3.4. Counterexamples to *reductio*s

3.4.0. Overview

All derivations that fail will now end in the failure of a *reductio*, and this produces some small changes in what we say about the failure of derivations.

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3.4.1. When *reductio*s fail

The system of derivations for negation can be shown to be adequate by establishing the three properties of sufficiency, conservativeness, and decisiveness discussed in 2.3. Here we will look at these properties in the case of the current system. To say that the system is conservative is to say that all its rules are sound and safe. Soundness and safety say more than do the basic laws of negation, but the natural way of establishing the basic laws for negation is enough to establish soundness and safety. The key to the argument in both cases is the fact that, when it comes to dividing a gap, having a given sentence (φ or ¬ φ) as a resource comes to the same thing as having a contradictory sentence (¬ φ or φ, respectively) as a goal. However, there is more to be said in the case of the properties of sufficiency and decisiveness.

The system is sufficient if it has enough rules to close any dead-end gaps that cannot be divided. Given the rules of the current system, a dead-end open gap must have ⊥ as its goal (since otherwise we could develop the gap with Cnj, RAA, or IP or close it with ENV), it cannot have a conjunction or a negated compound as a resource (since otherwise we could develop the gap with Ext or CR), it cannot have ⊥ among its resources (since otherwise we could close the gap using either QED or EFQ), and it cannot have both a sentence and its negation among its resources (since otherwise we could close the gap with Nc). So the proximate argument of a dead-end gap must be a *reductio* whose premises are limited to ⊥, atomic, and negated atomic sentences, with no sentence appearing both negated and unnegated among the premises. We can divide such an argument by making an atomic sentence true when it appears among the premises and false when its negation appears. We can assign truth values in this way since no sentence appears both negated and unnegated, such an assignment will make all premises true, and the conclusion is bound to be false since it is ⊥.

This argument for sufficiency tells us what we need to do in order to present a counterexample on the basis of a dead-end open gap. Here is an example of that.

	$\neg (A \wedge \neg (B \wedge C))$	2	
	$\neg B$	(7)	
	$\neg A$		
	$\neg A$		
	$\circ$		$\neg A, \neg B \not\Rightarrow \bot$
	$\bot$	4	
4 IP	A	3	
	$B \wedge C$	6	
6 Ext	B	(7)	
6 Ext	C		
	$\bullet$		
7 Nc	$\bot$	5	
5 RAA	$\neg (B \wedge C)$	3	
3 Cnj	$A \wedge \neg (B \wedge C)$	2	
2 CR	$\bot$	1	
1 IP	A		

A B C	$\neg (A \wedge \neg (B \wedge C))$, $\neg B$ / A
F F T	T F T F T F

(Although this derivation has been continued as far as possible, it could have been ended after the dead-end gap appeared at stage 4.)

The proximate argument of the dead-end gap is $\neg A, \neg B / \bot$. To divide this, we must make A and B false since their negations are active resources of the dead-end gap. The value assigned to C does not matter since neither it nor $\neg C$ appears among the premises of this argument. So, although C is assigned T in the counterexample presented above, an interpretation that made each of A, B, and C false would also be a counterexample.

Finally, recall that a system is decisive if we cannot go on applying its rules forever—i.e., if we must always reach a point at which no more rules can be applied. We argued that the system of derivations of the last chapter was decisive because we could not go on forever dropping and shortening sentences among the resources and goals. But we now have rules that can do things other than simplifying the resources and goals—in particular, adding resources while dropping goals and vice verse and, in the case of IP, doing this by adding a resource more complex than the goal that was dropped. The cases where we use IP and CR were restricted so that we would not go in circles, but some argument is needed to show that those restrictions were enough.

Decisiveness will follow if all our rules are progressive on some measure of distance from the end of a derivation. We cannot measure this simply by the length of the goal and resources of a gap since, in the first place, atomic sentences are as short as possible, but are a sign that a derivation has not reached its end when they appear as goals. And, on the other hand, negated atomic sentences are not as short as possible but can appear as resources at the end of a derivation. Let us say that the sort of sentences that may appear in a gap that cannot be narrowed further are *minimal*; that is, a minimal resources will be $\top$ or an atomic or negated atomic sentence and a minimal goal must be $\bot$. Notice that whether a given sentence counts as minimal depends on whether it appears as a resource or a goal.

