) rather than 2 boxes are such that (they are in the room and nothing else is), which says that two boxes are the only things in the room.

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8.3.s. Summary

1 When the word else appears as a modifier in a quantifier phrase, it is used to restrict the domain by excluding some previously mentioned object. An existential quantifier phrase modified by it thus claims the existence of a new example.

2 The same sort of restriction can be used to express a variety of numerical quantifier phrases. For example, at least 2 things amounts to something and something else, and at least 3 things amounts to something and something else and something other than those two. Still other numerical claims can be reached by truth-functional compounding—at most *n* by denying at least n+1 and exactly *n* by conjoining claims stated with at least *n* and at most *n*.

3 It is also possible to express Exactly 1 thing is such that (… it …) by Something is such that (… it … and nothing else does) or—equivalently, in a way that illustrates, among other things, a principle of contraposition)—by Something is such that (… it … and it is all that does).

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8.3.x. Exercise questions

1. Analyze the following in as much detail as possible.

a. If Oswald didn't shoot Kennedy, someone else did.

b. No one but Frank saw Sue.

c. Ed and only Ed was awake.

d. Everyone except Tom, Dick, and Harry arrived early.

e. Adam and another officer thanked everyone else.

f. At least two things went wrong.

g. Bill spoke to at most one person.

h. Just one thing will do.

i. Ann saw more than one assassin.

j. Ann saw exactly two assassins.

2. Synthesize idiomatic English sentences that express the propositions associated with the logical forms below using the intensional interpretations that follow them.

a. Fth ∧ (∃x: ¬ x = h) Ltx

F: [_ found _]; L: _ lost _ ; h: Tom's hat; t: Tom

b. (∃x: Px) (∃y: Py ∧ ¬ y = x) Sxy

P: [_ is a person]; S: _ spoke to _

c. (∀x: Px ∧ ¬ x = m) ¬ Rsx

P: [_ is a person]; R: _ recognized _ ; m: Mary; s: Sam

d. (∃x: Sx) Ox ∧ ¬ (∃x: Sx) (∃y: Sy ∧ ¬ y = x) (Ox ∧ Oy)

S: [_ is a store]; O: _ was open

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8.3.xa. Exercise answers

1. **a.** If Oswald didn't shoot Kennedy, someone else did

Oswald didn't shoot Kennedy → someone other than Oswald shot Kennedy

¬ Oswald shot Kennedy → (∃x: x is a person other than Oswald) x shot Kennedy

¬ Sok → (∃x: x is a person ∧ x is other than Oswald) x shot Kennedy

¬ Sok → (∃x: x is a person ∧ ¬ x = Oswald) x shot Kennedy

¬ Sok → (∃x: Px ∧ ¬ x = o) Sxk

¬ Sok → ∃x ((Px ∧ ¬ x = o) ∧ Sxk)

P: [_ is a person]; S: _ shot _ ; k: Kennedy; o: Oswald

b. No one but Frank saw Sue

¬ someone other than Frank saw Sue

¬ (∃x: x is a person ∧ ¬ x = Frank) x saw Sue

¬ (∃x: Px ∧ ¬ x = f) Sxs

¬ ∃x ((Px ∧ ¬ x = f) ∧ Sxs)

or:

No one but Frank saw Sue

(∀x: x is a person other than Frank) ¬ x saw Sue

(∀x: x is a person ∧ ¬ x = Frank) ¬ x saw Sue

(∀x: Px ∧ ¬ x = f) ¬ Sxs

∀x ((Px ∧ ¬ x = f) → ¬ Sxs)

P: [_ is a person]; S: _ saw _ ; f: Frank; s: Sue

c. Ed and only Ed was awake

Ed was awake ∧ only Ed was awake

Ed was awake ∧ (∀x: ¬ x is Ed) ¬ x was awake

Ae ∧ (∀x: ¬ x = e) ¬ Ax

Ae ∧ ∀x (¬ x = e → ¬ Ax)

A: [_ was awake]; e: Ed

d. Everyone except Tom, Dick, and Harry arrived early

(∀x: x is a person ∧ x is other than Tom, Dick, and Harry) x arrived early

(∀x: x is a person ∧ (¬ x = Tom ∧ ¬ x = Dick ∧ ¬ x = Harry))
x arrived early

$$(\forall x\!: Px \land (\neg\, x = t \land \neg\, x = d \land \neg\, x = h))\ Ex$$
$$\forall x\ (\ (Px \land (\neg\, x = t \land \neg\, x = d \land \neg\, x = h)) \rightarrow Ex\ )$$

