

## 4.2. Arguing from and for alternatives

### 4.2.0. Overview

Because a disjunction normally says less than its components while a conjunction says more, the two connectives play very different roles in entailment.

#### 4.2.1. Proofs by cases

Since a disjunction says only what is said by both its disjuncts, it entails only what is entailed by both of them.

#### 4.2.2. Proving disjunctions

Since a disjunction makes a relatively weak claim, it is easy to state a sound rule to plan for it, but a safe rule that will cover all cases where it holds is more complex.

#### 4.2.3. Further examples

There are now many choices to be regarding the order in which rules are applied and some differences in the length of derivations can result.

#### 4.2.4. The duality of conjunction and disjunction

Conjunction and disjunction are, in a certain formal sense, mirror images of one another.

Glen Helman 28 Aug 2008

### 4.2.1. Proofs by cases

The validity of the argument

Sam didn't praise the proposal without granting its significance  
Sam didn't condemn the proposal without granting its significance  
Sam either praised or condemned the proposal

---

Sam granted the proposal's significance.

can be accounted for by the validity of the following two arguments:

|   |   |
|---|---|
| Sam didn't praise the proposal without granting its significance  | Sam didn't praise the proposal without granting its significance  |
| Sam didn't condemn the proposal without granting its significance | Sam didn't condemn the proposal without granting its significance |
| <u>Sam praised the proposal</u>                                   | <u>Sam condemned the proposal</u>                                 |
| Sam granted the proposal's significance                           | Sam granted the proposal's significance                           |

Each replaces the disjunctive third premise of the original argument by one of its two components. This way of establishing an entailment is sometimes called a *proof by cases*. In this example, the two cases are Sam having praised the proposal and Sam having condemned it. Since the disjunction says all and only what is common to these two claims, what follows from the disjunction in isolation or in addition to other premises is what follows from each of these claims under similar circumstances.

More formally, the idea behind proofs by cases is captured by this principle:

LAW FOR DISJUNCTION AS A PREMISE.  $\Gamma, \phi \vee \psi \Rightarrow \chi$  if and only if both  $\Gamma, \phi \Rightarrow \chi$  and  $\Gamma, \psi \Rightarrow \chi$  (for any set  $\Gamma$  and sentences  $\phi, \psi$ , and  $\chi$ ).

To see why this law is true note that to divide the members of  $\Gamma$  and  $\phi \vee \psi$  on the one hand from  $\chi$  on the other, a possible world must make  $\phi \vee \psi$  and all members of  $\Gamma$  true while making  $\chi$  false. To do this it must make at least one of  $\phi$  and  $\psi$  true, so it must divide at least one of the arguments  $\Gamma, \phi / \chi$  and  $\Gamma, \psi / \chi$ . So, to say that the original argument is valid is to say that neither of these latter arguments can have its premises and alternatives divided—that is, that both are valid.

This idea appears in derivations by way of a rule we will call *Proof by Cases* (PC); it is shown in Figure 4.2.1-1.

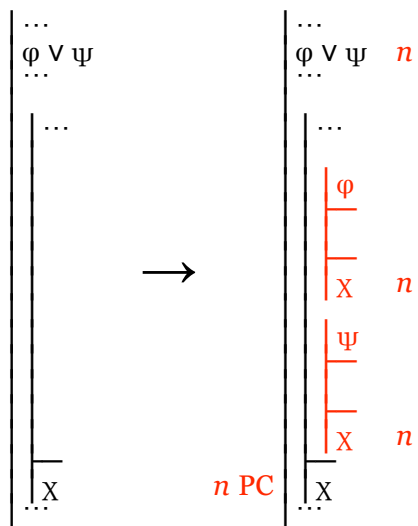
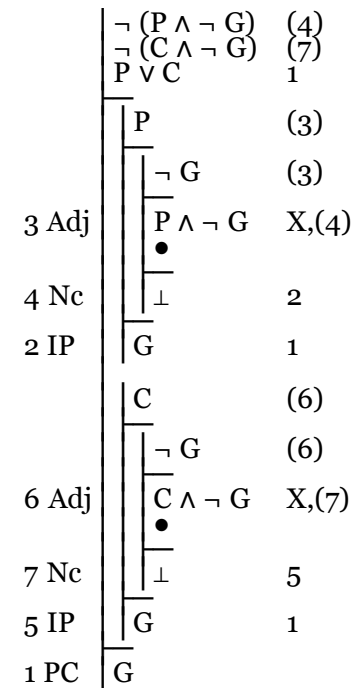


Fig. 4.2.1-1. Developing a derivation by exploiting a disjunction at stage  $n$ .

