2.3. Failed proofs and counterexamples

2.3.0. Overview

Derivations can also be used to tell when a claim of entailment does not follow from the principles for conjunction.

2.3.1. When enough is enough
: Our system has enough rules that, when no more rules can be applied to an open gap, we know its active resources do not entail its goal.

2.3.2. Sound and safe rules
: These rules are designed so that all gaps will close only if the initial argument is valid and so that we can reach a dead end developing a gap only if the initial argument is invalid.

2.3.3. Presenting counterexamples
: Because we have enough rules and the ones we have are well-behaved, any gap that reaches a dead end provides the basis for a table showing the argument for which we constructed the derivation can have true premises and a false conclusion.

2.3.4. Reaching decisions
: A derivation will always reach a point where we must stop either because all gaps are closed or because there is an open gap to which no more rules can be applied.

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2.3.1. When enough is enough

So far we have seen only derivations whose gaps all close, derivations which show that arguments are valid. But not all arguments are valid so, unless there is some problem with our system, there must be derivations whose gaps do not all close. If the gaps of a derivation will never all close, we will eventually have to give up work on it even though it still has open gaps. So we should ask what might lead us to give up work and what, if anything we can conclude if we do so.

We answered both questions in a preliminary way in 2.2.1 when considering tree-form proofs. We return to them now in order to consider the case of derivations more explicitly and to establish a framework for asking the same questions in later chapters. One byproduct of this discussion will be some ways of thinking about rules that will be useful when we consider some optional extra rules for derivations in the next section.

The short answer to the first of the two questions is that we must give up on a derivation when we run out of rules to apply, either to develop a gap or close it. Here's a simple example of a derivation for which that has happened.

	$(A \wedge \top) \wedge B$	1
1 Ext	$A \wedge \top$	6
1 Ext	B	(4)
	•	
4 QED	B	2
	•	
5 ENV	$\top$	3
6 Ext	A	
6 Ext	$\top$	
	○	$B, A, \top \nRightarrow C$
	C	3
3 Cnj	$\top \wedge C$	2
2 Cnj	$B \wedge (\top \wedge C)$	

The gap that is marked with a *white circle* ○ has C as its goal, and we currently have no rule to plan for such a goal. There are conjunctions among the available resources of the gap; but they were exploited in the course of developing this gap, so they are no longer active. Finally, since the only active resources of the gap are B, A, and ⊤ and its goal is not ⊤ nor among the resources, we have no rule for closing the gap. In short, no rule of any of the three sorts can be applied at this point. Notice that, while A ∧ ⊤ might have been exploited at any point after stage 1, its components were not needed to close the other gaps. As a result, its exploitation can be postponed until stage 6. However, even though its components do not enable us to close the open gap, it must be exploited before we end work on the derivation. It is only after it has been exploited that there is no rule for developing the gap further.

We will describe an open gap to which no more rules apply as a *dead-end gap*. (Although the qualification dead-end will be reserved for open gaps—indeed, a gap that has been closed is in one sense no longer a gap—we will often speak somewhat redundantly of "dead-end open gaps.") In these terms, we can say that we are forced to abandon a derivation when every open gap has reached a dead end. When we consider the significance of dead-end open gaps, we will see that we *may* abandon a derivation as soon as one open gap has reached a dead-end. As in the example above, we will use the white circle to mark open gaps that have reached a dead end. And, also as is done in that example, we will write the sign ⇏ (*rightwards double arrow with stroke*) between the active resources and the goal. This indicates, roughly, that the active resources do not entail the goal; but its precise significance is discussed more fully below.

Now, let's look more closely at what we can say in general about dead-end open gaps. First of all, such a gap must not have a conjunction either as its goal or among its active resources, for otherwise we could apply the rules Cnj or Ext. Moreover, it must not have ⊤ as a goal or ⊥ as a resource, or else we could apply the rules ENV or EFQ. Finally, its goal must not be among its resources because then we could apply the rule QED. So the active resources of dead-end gaps are limited to unanalyzed components and ⊤ and their goals are limited to unanalyzed components and ⊥; and no dead-end gap can contain an unanalyzed component both as an active resource and as its goal. And this means that we can assign truth-values to the unanalyzed components appearing in this gap in a way that makes its active resources true and its goal false. Since no unanalyzed component appears both as a resource and as the goal, we can make any that appears as a resource **T** and any that appears as the goal **F**. While we are not free to assign values to ⊤ and ⊥, the first can appear only as a resource and the second only as the goal so they will not interfere with having true resources and a false goal.