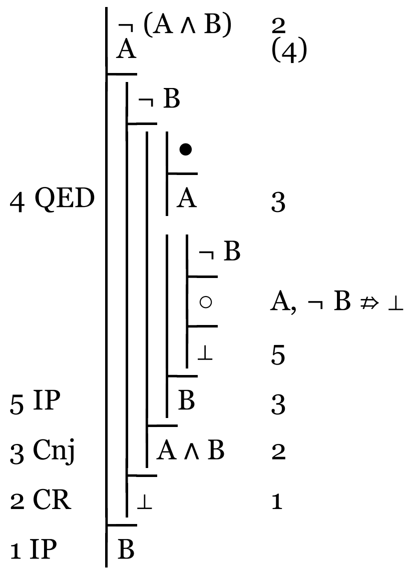
In order to measure distance from the end of a derivation, we will assign each resource and goal a *grade*. Minimal sentences form the lowest grade, and non-minimal sentences are graded according to their length. Now, consider the whole group of active resources and goals of every open gap of a derivation. If we look at each of the rules for narrowing gaps, we see that the effect of applying any one of them will always be to eliminate an active resource or a goal. It may also add resources or as goals, but any sentence that is added either is shorter than the sentence dropped or, in the case of IP, is a minimal sentence when the sentence dropped was not minimal. Either way, additions will be sentences of a lower grade, so eventually all active sentences will be minimal and the process must end. Notice that if, for example, we allowed CR to apply to negated atomic sentences as well as negated compounds, this would no longer be true since we would drop a minimal resource $\neg \varphi$ and add a non-minimal goal φ. However, when φ is compound, $\neg \varphi$ has a higher grade than φ because it is longer, so the change can be seen as progressive.

Glen Helman 28 Aug 2008

3.4.2. Some examples of consistency

The aim of this subsection is to consider a few examples, but its title makes a further general point. An interpretation that divides a dead-end open gap will divide a *reductio* argument and thus show that its premises can all be true together. That is, it will show that the active resources of a dead-end open gap form a consistent set. Counterexamples to arguments in chapter 2 did that, too, since they made all resources of the gap they divided true, but now that is the full significance of a counterexample since the goal of the gap it divides is ⊥ and is therefore automatically false.

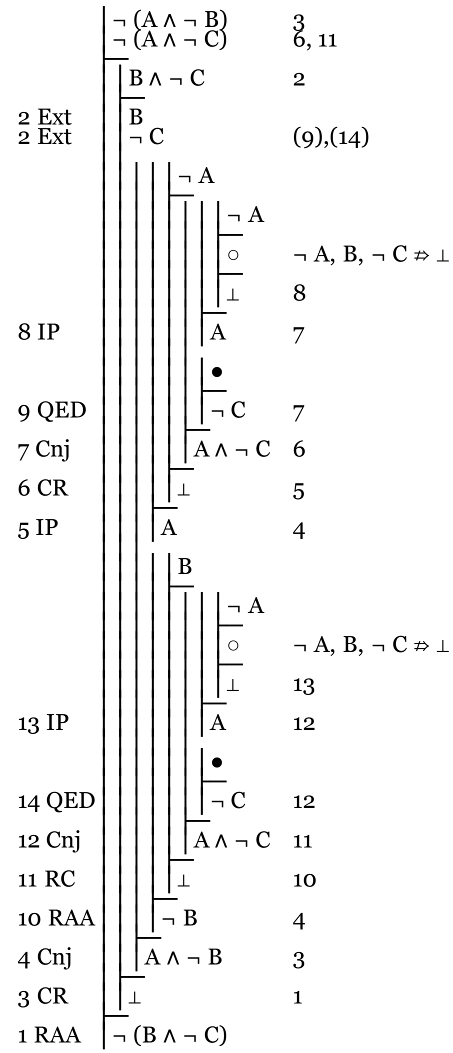
Here is a simple example that exhibits a common pattern.

