4.3. Detachment: eliminating alternatives

4.3.0. Overview

Since disjunctions (and negated conjunctions) make weak claims, the most general forms of reasoning about them are not simple; but there are simple patterns of argument involving them that work in special cases.

4.3.1. Detachment rules

If we add to a disjunction the information that one of its disjuncts is false, we can conclude the other disjunct; and a related principle applies to negated conjunctions.

4.3.2. More attachment rules

A disjunction is entailed by each of its disjuncts; and, while this does not provide a safe way of planning to reach a goal, it is a useful way of adding to inactive resources. Again, a similar principle applies to negated conjunctions.

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4.3.1. Detachment rules

When we exploit a disjunction using a proof by cases, we divide the parent gap into two children. Something like this is essential in any rule that allows us to exploit a disjunction by way of reasoning about its disjuncts, for the truth of a disjunction does not settle the truth values of its disjuncts. However, if we add to the disjunction information about the truth value of one disjunct, it can be possible to conclude something about the other.

In particular, if we know both that a disjunction is true and that one of its disjuncts is false, we can conclude that the other disjunct is true. This idea appears in a pattern of argument recognized long enough to have acquired a Latin name: *modus tollendo ponens*

$$\text{MTP}\ \frac{\varphi \lor \psi \quad \neg^{\pm}\varphi}{\psi} \qquad \text{MTP}\ \frac{\varphi \lor \psi \quad \neg^{\pm}\psi}{\varphi}$$

The name refers to what the second premise and conclusion say about the two disjuncts. It can be translated, very roughly, as way, by taking, of putting. That is, the argument enables you to put forth one component as the conclusion if you take away the other component by asserting a premise that negates or de-negates it.

The use of this idea in derivations will be based on a somewhat stronger pair of principles for which we will also use the name *modus tollendo ponens*.

$$\Gamma, \varphi \lor \psi, \neg^{\pm}\varphi \Rightarrow \chi \text{ if and only if } \Gamma, \psi, \neg^{\pm}\varphi \Rightarrow \chi$$
$$\Gamma, \varphi \lor \psi, \neg^{\pm}\psi \Rightarrow \chi \text{ if and only if } \Gamma, \varphi, \neg^{\pm}\psi \Rightarrow \chi$$

These tell that, in the presence of a sentence negating or de-negating one component of a disjunction, having the disjunction as a premise comes to the same thing as having its other component as a premise. These principles depend on the validity of the arguments above and also on the fact that a disjunction is entailed by each of this components individually.

The *modus tollendo ponens* principles describe grounds under which we can drop a disjunction from our active resources (and replace it by one of its disjuncts), so they justify a rule *Modus Tollendo Ponens* (MTP) that provides an added way of exploiting a disjunction.

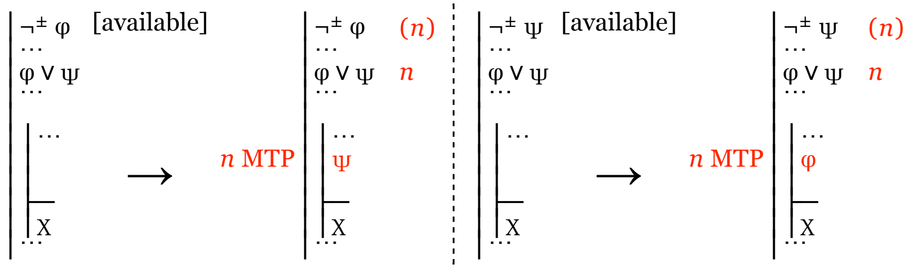


Fig. 4.3.1-1. Developing a derivation at stage *n* by exploiting a disjunction when a sentence negating or de-negating one component is also an active resource.

Notice that the negated or de-negated component is not exploited, so the stage number to its right is enclosed in parentheses. And, since we are not exploiting this resource, there is no need for it to be active. As is the case with the resources required by adjunction rules or rules for closing gaps, it is enough that this resource be available. On the other hand, the disjunction itself is exploited, so it must be active and the stage number added at its right is not parenthesized.