PC divides a gap into two new gaps. Each is a *case argument* that retains the original goal but adds one of the components of the disjunction as a supposition. The function of each supposition is to specify one of the two sorts of case in which the original disjunction is true. A supposition is required because, although our premises tell us that at least one of the disjuncts is true, we do not know which that is and the one that is true will vary among the possible worlds in which the premises are all true.

Here is a derivation which uses this rule to provide a proof for example with which we began.



C: Sam condemned the proposal; G: Sam granted the proposal's significance; P: Sam praised the proposal

In the two case arguments, we suppose first that Sam praised the proposal and then that he condemned it and, in each case, we show that he granted the proposal's significance (by showing that he could not have failed to grant it). Since at least one of these two cases must be true whenever the premises are all true, we know that the conclusion must be true also.

### 4.2.2. Proving disjunctions

Now let us look at disjunctions as conclusions. An entailment  $\Gamma \Rightarrow \varphi \vee \psi$  will hold if and only if  $\varphi \vee \psi$  is true in every possible world in which all members of  $\Gamma$  are true. But this is to say that at least one of  $\varphi$  and  $\psi$  is true in every such world, and that is a way of saying that  $\Gamma$  renders  $\varphi$  and  $\psi$  jointly exhaustive. So we can state the following principle:

$$\Gamma \Rightarrow \varphi \vee \psi \text{ if and only if } \Gamma \Rightarrow \varphi, \psi$$

Since the right-hand side has two alternatives, this is not a law concerning entailment alone, and we will not take the principle in this form as our account of the role of disjunctions as conclusions. However, we can use the basic law for relative exhaustiveness to restate the right-hand side as claim of entailment.

Indeed we have two ways of doing that. If  $\varphi$  and  $\bar{\varphi}$  are contradictory, we can say

$$\Gamma \Rightarrow \varphi \vee \psi \text{ if and only if } \Gamma, \bar{\varphi} \Rightarrow \psi$$

and if  $\psi$  and  $\bar{\psi}$  are contradictory, we can say

$$\Gamma \Rightarrow \varphi \vee \psi \text{ if and only if } \Gamma, \bar{\psi} \Rightarrow \varphi$$

In short, a disjunction is a valid conclusion from premises  $\Gamma$  if and only if adding to our premises a sentence contradictory to one disjunct enables us to validly conclude the other disjunct.

In stating a principle for disjunction we will limit ourselves to cases where a sentence and its negation are the pair of contradictory sentences. But, when the disjuncts are already negative, that leaves us with two choices for each of the pairs  $\varphi$  and  $\bar{\varphi}$  and  $\psi$  and  $\bar{\psi}$  since each of  $\bar{\varphi}$  and  $\bar{\psi}$  might be the result of either adding or dropping a negation. To avoid stating four principles to cover each of these possibilities, we will introduce some notation to capture the general idea of obtaining a contradictory sentence by either adding or dropping a negation. Let the notation  $\neg^\pm \varphi$  be the result of negating  $\varphi$  with an optional added step of deleting a double negation if  $\varphi$  was already negative. Then  $\neg^\pm \varphi$  will stand for  $\neg \varphi$  when  $\varphi$  is not a negation and, when  $\varphi$  is the negation  $\neg \chi$ , for either  $\neg \neg \chi$  or  $\chi$ . That is,  $\neg^\pm \varphi$  is the result of either negating or, perhaps, de-negating  $\varphi$ , and  $\neg^\pm \varphi$  will either be the negation of  $\varphi$  or have  $\varphi$  as its negation.

Then  $\neg^\pm \varphi$  and  $\varphi$  form a contradictory pair consisting of a sentence and its negation in one order or the other, so we may formulate a principle to account for conclusions that are disjunctions with only two

statements:

LAW FOR DISJUNCTION AS A CONCLUSION. (i)  $\Gamma \Rightarrow \varphi \vee \psi$  if and only if  $\Gamma, \neg^\pm \varphi \Rightarrow \psi$ , and (ii)  $\Gamma \Rightarrow \varphi \vee \psi$  if and only if  $\Gamma, \neg^\pm \psi \Rightarrow \varphi$  (for any set  $\Gamma$  and sentences  $\varphi, \psi$ , and  $\chi$ ).

When these are implemented as derivation rules, they give us two ways of planning for a disjunctive goal.