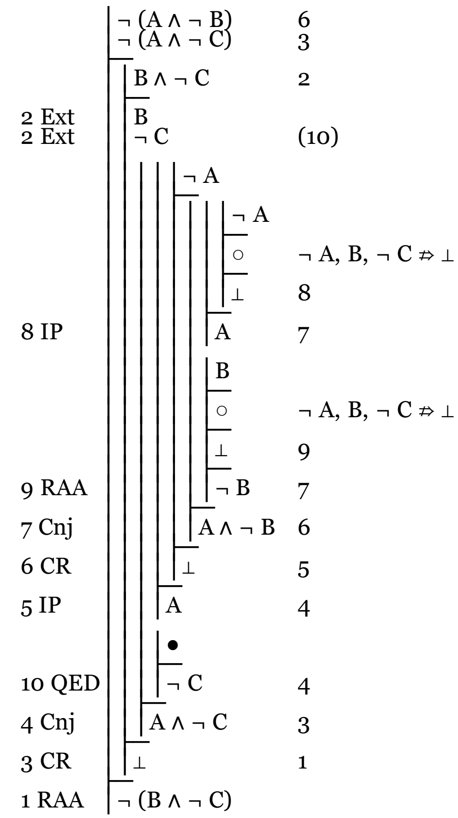


A	B	¬ (A ∧ B) ,	A	/ B
T	F	T F	T	F

It may seem odd to continue to stage 5 since, before IP is applied, the resources of the second gap are fully exploited and its goal is not among them. So, in this case, it is clear before stage 5 that the gap will not close. But, with enough thought, it would have been clear before stage 1 that some gap would not close so the simple fact that a dead-end gap can be foreseen is not grounds for declaring one. A dead-end gap is an indication of failure made fully explicit. What we count as fully explicit is a conventional matter, and we will treat as fully explicit only what cannot be made more explicit by the system of derivations. In this case, that requires the final use of IP (though the closure of the first gap at stage 4 might have been ignored).

Here is a somewhat longer example. The derivation on the left represents the most straightforward approach, in which resources are exploited in the order in which they appear when there is a choice while the derivation at the right exploits the premises in the opposite order.





A	B	C	¬ (A ∧ ¬ B) ,	¬ (A ∧ ¬ C)	/ ¬ (B ∧ ¬ C)
F	T	F	T F F	T F T	F T T

Although the full on the right derivation is significantly shorter than the one on the left, neither derivation needed to continue beyond stage 8, so the difference in this case was unimportant. However, in other cases, the order in which premises are exploited could matter more for length (though it will never matter for the final outcome). Notice also that the *reductio* we attempt to complete at stage 8 has the same supposition as the one we attempted at stage 5 and has fewer active resources (since another premise was exploited at a stage between the two). This is a consequence of the fact that the requirements for the second premise to be true are already met by other active resources when it is exploited at stage 6, so the exploitation ends up adding nothing to the resources. The repetition in the derivation is just the way this repetition in the content of the resources is made explicit.

Glen Helman 30 Sep 2008

3.4.s. Summary

1 The adequacy of our current system is established by showing that it is sufficient, conservative, and decisive. The arguments for sufficiency and decisiveness take a slightly different form from those used in the last chapter. A gap that remains open at a dead end will now always have $\bot$ as its goal and its resources are limited to $\top$, atomic sentences, and negated atomic sentences, with no resource being the negation of another. Any such gap can be divided by an interpretation that makes all its active resources true, so the rules are sufficient to close any gap that cannot be divided. Also, we can show that our new rules will not lead us on forever by showing that they are progressive by leading us always to replace goals or resources by others of a lower grade eventually leading us to goals and resources that are minimal, a class that includes $\top$, atomic sentences and negated atomic sentences in the case of resources and $\bot$ alone in the case of goals.

2 Dead-end gaps will now have proximate arguments that are *reductio*s, so the failure of a derivation will turn on the failure of a *reductio* and thus on the fact that the premises of the *reductio* form a consistent set. Thus any example of the failure of entailment will henceforth be due to the consistency of some set.

Glen Helman 28 Aug 2008

3.4.x. Exercise questions

1. The following arguments are not formally valid. In each case, use a derivation to show this and present a counterexample that the derivation leads you to.

- **a.** ¬ B / ¬ (A ∧ ¬ B)
- **b.** ¬ (A ∧ B) / ¬ A ∧ ¬ B
- **c.** ¬ (A ∧ B), ¬ (B ∧ C) / ¬ (A ∧ C)

2. Use derivations to check the following claims of entailment. If the claim fails, present a counterexample that the derivation leads you to.