This is only the first of a number of rules that will enable us to exploit weak compounds in the presence of information about a component. We will label as *detachment rules* these rules along with others that enable us to exploit resources given certain further information. The resource that is exploited by such a rule will be called the *main resource* while the resource that must be available but is not exploited will be called the *auxiliary resource*. In the case of MTP, the disjunction is the main resource and the sentence negating or de-negating one of its disjuncts is the auxiliary resource.

The second detachment rule we will add concerns the not-both form. De Morgan's laws tell us that the form $\neg(\varphi \land \psi)$ is equivalent to the disjunction $\neg^{\pm}\varphi \lor \neg^{\pm}\psi$, so we should expect some appropriate modification of *modus tollendo ponens* to be valid. The proper form is this:

$$\text{MPT}\ \frac{\neg(\varphi \land \psi) \quad \varphi}{\neg^{\pm}\psi} \qquad \text{MPT}\ \frac{\neg(\varphi \land \psi) \quad \psi}{\neg^{\pm}\varphi}$$

These arguments are called *modus ponendo tollens*: they are a way of, by putting, taking. That is, if we know that φ and ψ are not both true, adding the information that one of them is true (i.e., putting it forth), enables us to conclude that the other is not true (i.e., we can take it away). The corresponding principles called *modus ponendo tollens* are these:

$$\Gamma, \neg(\varphi \land \psi), \varphi \Rightarrow \chi \text{ if and only if } \Gamma, \neg^{\pm}\psi, \varphi \Rightarrow \chi$$
$$\Gamma, \neg(\varphi \land \psi), \psi \Rightarrow \chi \text{ if and only if } \Gamma, \neg^{\pm}\varphi, \psi \Rightarrow \chi$$

They are based on the *modus ponendo tollens* arguments and also on the fact that a not-both form $\neg(\varphi \land \psi)$ is entailed by a sentence negating or de-negating either φ or ψ. That is, in the presence of a premise asserting φ or ψ, the not-both $\neg(\varphi \land \psi)$ can be replaced by a sentence that negates or de-negates the other component.

The rule *Modus Ponendo Tollens* (MPT) is this:

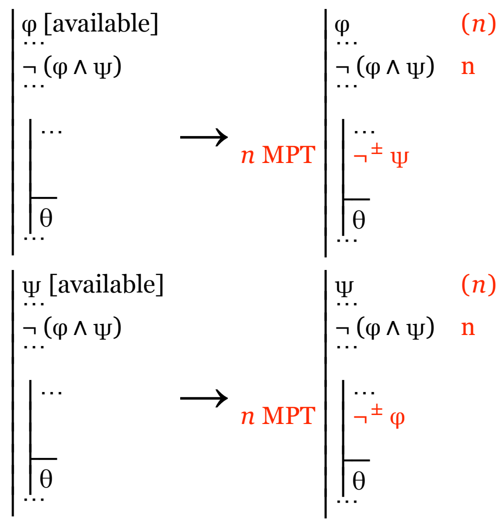
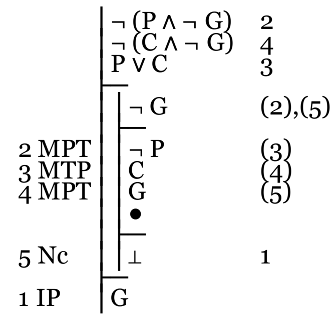


Fig. 4.3.1-2. Developing a derivation at stage *n* by exploiting a negated conjunction when a conjunct is also an active resource.

As with MTP, one resource, the main resource, is exploited (and should be active) while the other, auxiliary resource, is not exploited and need only be available.

As an example of these new rules, here is an alternative version of the derivation at the end of 4.2.1:



This is far from the only way of using the new rules to complete the derivation. To choose only the most minor variation on the one above, notice that in the second use of MPT either G or ¬ ¬ G could be concluded (since both can be described as $\neg^{\pm}$ ¬ G). And either could be used along with ¬ G to conclude ⊥ by Nc.