The two rules are shown as alternative developments in Figure 4.2.2-1. We will refer to both forms of the rule as *Proof of Exhaustion* (PE) since it is a way of showing that  $\varphi$  and  $\psi$ , taken together, exhaust all possibilities left open by the premises.

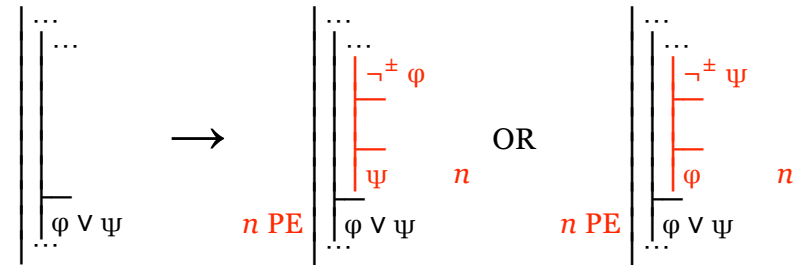


Fig. 4.2.2-1. Alternative ways of developing a derivation by planning for a disjunction at stage  $n$ .

In each way of developing a gap, we set one of the components of the disjunction as a new goal and add the negation or de-negation of the other component as a supposition. In each way of developing a gap, we set one of the components of the disjunction as a new goal.

Both forms of planning will lead to the same answer in the end, but one or the other may be more efficient in a particular case. There is no simple way of predicting which choice is best but the following rules of thumb may help:

- (i) if only one component is a negation, choose it to form the supposition (by dropping its negation);
- (ii) if only one component is a non-negative compound choose it as the goal;
- (iii) if only one component seems likely to figure in closing the gap and it is not a negation, choose it as the goal.

In many cases none of these suggestions will apply; but, in most such cases, neither one of the two forms of the rule is better than the other.

The supposition in PE may be described as *hypothetical*, and this indicates a third role that suppositions may play. In *reductio* arguments



### 4.2.3. Further examples

Both disjunction rules are illustrated by the derivation at the right, in which one grouping of a three-part disjunction is shown to entail the other. Choices between the two ways of planning for a goal disjunction were made at stages 2, 3, 5, 6, and 7 in accordance with the rules of thumb given above. Each choice helped to shorten the derivation—though only by a few steps. The derivation is contrived to provide several examples of this rule; we might have instead planned for initial the goal at stage 1 before exploiting the premise rather than planning for it separately in each of three gaps.

|        |                     |      |  |
|--------|---------------------|------|--|
|        | $A \vee (B \vee C)$ | 1    |  |
|        | $A$                 | (4)  |  |
|        | $\neg C$            |      |  |
|        | $\neg B$            |      |  |
|        | •                   |      |  |
| 4 QED  | $A$                 | 3    |  |
| 3 PE   | $A \vee B$          | 2    |  |
| 2 PE   | $(A \vee B) \vee C$ | 1    |  |
|        | $B \vee C$          | 5    |  |
|        | $B$                 | (8)  |  |
|        | $\neg C$            |      |  |
|        | $\neg A$            |      |  |
|        | •                   |      |  |
| 8 QED  | $B$                 | 7    |  |
| 7 PE   | $A \vee B$          | 6    |  |
| 6 PE   | $(A \vee B) \vee C$ | 5    |  |
|        | $C$                 | (10) |  |
|        | $\neg (A \vee B)$   |      |  |
|        | •                   |      |  |
| 10 QED | $C$                 | 9    |  |
| 9 PE   | $(A \vee B) \vee C$ | 5    |  |
| 5 PC   | $(A \vee B) \vee C$ | 1    |  |
| 1 PC   | $(A \vee B) \vee C$ |      |  |

The scale of the difference you can expect a choice between the two forms of PE to make is illustrated by the two derivations below.

|       |                     |     |  |       |                     |     |  |
|-------|---------------------|-----|--|-------|---------------------|-----|--|
|       | $B$                 | (3) |  |       | $B$                 | (5) |  |
|       | $\neg A$            |     |  |       | $\neg (B \vee C)$   | 3   |  |
|       | $\neg C$            |     |  |       | $\neg A$            |     |  |
|       | •                   |     |  |       | $\neg C$            |     |  |
| 3 QED | $B$                 | 2   |  |       | •                   |     |  |
| 2 PE  | $B \vee C$          | 1   |  | 5 QED | $B$                 | 4   |  |
| 1 PE  | $A \vee (B \vee C)$ |     |  | 4 PE  | $B \vee C$          | 3   |  |
|       |                     |     |  | 3 CR  | $\perp$             | 2   |  |
|       |                     |     |  | 2 IP  | $A$                 | 1   |  |
|       |                     |     |  | 1 PE  | $A \vee (B \vee C)$ |     |  |

Each chooses a different way of planning for the initial goal at stage 1. Notice that in the second, which makes the less efficient choice, we are led back to the goal  $B \vee C$  in a couple of stages.