E: [_ arrived early]; P: _ is a person; d: Dick; h: Harry; t: Tom

e. Adam and another officer thanked everyone else

(∃x: x is a officer other than Adam) Adam and x thanked everyone else

(∃x: x is a officer ∧ x is other than Adam) everyone other than Adam and x is such that (Adam and x thanked him or her)

(∃x: Ox ∧ ¬ x = Adam) (∀y: y is a person other than Adam and x) Adam and x both thanked y

(∃x: Ox ∧ ¬ x = Adam) (∀y: y is a person ∧ y is other than Adam and x) (Adam thanked y ∧ x thanked y)

(∃x: Ox ∧ ¬ x = a) (∀y: Py ∧ (¬ y = Adam ∧ ¬ y = x)) (Tat ∧ Txy)

$$(\exists x\!: Ox \land \neg\, x = a)\ (\forall y\!: Py \land (\neg\, y = a \land \neg\, y = x))\ (Tay \land Txy)$$
$$\exists x\ ((Ox \land \neg\, x{=}a) \land \forall y\ ((Py \land (\neg\, y{=}a \land \neg\, y{=}x)) \rightarrow (Tay \land Txy)))$$

O: [_ is an officer]; P: _ is a person; T: [_ thanked _]; a: Adam

or:

Adam and another officer thanked everyone else

Adam thanked everyone else
∧ an officer other than Adam thanked everyone else

everyone other than Adam is such that (Adam thanked him or her)
∧ (∃x: x is a officer other than Adam) x thanked everyone else

(∀y: y is a person other than Adam) Adam thanked y
∧ (∃x: Ox ∧ ¬ x = Adam) everyone other than x is such that (x thanked him or her)

(∀y: Py ∧ ¬ y = Adam) Tay
∧ (∃x: Ox ∧ ¬ x = a) (∀y: y is a person other than x) x thanked y

$$(\forall y\!: Py \land \neg\, y = a)\ Tay \land (\exists x\!: Ox \land \neg\, x = a)\ (\forall y\!: Py \land \neg\, y = x)\ Txy$$
$$\forall y\ ((Py \land \neg\, y{=}a) \rightarrow Tay) \land \exists x\ ((Ox \land \neg\, x{=}a) \land \forall y\ ((Py \land \neg\, y{=}x) \rightarrow Txy))$$

The logical forms produced by these two analyses are not equivalent. It could be said that the first interprets else as referring to Adam and the other officer collectively while the second interprets it as referring to them individually. The latter interpretation produces a pair of generalizations each of whose domains excludes only one of the two rather than both together. That means that the second together with the assumption that Adam and the other officer are both people entails that they thanked each other.

f. At least two things went wrong

∃x (∃y: ¬ y = x) (x and y went wrong)

∃x (∃y: ¬ y = x) (x went wrong ∧ y went wrong)

$$\exists x\ (\exists y\!: \neg\, y = x)\ (Wx \land Wy)$$
$$\exists x\ \exists y\ (\neg\, y = x \land (Wx \land Wy))$$

W: [_ went wrong]

g. Bill spoke to at most one person

¬ Bill spoke to at least two people

¬ at least two people are such that (Bill spoke to them)

¬ (∃x: x is a person) (∃y: y is a person ∧ ¬ y = x) (Bill spoke to x and y)

¬ (∃x: Px) (∃y: Py ∧ ¬ y = x) (Bill spoke to x ∧ Bill spoke to y)

$$\neg\ (\exists x\!: Px)\ (\exists y\!: Py \land \neg\, y = x)\ (Sbx \land Sby)$$
$$\neg\ \exists x\ (Px \land \exists y\ ((Py \land \neg\, y = x) \land (Sbx \land Sby)))$$

S: [_ spoke to _]; b: Bill

h. At least one thing will do ∧ at most one thing will do

∃x x will do ∧ ¬ at least 2 things will do

∃x Dx ∧ ¬ ∃x (∃y: ¬ y = x) (x and y will do)

∃x Dx ∧ ¬ ∃x (∃y: ¬ y = x) (x will do ∧ y will do)

$$\exists x\ Dx \land \neg\ \exists x\ (\exists y\!: \neg\, y = x)\ (Dx \land Dy)$$
$$\exists x\ Dx \land \neg\ \exists x\ \exists y\ (\neg\, y = x \land (Dx \land Dy))$$

D: [_ will do]

or:

∃x (x will do ∧ nothing other than x will do)

∃x (Dx ∧ (∀y: ¬ y = x) ¬ y will do)

$$\exists x\ (Dx \land (\forall y\!: \neg\, y = x)\ \neg\, Dy)$$
$$\exists x\ (Dx \land \forall y\ (\neg\, y = x \rightarrow \neg\, Dy))$$

or:

∃x (x will do ∧ x is all that will do)

∃x (Dx ∧ everything that will do is such that (x is it))

∃x (Dx ∧ (∀y: y will do) x is y)

$$\exists x\ (Dx \land (\forall y\colon Dy)\ x = y)$$
$$\exists x\ (Dx \land \forall y\ (Dy \to x = y))$$

i. Ann saw more than one assassin
Ann saw at least two assassins
At least two assassins are such that (Ann saw them)
(∃x: x is an assassin) (∃y: y is an assassin ∧ ¬ y = x) (Ann saw x and y)
(∃x: Ax) (∃y: Ay ∧ ¬ y = x) (Ann saw x ∧ Ann saw y)

$$(\exists x\colon Ax)\ (\exists y\colon Ay \land \neg\, y = x)\ (Sax \land Say)$$
$$\exists x\ (Ax \land \exists y\ ((Ay \land \neg\, y = x) \land (Sax \land Say)))$$

A: [_ is an assassin]; S: _ saw _ ; a: Ann

j. Ann saw exactly two assassins
Exactly two assassins are such that (Ann saw them)
Two assassins are such that (Ann saw them and no other assassins)
(∃x: x is an assassin) (∃y: y is an assassin ∧ ¬ y = x) (Ann saw x and y and no other assassins)
(∃x: Ax) (∃y: Ay ∧ ¬ y = x) (Ann saw x ∧ Ann saw y ∧ Ann saw no assassin other than x and y)
(∃x: Ax) (∃y: Ay ∧ ¬ y = x) ((Sax ∧ Say) ∧ no assassin other than x and y is such that (Ann saw him or her))
(∃x: Ax) (∃y: Ay ∧ ¬ y = x) ((Sax ∧ Say) ∧ (∀z: z is an assassin ∧ (¬ z = x ∧ ¬ z = y)) ¬ Ann saw z)

$$(\exists x\colon Ax)\ (\exists y\colon Ay \land \neg\, y = x)\ ((Sax \land Say) \land (\forall z\colon Az \land (\neg\, z = x \land \neg\, z = y))\ \neg\, Saz)$$
$$\exists x(Ax \land \exists y((Ay \land \neg\, y{=}x) \land ((Sax \land Say) \land \forall z((Az \land (\neg\, z{=}x \land \neg\, z{=}y)) \to \neg\, Saz))))$$

A: [_ is an assassin]; S: _ saw _ ; a: Ann

or:

$$(\exists x\colon Ax)\ (\exists y\colon Ay \land \neg\, y = x)\ ((Sax \land Say) \land (\forall z\colon Az \land Saz)\ (x = z \lor y = z))$$

The formula (∀z: Az ∧ Saz) (x = z ∨ y = z)) used here amounts to x and y together account for all the assassins Ann saw.

2. a. Tom found Tom's hat ∧ (∃x: ¬ x = Tom's hat) Tom lost x
Tom found his hat ∧ (∃x: x is other than Tom's hat) Tom lost x
Tom found his hat ∧ something other than Tom's hat is such that (Tom lost it)
Tom found his hat ∧ Tom lost something other than his hat
Tom found his hat but he lost something else

b. (∃x: x is a person) (∃y: y is a person ∧ ¬ y = x) x spoke to y
(∃x: x is a person) (∃y: y is a person ∧ y is other than x) x spoke to y
(∃x: x is a person) (∃y: y is a person other than x) x spoke to y
(∃x: x is a person) someone other than x is such that (x spoke to him or her)
(∃x: x is a person) x spoke to someone else
Someone is such that (he or she spoke to someone else)
Someone spoke to someone else

c. (∀x: x is a person ∧ ¬ x = Mary) ¬ Sam recognized x
(∀x: x is a person ∧ x is other than Mary) ¬ Sam recognized x
(∀x: x is a person other than Mary) ¬ Sam recognized x
No one other than Mary is such that (Sam recognized him or her)
Sam recognized no one other than Mary
or: Sam didn't recognize anyone other than Mary

d. (∃x: x is a store) x was open ∧ ¬ (∃x: x is a store) (∃y: y is a store ∧ ¬ y = x) (x was open ∧ y was open)
At least one store was open ∧ ¬ (∃x: x is a store) (∃y: y is a store ∧ ¬ y = x) (x and y were open)
At least one store was open ∧ ¬ at least two stores are such that (they were open)
At least one store was open ∧ ¬ at least 2 stores were open
At least one store was open ∧ at most 1 store was open
Just one store was open