Notice that the supposition ¬ G (Sam didn't grant the proposal's significance) enables us to conclude first that ¬ P (Sam didn't praise the proposal), then C (Sam condemned the proposal), and finally G itself. An argument by which a claim is shown to follow from its own denial is traditionally called a *consequentia mirabilis* (an amazing consequence) and has been a standard form of philosophical argumentation since antiquity. (For example, a common way of arguing against a skeptic who denies the existence of knowledge is show that this claim, that there is no knowledge, in fact implies that there is knowledge, which leads to the conclusion that knowledge must exist. Any reply to this argument must question the moves by which one is supposed to get from the claim that there is no knowledge to the consequence that there is knowledge because, if this transition is valid, an indirect proof will show that knowledge does exist.)

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4.3.2. Attachment rules

The principles that lie behind the rules MTP and MPT were based in part on the fact that the weak compounds φ ∨ ψ and ¬ (φ ∧ ψ) are entailed by certain information about their components. We will refer to the principles asserting these entailments as *weakening principles*:

$$\varphi \Rightarrow \varphi \vee \psi$$
$$\psi \Rightarrow \varphi \vee \psi$$
$$\neg^{\pm} \varphi \Rightarrow \neg (\varphi \wedge \psi)$$
$$\neg^{\pm} \psi \Rightarrow \neg (\varphi \wedge \psi)$$

They provide the basis for further attachment rules (in addition to Adj). These rules allow us to enter the conclusions of the weakening principles as inactive resources when their premises are already available.

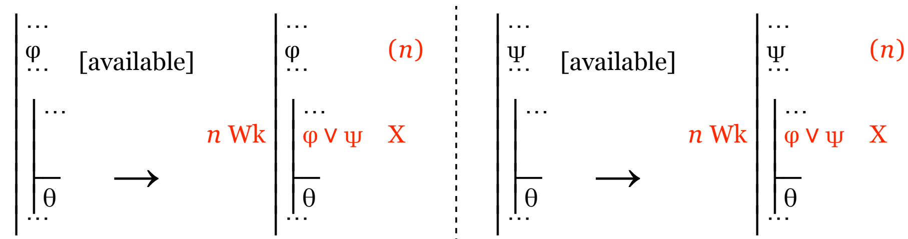

Fig. 4.3.2-1. Developing a derivation at stage *n* by adding an inactive disjunction that weakens one of the available resources.

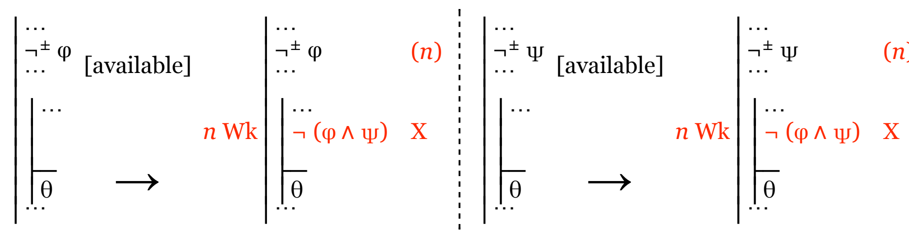

Fig. 4.3.2-2. Developing a derivation at stage *n* by adding an inactive negated conjunction that weakens one of the available resources.

These rules can be used, as we have used Adj, to provide material for closing gaps. But the rules MTP and MPT now provide a further way of using inactive resources, and Wk can provide material for them, too (as can Adj). For example, below are two approaches to the same argument. The argument is designed as an illustration but can be given an English interpretation as follows:

Suppose we know in general that either Ann and Bill were both at the party or Carol and Dave were both there. And also that it is not the case that both Bill and Ed were there along with either Fred or Gail. Then, assuming we know in particular that Ed and Fred were both there, we can conclude that Carol was, too.