Glen Helman 28 Aug 2008

#### 4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term **dual**.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. For example, let us again take **X was home** and **X was out** to be contradictories. Then note that to get a sentence contradictory to **Ann and Bill were home**, we cannot take **Ann and Bill were out** since both would be false if one of Ann and Bill was home and the other out. To get a contradictory to we need to cover both of those possibilities as well, and **Ann or Bill was out** will do this. That is, **Ann and Bill were home** is contradictory to **Ann or Bill was out** and, similarly, **Ann or Bill was home** is contradictory to **Ann and Bill were out**. And this is to say that  $\neg$  **Ann and Bill were home**  $\Leftrightarrow$  **Ann or Bill was out** and that  $\neg$  **Ann or Bill was home**  $\Leftrightarrow$  **Ann and Bill were out**.

In cases of contradictoriness captured by the  $\neg^\pm$  notation, these patterns of equivalence are know as *De Morgan's laws*:

$$\begin{aligned} \neg(\varphi \wedge \psi) &\Leftrightarrow \neg^\pm \varphi \vee \neg^\pm \psi \\ \neg(\varphi \vee \psi) &\Leftrightarrow \neg^\pm \varphi \wedge \neg^\pm \psi \end{aligned}$$

Although these are named after Augustus De Morgan (1806-1871), they were known well before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles of relative exhaustiveness. The table below follows the pattern of the one given for  $\perp$  and  $\top$  in 1.4.8.

|             | as a premise   | as an alternative  |
|-------------|--|--|
| Conjunction | $\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ iff $\Gamma, \varphi, \psi \Rightarrow \Delta$                                    | $\Gamma \Rightarrow \varphi \wedge \psi, \Delta$ iff both $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$ |
| Disjunction | $\Gamma, \varphi \vee \psi \Rightarrow \Delta$ iff both $\Gamma, \varphi \Rightarrow \Delta$ and $\Gamma, \psi \Rightarrow \Delta$ | $\Gamma \Rightarrow \varphi \vee \psi, \Delta$ iff $\Gamma \Rightarrow \varphi, \psi, \Delta$  |

(Here *iff* is used as an abbreviation of *if and only if*.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise much as disjunction behaves as an alternative and disjunction behaves as premise much as conjunction behaves as an alternative.

Since  $\perp$  and  $\top$  are paired as duals and so are conjunction and disjunction, you might wonder about negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to

the two contradictory sentences in order for the results to be contradictory. And negations behavior as a premise is analogous to its behavior as an alternative.

$$\begin{aligned} \Gamma, \neg \varphi \Rightarrow \Delta &\text{ iff } \Gamma \Rightarrow \varphi, \Delta \\ \Gamma, \varphi \Rightarrow \Delta &\text{ iff } \Gamma \Rightarrow \neg \varphi, \Delta \end{aligned}$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

The term **duality** points to a certain sort of two-for-one principle. It is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

$$\begin{array}{c} \text{premise} \text{---} \text{alternative} \\ \perp \text{---} \top \\ \text{negation} \text{---} \text{negation} \\ \text{conjunction} \text{---} \text{disjunction} \end{array}$$

So, for example (and to deal only with informal statements of the principles), the principle **A conjunction as a premise may be replaced by its components as separate premises** (the upper left in the table above) turns into **A disjunction as an alternative may be replaced by its components as separate alternatives** (the lower right). And the principle **A negation as a premise may be replaced by its immediate component as an alternative** (the first of the principles for negation above) turns into **A negation as an alternative may be replaced by its immediate component as a premise** (the second of the principles). We will see more examples of such transformations in the next section but we have already seen some further ones: each of the two forms of De Morgan's laws may be transformed into the other by this association.

Since these transformations treat premises and alternatives in a parallel way, not all will apply to entailment, which allows multiple premises but only a single alternative. However, we have also seen that principles for relative exhaustiveness may be transformed still further into principles of entailment by the basic law for relative exhaustiveness (of which the two principles for negation above are special cases) since that law enables us to replace alternatives by premises that are their contradictories.