- **a.** ¬ (A ∧ ¬ B) ⇒ B
- **b.** ¬ (A ∧ B) ⇒ ¬ (B ∧ A)
- **c.** ¬ (A ∧ ¬ B) ⇒ ¬ (B ∧ ¬ A)
- **d.** ¬ (A ∧ B), ¬ (B ∧ C), B ⇒ ¬ A ∧ ¬ C
- **e.** ¬ (A ∧ ¬ (B ∧ ¬ (C ∧ ¬ D))) ⇒ ¬ (A ∧ ¬ (B ∧ D))

For more exercises, use the exercise machine.

Glen Helman 28 Aug 2008

3.4.xa. Exercise answers

1. a.

	¬ B	
	A ∧ ¬ B	2
2 Ext	A	
2 Ext	¬ B	
	○	A, ¬ B ⇏ ⊥
	⊥	1
1 RAA	¬ (A ∧ ¬ B)	

A	B	¬	B	/	¬	(A	∧	¬	B)
T	F	T			F	T	T		

b.

	¬ (A ∧ B)	3,8
	A	(5)
	•	
5 QED	A	4
	¬ B	
	○	A, ¬ B ⇏ ⊥
	⊥	6
6 IP	B	4
4 Cnj	A ∧ B	3
3 CR	⊥	2
2 RAA	¬ A	1
	B	(11)
	¬ A	
	○	¬ A, B ⇏ ⊥
	⊥	10
10 IP	A	9
	•	
11 QED	B	9
9 Cnj	A ∧ B	8
8 CR	⊥	7
7 RAA	¬ B	1
1 Cnj	¬ A ∧ ¬ B	

A	B	¬	(A	∧	B)	/	¬	A	∧	¬	B
T	F	T		F			F		F	T	
F	T	T		F			T		F	F	

The first row is an interpretation that divides the first gap; and the second row is an interpretation that divides the second gap.

c.

```
           | ¬ (A ∧ B)                      3
           | ¬ (B ∧ C)                      7
           |------
           | | A ∧ C                        2
           | |------
2 Ext      | | A
2 Ext      | | C                            (10)
           | | | | •
           | | | |------
5 QED      | | | | A                        4
           | | | | | ¬ B
           | | | | |------
           | | | | | | | | ¬ B
           | | | | | | | |------
           | | | | | | | | ○                A, ¬ B, C ⇏ ⊥
           | | | | | | | |------
           | | | | | | | | ⊥                9
           | | | | | | |------
9 IP       | | | | | | | B                  8
           | | | | | | | •
           | | | | | | |------
10 QED     | | | | | | | C                  8
           | | | | | |------
8 Cnj      | | | | | | B ∧ C              7
           | | | | |------
7 CR       | | | | | ⊥                    6
           | | | |------
6 IP       | | | | B                      4
           | | |------
4 Cnj      | | | A ∧ B                    3
           | |------
3 CR       | | ⊥                          1
           |------
1 RAA      | ¬ (A ∧ C)
```

A B C	¬ (A ∧ B) , ¬ (B ∧ C) / ¬ (A ∧ C)
T F T	Ⓣ F Ⓣ F Ⓕ T

2. a.

```
           | ¬ (A ∧ ¬ B)                    2
           |------
           | | ¬ B                          (5)
           | |------
           | | | | | ¬ A
           | | | | |------
           | | | | | ○                      ¬ A, ¬ B ⇏ ⊥
           | | | | |------
           | | | | | ⊥                      4
           | | | |------
4 IP       | | | | A                        3
           | | | | •
           | | | |------
5 QED      | | | | ¬ B                      3
           | | |------
3 Cnj      | | | A ∧ ¬ B                  2
           | |------
2 CR       | | ⊥                          1
           |------
1 RAA      | B
```