	$(A \land B) \lor (C \land D)$	6
	$\neg ((B \land E) \land (F \lor G))$	3
	$E \land F$	1
1 Ext	E	(4)
1 Ext	F	(2)
2 Wk	$F \lor G$	X,(3)
3 MPT	$\neg (B \land E)$	4
4 MPT	$\neg B$	(5)
5 Wk	$\neg (A \land B)$	X,(6)
6 MTP	$C \land D$	7
7 Ext	C	(8)
7 Ext	D	
	•	
8 QED	C	

	$(A \land B) \lor (C \land D)$	4
	$\neg ((B \land E) \land (F \lor G))$	7
	$E \land F$	1
1 Ext	E	(6)
1 Ext	F	(8)
	$\neg C$	
3 Wk	$\neg (C \land D)$	X,(4)
4 MTP	$A \land B$	5
5 Ext	A	
5 Ext	B	(6)
6 Adj	$B \land E$	X,(7)
7 MPT	$\neg (F \lor G)$	(9)
8 Wk	$F \lor G$	X,(9)
	•	
9 Nc	$\bot$	
		2
2 IP	C	

Both begin with the third premise, but they use the other two premises in a different order. The derivation on the left produces a direct proof of the conclusion C while the one of the right reaches C by an indirect proof showing that ¬ C is incompatible with the premises.

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4.3.s. Summary

1 While a disjunction does not settle the truth values of its disjuncts, it says enough about them that adding the information that one is false will tell us that the other is true. This principle is known traditionally as *modus tollendo ponens*. Since each disjunct entails the disjunction, we know that, if one disjunct is false, then the disjunction and the other disjunct provide the same information. This idea is implemented in a further rule for exploiting disjunctions, also known as *Modus Tollendo Ponens* (MTP). The not-both form $\neg (\varphi \land \psi)$ is analogous to disjunction and analogous principles apply. Specifically, a principle *modus ponendo tollens* tells us that $\neg (\varphi \land \psi)$ together with the assertion of one of φ and ψ entails the denial of the other. And, since the denial of either φ or ψ entails $\neg (\varphi \land \psi)$, we can have a rule *Modus Ponendo Tollens* (MPT) for exploiting not-both forms. The rules MTP and MPT are examples of detachment rules. The resource exploited in each is its main resource and the additional resource that must be available is the auxiliary resource.

2 We will refer to as weakening the principle that disjunctions and not-both forms are entailed by assertions of components (in the case of disjunctions) or their denials (in the case of the not-both form). This principle provides the basis for two further attachment rules, both called Weakening (Wk), that license the addition of inactive resources. Since the second resource we must have in order to apply a detachment rule need only be available, attachment rules can be used to prepare for the use of detachment rules as well to prepare for the use of rules that close gaps.

We now have examples of all the types of rules we will employ:

Rules for developing gaps	*for resources*	*for goals*
atomic sentence		IP
negation $\neg\,\varphi$	CR (if φ is not atomic and the goal is $\bot$)	RAA
conjunction $\varphi \wedge \psi$	Ext	Cnj
disjunction $\varphi \vee \psi$	PC	PE

Rules for closing gaps — *when to close*	*rule*
the goal is also a resource	QED
sentences φ and $\neg\,\varphi$ are resources & the goal is $\bot$	Nc
$\top$ is the goal	ENV
$\bot$ is a resource	EFQ

Basic system

Detachment rules (optional)

main resource	*auxiliary resource*	*rule*
$\varphi \vee \psi$	$\neg^{\pm}\,\varphi$ or $\neg^{\pm}\,\psi$	MTP
$\neg\,(\varphi \wedge \psi)$	φ or ψ	MPT

Attachment rules

added resource	*rule*
$\varphi \wedge \psi$	Adj
$\varphi \vee \psi$	Wk
$\neg\,(\varphi \wedge \psi)$	Wk

Rule for lemmas

prerequisite	*rule*
the goal is $\bot$	LFR

Added rules (optional)

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4.3.x. Exercises

Redo the exercises of 4.2.x, looking for opportunities to use the new rules. (Each of the answers in 4.2.xa has as least one alternative using the new rules; and, in most cases, the alternative is much shorter than the one given there.)

Since the exercise machine incorporates detachment rules but not attachment rules, it can be used to produce only some of the alternative derivations that are possible using the rules of this section.

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4.3.xa. Exercise answers

1.

a.

	A ∧ B	1
1 Ext	A	(2)
1 Ext	B	
2 Wk	A ∨ B	X,(3)
	•	
3 QED	A ∨ B	

b.