Glen Helman 28 Aug 2008

#### 4.2.s. Summary

- 1 A disjunction  $\phi \vee \psi$  is false only when its disjuncts are both false, and it thus says only what both of them say. The law for disjunction as a premise tell us that we can establish a conclusion using such a premise by showing that it is entailed by each of the disjuncts (given our other premises). This way of exploiting a disjunction is known as an proof by cases and it appears in our system of derivations as the rule Proof by Cases (PC) that leads us to divide a gap into two case arguments, each of which takes over the original goal and adds one of the two disjuncts as a supposition.
- 2 To show that a disjunction is a valid conclusion, we must show that its disjuncts are rendered jointly exhaustive by the premises. We can do this by showing that one of the disjuncts will follow if we add the contradictory of the other to our premises. We use the notation  $\neg^\pm \phi$  to indicate the result of either negating or de-negating  $\phi$ . The law for disjunction as a conclusion then tells us that we can conclude a disjunction if we can conclude one disjunct provided we take the negation or de-negation of the other disjunct as a premise. The rule implementing this idea is Proof of Exhaustion; it enables us to conclude a disjunction from an argument that may be called hypothetical since it bases disjunct on an assumption (of the negation or de-negation of the other disjunct) that we may not be prepared to assert categorically. It does not matter for the soundness or safety of PE which disjunct figures as the goal of this hypothetical argument and which is negated or de-negated in its supposition.
- 3 Derivations, especially those that have a disjunction as a goal as well as a premise can often be developed in different ways. Some of these can be significantly longer than others but the choice between forms of PE will usually have only a limited impact on the length.
- 4 Conjunction and disjunction are opposite in the sense of being dual. One manifestation of this relation is in De Morgan's laws, which tell how to restate the denial of a conjunction or disjunction as an assertion of the other form of compound. Another manifestation is a pattern in laws of relative exhaustiveness which allows us to interchange conjunctions and disjunctions if at the same time we interchange  $\perp$  and  $\top$  and also premises and alternatives.

Glen Helman 28 Aug 2008

#### 4.2.x. Exercises

1. Use derivations to establish each of the claims of entailment and equivalence shown below. (Remember that claims of equivalence require derivations in both directions.)
  - a.  $A \wedge B \Rightarrow A \vee B$
  - b.  $A \wedge B \Rightarrow B \vee C$
  - c.  $A \vee B, \neg A \Rightarrow B$
  - d.  $A \vee (A \wedge B) \Rightarrow A$
  - e.  $A \vee B, \neg (A \wedge C), \neg (B \wedge C) \Rightarrow \neg C$
  - f.  $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee C$
  - g.  $A \vee B, C \Rightarrow (A \wedge C) \vee (B \wedge C)$
  - h.  $A \vee B, \neg A \vee C \Rightarrow B \vee C$
  - i.  $A \Leftrightarrow (A \wedge B) \vee (A \wedge \neg B)$
2. Use derivations to establish each of the claims of equivalence below.
  - a.  $A \vee A \Leftrightarrow A$
  - b.  $A \vee B \Leftrightarrow B \vee A$
  - c.  $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$
  - d.  $A \vee (B \wedge \neg B) \Leftrightarrow A$
  - e.  $\neg (A \vee B) \Leftrightarrow \neg A \wedge \neg B$
  - f.  $\neg (A \wedge B) \Leftrightarrow \neg A \vee \neg B$
3. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap.
  - a.  $A \vee B, A \Rightarrow \neg B$
  - b.  $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge C$
  - c.  $\neg (A \vee B) \Leftrightarrow \neg A \vee \neg B$

For more exercises, use the [exercise machine](#).

Glen Helman 28 Aug 2008

### 4.2.xa. Exercise answers

1. a.

|       |              |     |
|-------|--------------|-----|
|       | $A \wedge B$ | 1   |
| 1 Ext | $A$          |     |
| 1 Ext | $B$          | (3) |
|       | $\neg A$     |     |
|       | •            |     |
| 3 QED | $B$          | 2   |
| 2 PE  | $A \vee B$   |     |

b.

|       |              |     |
|-------|--------------|-----|
|       | $A \wedge B$ | 1   |
| 1 Ext | $A$          |     |
| 1 Ext | $B$          | (3) |
|       | $\neg C$     |     |
|       | •            |     |
| 3 QED | $B$          | 2   |
| 2 PE  | $B \vee C$   |     |

c.