A B	¬ (A ∧ ¬ B) / B
F F	Ⓣ F T Ⓕ

b.

```
           | ¬ (A ∧ B)                      3
           |------
           | | B ∧ A                        2
           | |------
2 Ext      | | B                            (6)
2 Ext      | | A                            (5)
           | | | | •
           | | | |------
5 QED      | | | | A                        4
           | | | | •
           | | | |------
6 QED      | | | | B                        4
           | | |------
4 Cnj      | | | A ∧ B                    3
           | |------
3 CR       | | ⊥                          1
           |------
1 RAA      | ¬ (B ∧ A)
```

c.

```
           | ¬ (A ∧ ¬ B)                    3
           |------
           | | B ∧ ¬ A                      2
           | |------
2 Ext      | | B
2 Ext      | | ¬ A
           | | | | | ¬ A
           | | | | |------
           | | | | | ○                      ¬ A, B ⇏ ⊥
           | | | | |------
           | | | | | ⊥                      5
           | | | |------
5 IP       | | | | A                        4
           | | | | | B
           | | | | |------
           | | | | | ○                      ¬ A, B ⇏ ⊥
           | | | | |------
           | | | | | ⊥                      6
           | | | |------
6 IP       | | | | ¬ B                      4
           | | |------
4 Cnj      | | | A ∧ ¬ B                  3
           | |------
3 CR       | | ⊥                          1
           |------
1 RAA      | ¬ (B ∧ ¬ A)
```

A B	¬ (A ∧ ¬ B) / ¬ (B ∧ ¬ A)
F T	Ⓣ F F Ⓕ T T

d.

	$\neg (A \wedge B)$	3,8
	$\neg (B \wedge C)$	11
	B	(6),(10),(14)
	A	(5)
	$\bullet$	
5 QED	A	4
	$\bullet$	
6 QED	B	4
4 Cnj	$A \wedge B$	3
3 CR	$\bot$	2
2 RAA	$\neg A$	1
	C	(15)
	$\neg A$	
	$\bullet$	
14 QED	B	13
	$\bullet$	
15 QED	C	13
13 Cnj	$B \wedge C$	12
12 CR	$\bot$	11
11 IP	A	9
	$\bullet$	
10 QED	B	9
9 Cnj	$A \wedge B$	8
8 CR	$\bot$	7
7 RAA	$\neg C$	1
1 Cnj	$\neg A \wedge \neg C$	

	$\neg (A \wedge B)$	3
	$\neg (B \wedge C)$	8
	B	(6),(10)
	A	(5)
	$\bullet$	
5 QED	A	4
	$\bullet$	
6 QED	B	4
4 Cnj	$A \wedge B$	3
3 CR	$\bot$	2
2 RAA	$\neg A$	1
	C	(11)
	$\bullet$	
10 QED	B	9
	$\bullet$	
11 QED	C	9
9 Cnj	$B \wedge C$	8
8 CR	$\bot$	7
7 RAA	$\neg C$	1
1 Cnj	$\neg A \wedge \neg C$	

The derivation on the left exploits resources in their order of appearance; while the one above chooses, at stage 8, the resource that is most closely connected with other resources of the gap in which it is exploited. Notice that derivation on the left is eventually led to exploit the same resource to the same effect.

e.

	$\neg (A \wedge \neg (B \wedge \neg (C \wedge \neg D)))$	3
	$A \wedge \neg (B \wedge D)$	2
2 Ext	A	(5)
2 Ext	$\neg (B \wedge D)$	8
	$\bullet$	
5 QED	A	4
	$B \wedge \neg (C \wedge \neg D)$	7
7 Ext	B	(10)
7 Ext	$\neg (C \wedge \neg D)$	12
	$\bullet$	
10 QED	B	9
	$\neg D$	(15)
	$\neg C$	
	$\circ$	$A, B, \neg C, \neg D \nRightarrow \bot$
	$\bot$	14
14 IP	C	13
	$\bullet$	
15 QED	$\neg D$	13
13	$C \wedge \neg D$	12
12 CR	$\bot$	11
11 IP	D	9
9 Cnj	$B \wedge D$	8
8 CR	$\bot$	6
6 RAA	$\neg (B \wedge \neg (C \wedge \neg D))$	4
4 Cnj	$A \wedge \neg (B \wedge \neg (C \wedge \neg D))$	3
3 CR	$\bot$	1
1 RAA	$\neg (A \wedge \neg (B \wedge D))$	

A B C D	$\neg (A \wedge \neg (B \wedge \neg (C \wedge \neg D)))$ / $\neg (A \wedge \neg (B \wedge D))$
T T F F	T F F T T F T F T T F

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