	A ∧ B	1
1 Ext	A	
1 Ext	B	(2)
2 Wk	B ∨ C	X,(3)
	•	
3 QED	B ∨ C	

c.

	A ∨ B	1
	¬ A	(1)
1 MTP	B	(2)
	•	
2 QED	B	

d. Although the following is a possible approach, the derivation in 4.2.xa is probably more natural:

	A ∨ (A ∧ B)	2
	¬ A	(2),(4)
2 MTP	A ∧ B	3
3 Ext	A	(4)
3 Ext	B	
	•	
4 Nc	⊥	1
1 IP	A	

e.

	A ∨ B	3
	¬ (A ∧ C)	2
	¬ (B ∧ C)	4
	C	(2),(5)
2 MPT	¬ A	(3)
3 MTP	B	(4)
4 MPT	¬ C	(5)
	•	
5 Nc	⊥	1
1 RAA	¬ C	

f.

	A ∧ (B ∨ C)	1
1 Ext	A	(4)
1 Ext	B ∨ C	3
	¬ C	(3)
3 MTP	B	(4)
4 Adj	A ∧ B	X,(5)
	•	
5 QED	A ∧ B	2
2 PE	(A ∧ B) ∨ C	

or

	A ∧ (B ∨ C)	1
1 Ext	A	(3)
1 Ext	B ∨ C	2
	B	(3)
3 Adj	A ∧ B	X,(4)
4 Wk	(A ∧ B) ∨ C	X,(5)
	•	
5 QED	(A ∧ B) ∨ C	2
	C	(6)
6 Wk	(A ∧ B) ∨ C	X,(7)
	•	
7 QED	(A ∧ B) ∨ C	2
2 PC	(A ∧ B) ∨ C	

g.

	A ∨ B	1
	C	(2),(5)
	A	(2)
2 Adj	A ∧ C	X,(3)
3 Wk	(A ∧ C) ∨ (B ∧ C)	X,(4)
	•	
4 QED	(A ∧ C) ∨ (B ∧ C)	1
	B	(5)
5 Adj	B ∧ C	X,(6)
6 Wk	(A ∧ C) ∨ (B ∧ C)	X,(7)
	•	
7 QED	(A ∧ C) ∨ (B ∧ C)	1
1 PC	(A ∧ C) ∨ (B ∧ C)	

or

	A ∨ B	1
	C	(2),(4)
	¬ (A ∧ C)	2
2 MPT	¬ A	(3)
3 MTP	B	(4)
4 Adj	B ∧ C	X,(5)
	•	
5 QED	B ∧ C	1
1 PE	(A ∧ C) ∨ (B ∧ C)	

h.

	A ∨ B	1
	¬ A ∨ C	2
	A	(2)
2 MTP	C	(3)
3 Wk	B ∨ C	X,(4)
	•	
4 QED	B ∨ C	1
	B	(5)
5 Wk	B ∨ C	X,(6)
	•	
6 QED	B ∨ C	1
1 PC	B ∨ C	

or

	A ∨ B	2
	¬ A ∨ C	3
	¬ B	(2)
2 MTP	A	(3)
3 MTP	C	(4)
	•	
4 QED	C	1
1 PE	B ∨ C	

i.

	A	(2),(3)
	¬ (A ∧ B)	2
2 MPT	¬ B	(3)
3 Adj	A ∧ ¬ B	X,(4)
	•	
4 QED	A ∧ ¬ B	1
1 PE	(A ∧ B) ∨ (A ∧ ¬ B)	

	(A ∧ B) ∨ (A ∧ ¬ B)	3
	¬ A	(2),(5)
2 Wk	¬ (A ∧ B)	X,(3)
3 MTP	A ∧ ¬ B	4
4 Ext	A	(5)
4 Ext	¬ B	
	•	
5 Nc	⊥	1
1 IP	A	

Although the derivation above for the second entailment is possible, the derivation for it in 4.2.xa is probably more natural