|       |            |     |
|-------|------------|-----|
|       | $A \vee B$ | 1   |
|       | $\neg A$   | (3) |
|       | $A$        | (3) |
|       | $\neg B$   |     |
|       | •          |     |
| 3 Nc  | $\perp$    | 2   |
| 2 IP  | $B$        | 1   |
|       | $B$        | (4) |
|       | •          |     |
| 4 QED | $B$        | 1   |
| 1 PC  | $B$        |     |

d.

|       |                       |     |
|-------|-----------------------|-----|
|       | $A \vee (A \wedge B)$ | 1   |
|       | $A$                   | (2) |
|       | •                     |     |
| 2 QED | $A$                   | 1   |
|       | $A \wedge B$          | 3   |
| 3 Ext | $A$                   | (4) |
| 3 Ext | $B$                   |     |
|       | •                     |     |
| 4 QED | $A$                   | 1   |
| 1 PC  | $A$                   |     |

e.

|        |                     |          |
|--------|---------------------|----------|
|        | $A \vee B$          | 2        |
|        | $\neg (A \wedge C)$ | 3        |
|        | $\neg (B \wedge C)$ | 7        |
|        | $C$                 | (6),(10) |
|        | $A$                 | (5)      |
|        | •                   |          |
| 5 QED  | $A$                 | 4        |
|        | •                   |          |
| 6 QED  | $C$                 | 4        |
| 4 Cnj  | $A \wedge C$        | 3        |
| 3 CR   | $\perp$             | 2        |
|        | $B$                 | (9)      |
|        | •                   |          |
| 9 QED  | $B$                 | 8        |
|        | •                   |          |
| 10 QED | $C$                 | 8        |
| 8 Cnj  | $B \wedge C$        | 7        |
| 7 CR   | $\perp$             | 2        |
| 2 PC   | $\perp$             | 1        |
| 1 RAA  | $\neg C$            |          |

f.

|       |                       |     |
|-------|-----------------------|-----|
|       | $A \wedge (B \vee C)$ | 1   |
| 1 Ext | $A$                   | (5) |
| 1 Ext | $B \vee C$            | 2   |
|       | $B$                   | (6) |
|       | $\neg C$              |     |
|       | •                     |     |
| 5 QED | $A$                   | 4   |
|       | •                     |     |
| 6 QED | $B$                   | 4   |
| 4 Cnj | $A \wedge B$          | 3   |
| 3 PE  | $(A \wedge B) \vee C$ | 2   |
|       | $C$                   | (8) |
|       | $\neg (A \wedge B)$   |     |
|       | •                     |     |
| 8 QED | $C$                   | 7   |
| 7 PE  | $(A \wedge B) \vee C$ | 2   |
| 2 PC  | $(A \wedge B) \vee C$ |     |



**g.**

|       |  |         |           |                                  |     |
|-------|--|---------|-----------|----------------------------------|-----|
|       | $\frac{A \vee B}{C}$                             | 1       | <b>h.</b> | $\frac{A \vee B}{\neg A \vee C}$ | 1   |
|       | $\frac{A}{\neg(B \wedge C)}$                     | (5),(9) |           | $\frac{A}{\neg A}$               | 2   |
|       | $\frac{A}{\bullet}$                              | (4)     |           | $\frac{A}{\neg B}$               | (5) |
| 4 QED | $\frac{A}{\bullet}$                              | 3       |           | $\frac{\neg C}{\bullet}$         | (5) |
|       | $\frac{C}{\bullet}$                              |         | 5 Ne      | $\perp$                          | 4   |
| 5 QED | $\frac{A \wedge C}{\bullet}$                     | 3       | 4 IP      | $C$                              | 3   |
| 3 Cnj | $\frac{(A \wedge C) \vee (B \wedge C)}{\bullet}$ | 2       | 3 PE      | $B \vee C$                       | 2   |
| 2 PE  | $\frac{B}{\bullet}$                              | 1       |           | $C$                              | (7) |
|       | $\frac{\neg(A \wedge C)}{\bullet}$               | (8)     |           | $\neg B$                         |     |
| 8 QED | $\frac{B}{\bullet}$                              | 7       | 7 QED     | $C$                              | 6   |
|       | $\frac{C}{\bullet}$                              |         | 6 PE      | $B \vee C$                       | 2   |
| 9 QED | $\frac{B \wedge C}{\bullet}$                     | 7       | 2 PC      | $B \vee C$                       | 1   |
| 7 Cnj | $\frac{(A \wedge C) \vee (B \wedge C)}{\bullet}$ | 6       |           | $B$                              | (9) |
| 6 PE  | $\frac{(A \wedge C) \vee (B \wedge C)}{\bullet}$ | 1       |           | $\neg C$                         |     |
| 1 PC  | $(A \wedge C) \vee (B \wedge C)$                 |         | 9 QED     | $B$                              | 8   |
|       |  |         | 8 PE      | $B \vee C$                       | 1   |
|       |  |         | 1 PC      | $B \vee C$                       |     |

**i.**

|       |                                       |         |   |           |     |
|-------|---------------------------------------|---------|---|-----------|-----|
|       | $A$                                   | (3),(7) | $\frac{(A \wedge B) \vee (A \wedge \neg B)}{\bullet}$ | 1         |     |
|       | $\neg(A \wedge B)$                    | 5       | $\frac{A \wedge B}{\bullet}$                          | 2         |     |
| 3 QED | $\frac{A}{\bullet}$                   | 2       | 2 Ext   | $B$       | (3) |
|       | $B$                                   | (8)     | 2 Ext   | $\bullet$ |     |
| 7 QED | $\frac{A}{\bullet}$                   | 6       | 3 QED   | $A$       | 1   |
|       | $B$                                   |         | $\frac{A \wedge \neg B}{\bullet}$                     | 4         |     |
| 8 QED | $\frac{A \wedge B}{\bullet}$          | 6       | 4 Ext   | $A$       | (5) |
| 6 Cnj | $\frac{A \wedge \neg B}{\bullet}$     | 5       | 4 Ext   | $\neg B$  |     |
| 5 CR  | $\perp$                               | 4       | 5 QED   | $A$       | 1   |
| 4 RAA | $\neg B$                              | 2       | 1 PC  | $A$       |     |
| 2 Cnj | $A \wedge \neg B$                     | 1       |   |           |     |
| 1 PE  | $(A \wedge B) \vee (A \wedge \neg B)$ |         |   |           |     |

**2. a.**

|       |                      |     |                     |                            |
|-------|----------------------|-----|---------------------|----------------------------|
|       | $\frac{A \vee A}{A}$ | 1   | $\frac{A}{\neg A}$  | (2)                        |
|       | $\frac{A}{\bullet}$  | (2) | $\frac{\bullet}{A}$ | 1                          |
| 2 QED | $\frac{A}{\bullet}$  | 1   | 2 QED               | $\frac{A \vee A}{\bullet}$ |
|       | $\frac{A}{\bullet}$  | (3) | 1 PE                |                            |
| 3 QED | $\frac{A}{\bullet}$  | 1   |                     |                            |
| 1 PC  | $A$                  |     |                     |                            |

|       |            |      |            |     |
|-------|------------|------|------------|-----|
|       | $A \vee B$ | 1    | $B \vee A$ | 2   |
|       | A          | (3)  | $\neg A$   | (5) |
|       | $\neg B$   |      | B          | (3) |
|       | •          |      | •          |     |
| 3 QED | A          | 2    | B          | 2   |
| 2 PE  | $B \vee A$ | 1    | A          | (5) |
|       | B          |      | $\neg B$   |     |
|       | $\neg A$   | (5)  | •          |     |
|       | •          |      | $\perp$    | 4   |
| 5 QED | B          | 4    | B          | 2   |
| 4 PE  | $B \vee A$ | 1    | B          | 1   |
| 1 PC  | $B \vee A$ | 1 PE | $A \vee B$ |     |

**c.**

|       |                     |     |  |
|-------|---------------------|-----|--|
|       | $(A \vee B) \vee C$ | 3   | This is the second of the two derivations needed; the first appears in 4.2.3. In that one, disjunctive resources are exploited before disjunctive goals are planned for while the derivation at the left here illustrates the opposite approach. |
|       | $\neg A$            | (6) |  |
|       | $\neg B$            | (8) |  |
|       | $A \vee B$          | 4   |  |
|       | A                   | (6) |  |
|       | $\neg C$            |     |  |
|       | •                   |     |  |
| 6 Nc  | $\perp$             | 5   |  |
| 5 IP  | C                   | 4   |  |
|       | B                   | (8) |  |
|       | $\neg C$            |     |  |
|       | •                   |     |  |
| 8 Nc  | $\perp$             | 7   |  |
| 7 IP  | C                   | 4   |  |
| 4 PC  | C                   | 3   |  |
|       | C                   | (9) |  |
|       | •                   |     |  |
| 9 QED | C                   | 3   |  |
| 3 PC  | C                   | 2   |  |
| 2 PE  | $B \vee C$          | 1   |  |
| 1 PE  | $A \vee (B \vee C)$ |     |  |