2. **a.**

```
        | A ∨ A      2
        |—
        | | ¬ A      (2),(3)
        | |—
2 MTP   | | A        (3)
        | | •
        | |—
3 Nc    | | ⊥        1
        |—
1 IP    | A
```

```
        | A          (1)
        |—
1 Wk    | A ∨ A      X,(2)
        | •
        |—
2 QED   | A ∨ A
```

Another somewhat artificial approach.

b.

```
        | A ∨ B      1
        |—
        | | A        (2)
        | |—
2 Wk    | | B ∨ A    X,(3)
        | | •
        | |—
3 QED   | | B ∨ A    1
        |
        | | B        (4)
        | |—
4 Wk    | | B ∨ A    X,(5)
        | | •
        | |—
5 QED   | | B ∨ A    1
        |—
1 PC    | B ∨ A
```

```
        | B ∨ A      2
        |—
        | | ¬ A      (2)
        | |—
2 MTP   | | B        (3)
        | | •
        | |—
3 QED   | | B        1
        |—
1 PE    | A ∨ B
```

As was the case with the derivations in 4.2.xa, each of the above approaches could have been used for both entailments.

c.

```
        | (A ∨ B) ∨ C        3
        |—
        | | ¬ A              (4)
        | |—
        | | | ¬ C            (3)
        | | |—
3 MTP   | | | A ∨ B          4
4 MTP   | | | B              (5)
        | | | •
        | | |—
5 QED   | | | B              2
        | |—
2 PE    | | B ∨ C            1
        |—
1 PE    | A ∨ (B ∨ C)
```

The derivation at the right can be compared to the one in 4.2.3

```
        | A ∨ (B ∨ C)            1
        |—
        | | A                    (2)
        | |—
2 Wk    | | A ∨ B                X,(3)
3 Wk    | | (A ∨ B) ∨ C          X,(4)
        | | •
        | |—
4 QED   | | (A ∨ B) ∨ C          1
        |
        | | B ∨ C                5
        | |—
        | | | B                  (6)
        | | |—
6 Wk    | | | A ∨ B              X,(7)
7 Wk    | | | (A ∨ B) ∨ C        X,(8)
        | | | •
        | | |—
8 QED   | | | (A ∨ B) ∨ C        5
        | |
        | | | C                  (9)
        | | |—
9 Wk    | | | (A ∨ B) ∨ C        (10)
        | | | •
        | | |—
10 QED  | | | (A ∨ B) ∨ C        5
        | |—
5 PC    | | (A ∨ B) ∨ C          1
        |—
1 PC    | (A ∨ B) ∨ C
```

d.

```
        | A ∨ (B ∧ ¬ B)      2
        |—
        | | ¬ A              (2)
        | |—
2 MTP   | | B ∧ ¬ B          3
3 Ext   | | B                (4)
3 Ext   | | ¬ B              (4)
        | | •
        | |—
4 Nc    | | ⊥                4
        |—
1 IP    | A
```

```
        | A                  (1)
        |—
1 Wk    | A ∨ (B ∧ ¬ B)      X,(2)
        | •
        |—
2 QED   | A ∨ (B ∧ ¬ B)
```

e.

```
        | ¬ (A ∨ B)      (4),(7)
        |—
        | | | A          (3)
        | | |—
3 Wk    | | | A ∨ B      X,(4)
        | | | •
        | | |—
4 Nc    | | | ⊥          2
        | |—
2 RAA   | | ¬ A          1
        |
        | | | B          (6)
        | | |—
6 Wk    | | | A ∨ B      X,(7)
        | | | •
        | | |—
7 Nc    | | | ⊥          5
        | |—
5 RAA   | | ¬ B          1
        |—
1 Cnj   | ¬ A ∧ ¬ B
```

```
        | ¬ A ∧ ¬ B      1
        |—
1 Ext   | ¬ A            (3)
1 Ext   | ¬ B            (4)
        | | A ∨ B        3
        | |—
3 MTP   | | B            (4)
        | | •
        | |—
4 Nc    | | ⊥            2
        |—
2 RAA   | ¬ (A ∨ B)
```

f.

```
        | ¬ (A ∧ B)      2
        |—
        | | A            (2)
        | |—
2 MPT   | | ¬ B          (3)
        | | •
        | |—
3 QED   | | ¬ B          1
        |—
1 PE    | ¬ A ∨ ¬ B
```

```
        | ¬ A ∨ ¬ B      3
        |—
        | | A ∧ B        2
        | |—
2 Ext   | | A            (3)
2 Ext   | | B            (4)
3 MTP   | | ¬ B          (4)
        | | •
        | |—
4 Nc    | | ⊥            1
        |—
1 RAA   | ¬ (A ∧ B)
```