**d.**

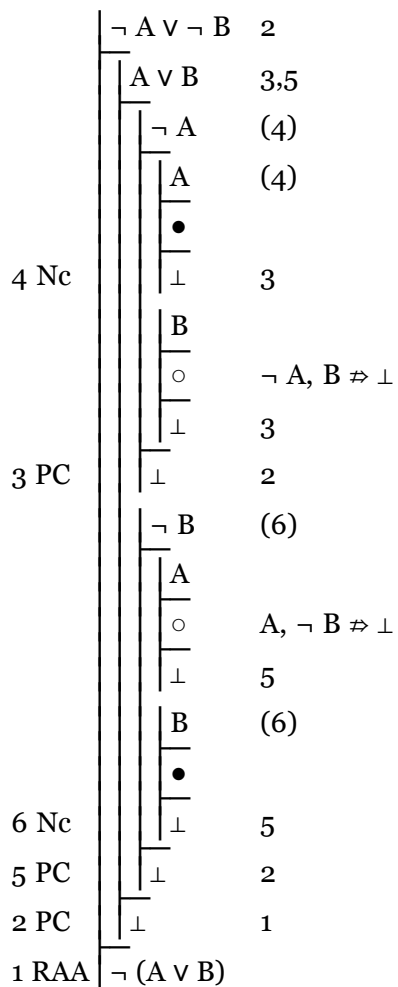
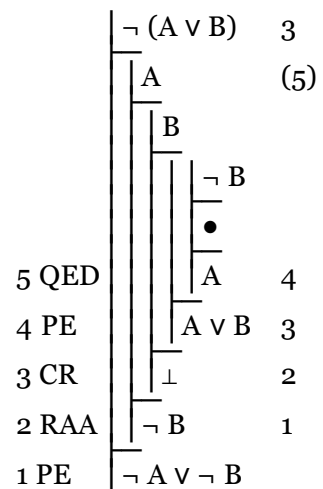
|       |                            |     |                            |     |
|-------|----------------------------|-----|----------------------------|-----|
|       | $A \vee (B \wedge \neg B)$ | 1   | $A$                        | (2) |
|       | A                          | (2) | $\neg (B \wedge \neg B)$   |     |
|       | •                          |     | •                          |     |
| 2 QED | A                          | 1   | A                          | 1   |
|       | $B \wedge \neg B$          | 3   | $A \vee (B \wedge \neg B)$ |     |
| 3 Ext | B                          | (5) |                            |     |
| 3 Ext | $\neg B$                   | (5) |                            |     |
|       | $\neg A$                   |     |                            |     |
|       | •                          |     |                            |     |
| 5 Nc  | $\perp$                    | 4   |                            |     |
| 4 IP  | A                          | 1   |                            |     |
| 1 PC  | A                          |     |                            |     |

**e.**

|       |                        |     |                        |                   |
|-------|------------------------|-----|------------------------|-------------------|
|       | $\neg (A \vee B)$      | 3,7 | $\neg A \wedge \neg B$ | 1                 |
|       | A                      | (5) | $\neg A$               | (4)               |
|       | $\neg B$               |     | $\neg B$               | (5)               |
|       | •                      |     | $A \vee B$             | 3                 |
| 5 QED | A                      | 4   | A                      | (4)               |
| 4 PE  | $A \vee B$             | 3   | •                      |                   |
| 3 CR  | $\perp$                | 2   | $\perp$                | 3                 |
| 2 RAA | $\neg A$               | 1   | B                      | (5)               |
|       | B                      | (9) | •                      |                   |
|       | $\neg A$               |     | $\perp$                | 3                 |
|       | •                      |     | $\perp$                | 2                 |
| 9 QED | B                      | 8   | 3 PC                   |                   |
| 8 PE  | $A \vee B$             | 7   | 2 RAA                  | $\neg (A \vee B)$ |
| 7 CR  | $\perp$                | 6   |                        |                   |
| 6 RAA | $\neg B$               | 1   |                        |                   |
| 1 Cnj | $\neg A \wedge \neg B$ |     |                        |                   |



c.



The following divide the first and second open gap, respectively:

| A | B | $\neg A \vee \neg B$ | ⊕ | ⊖ | T |
|---|---|----------------------|---|---|---|
| F | T | T                    | ⊕ | F | ⊖ |
| T | F | F                    | ⊖ | T | ⊕ |