3. **a.** This derivation is unchanged from 4.2.xa

```
        │A ∨ B        2
        │A
        ├─
        │ │B
        │ ├─
        │ │ │A
        │ │ │○          A, B ⇏ ⊥
        │ │ ├─
        │ │ │⊥          2
        │ │
        │ │ │B
        │ │ │○          A, B ⇏ ⊥
        │ │ ├─
        │ │ │⊥          2
        │ ├─
2 PC    │ │⊥            1
1 RAA   ├─
        │¬ B
```

A B	A ∨ B , A / ¬ B		
T T	Ⓣ	Ⓣ	Ⓕ

b.

```
        │A ∨ (B ∧ C)      3,8
        │ │ │¬ A          (3)
        │ │ ├─
3 MTP   │ │ │B ∧ C        4
4 Ext   │ │ │B            (5)
4 Ext   │ │ │C
        │ │ │●
        │ │ ├─
5 QED   │ │ │B            2
        │ ├─
2 PE    │ │A ∨ B          1
        │
        │ │ │¬ C          (7)
        │ │ ├─
7 Wk    │ │ │¬ (B ∧ C)    X,(8)
8 MTP   │ │ │A
        │ │ │○            A, ¬ C ⇏ ⊥
        │ │ ├─
        │ │ │⊥            9
        │ ├─
6 IP    │ │C              1
        ├─
1 Cnj   │(A ∨ B) ∧ C
```

```
        │(A ∨ B) ∧ C      1
1 Ext   │A ∨ B            3
1 Ext   │C                (4)
        │ │¬ A            (3)
        │ ├─
3 MTP   │ │B              (4)
4 Adj   │ │B ∧ C          X,(5)
        │ │●
        │ ├─
5 QED   │ │B ∧ C          2
        ├─
2 PE    │A ∨ (B ∧ C)
```

Each of the following divides the one open gap:

A B C	A ∨ (B ∧ C) / (A ∨ B) ∧ C			
T T F	Ⓣ	F	T	Ⓕ
T F F	Ⓣ	F	T	Ⓕ

Although the use of Wk and MTP shortens the whole first derivation, it actually delays the dead end, which would have been reached after stage 7 if the first premise had been exploited by PC in the second gap. As in 4.2.xa, the second derivation is unnecessary once a dead-end gap is found in the first.

c.

```
        │¬ (A ∨ B)        (4)
        ├─
        │ │A              (3)
        │ ├─
        │ │ │B
        │ │ ├─
3 Wk    │ │ │A ∨ B        X,(4)
        │ │ │●
        │ │ ├─
4 Nc    │ │ │⊥            2
        │ ├─
2 RAA   │ │¬ B            1
        ├─
1 PE    │¬ A ∨ ¬ B
```

The following divide the first and second open gap, respectively:

A B	¬ A ∨ ¬ B / ¬ (A ∨ B)			
F T	T	Ⓣ F	Ⓕ	T
T F	F	Ⓣ T	Ⓕ	T

```
        │¬ A ∨ ¬ B        2
        ├─
        │ │A ∨ B          3,4
        │ ├─
        │ │ │¬ A          (3)
        │ │ ├─
3 MTP   │ │ │B
        │ │ │○            ¬ A, B ⇏ ⊥
        │ │ ├─
        │ │ │⊥            2
        │ │
        │ │ │¬ B          (4)
        │ │ ├─
4 MTP   │ │ │A
        │ │ │○            A, ¬ B ⇏ ⊥
        │ │ ├─
        │ │ │⊥            2
        │ ├─
2 PC    │ │⊥              1
        ├─
1 RAA   │¬ (A ∨ B)
```

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