Such an assignment of truth values is an extensional interpretation in the sense defined in 2.1.8. In the case of the derivation above, an interpretation making the active resource of the dead-end gap true and its goal false is displayed in the table below.

A	B	C	B,	A,	⊤	/	C
T	T	F	Ⓣ	Ⓣ	Ⓣ		Ⓕ

The extensional interpretation of unanalyzed components appears on the left of the table. On the right are the resulting truth values of resources and goals (which mainly just repeat the assignments).

We will extend the use of the term divide that was introduced in 1.4.2 to describe what this sort of interpretation does. We will say that an extensional interpretation like this *divides* the active resources of a gap from its goal; and, when it does this, we will say that it divides the gap.

This terminology was originally introduced for arguments; and, in applying it here, we are thinking of the resources and goal of a gap as forming an argument. However, this is not the argument for which the derivation was originally constructed. From one point of view, the function of a derivation is to transform the question whether an argument is valid into an analogous question about one or more simpler arguments. The argument formed from the active resources and goals of a dead-end open gap is the end of the line in this process. We will call the argument for which the derivation was originally constructed the *ultimate argument* of the derivation. When working on a particular gap, we are most immediately trying to show that the active resources of the gap entail its goal, so we are trying to show that the argument with these resources as premises and the goal as its conclusion is a valid one. We will call this argument the *proximate argument* of the gap. The proximate argument of a gap is "nearby" in the sense of being our immediate concern while our final goal is to decide whether the ultimate argument is valid. Notice that the ultimate argument of a derivation is the proximate argument of its initial gap.

We will refer to the extensional interpretation which divides the gap as a *counterexample* to the proximate argument of the gap. And, in writing B, A, ⊤ ⇏ C to the right of the gap, we say that the proximate argument of the gap is not valid. However, these references to counterexamples and invalidity require some qualification. In the context of derivations as in the context of analyses, Roman capital letters are used to stand for particular sentences that are not analyzed further and, in principle, such sentences need not be logically independent. That means that a given extensional interpretation of such sentences need not be realized in any possible world. So in the example above, it might be that the sentences A and B do together entail C and it could even be that C is tautology or that A or B is absurd. In short, knowing that there is an extensional interpretation of analyzed sentences that makes certain ones of them true and others false does not show that it is logically possible for the sentences to have these truth values.

On the other hand, our interest in derivations and tree-form proofs is as a way of applying general principles of entailment. And, even though these principles are applied to particular sentences, their application depends only on the features of these sentences that are displayed in the analysis of them that is shown by the symbolic notation. In particular, the use of rules does not depend on the specific identity of unanalyzed components. This means that when the gaps of a derivation all close we know not only that its premises entail its conclusion but also that the same is true for any argument having the same form. One way of putting this is to say that the argument is *formally valid* or, more precisely, is valid in virtue of the form exhibited in its analysis. The idea of validity in virtue of form can itself be spelled out by saying that an argument is formally valid with respect to a given analysis when any way of associating sentences with its unanalyzed components produces a valid argument. This sort of association of sentences with unanalyzed components is an intensional interpretation as defined in 2.1.8, so we can say that an analyzed argument is formally valid when every intensional interpretation of it is valid. We usually will not know the identity of the unanalyzed components of a symbolic argument, so formal validity is all that we will be in a position to judge; and we will often drop the qualification formal.

When a derivation fails, what we know, speaking most strictly, is that it's ultimate argument is not formally valid. That is because one test of formal validity is whether there is an extensional interpretation of the argument that divides its premises from it conclusion. If there is such a dividing interpretation, we can construct an intensional interpretation by assigning to each component an actual sentence with the truth assigned by the extensional interpretation, and this will yield an actual argument having the same form as the original one but with actually true premises and an actually true conclusion. In example above, we might associate sentences with unanalyzed components as follows:

A: Atlanta is in Georgia
B: Boston is in Massachusetts
C: Chicago is in Indiana

If so, we will have the invalid argument

$$\frac{\begin{array}{c}\text{Boston is in Massachusetts}\\ \text{Atlanta is in Georgia}\\ \top\end{array}}{\text{Chicago is in Indiana}}$$

which has a false conclusion along with true premises not merely in *some* possible world but, indeed, in the actual world. And, because this argument is invalid and has the same form as the proximate argument of the gap, the latter argument is not valid with respect to the form displayed in its analysis. This sort of thing will work with any example, so we know that, if the premises of an argument are divided from its conclusion by an extensional interpretation, the argument is not formally valid.

It is also true that, if an argument is not formally valid, its premises are divided from its conclusion by an extensional interpretation. A claim that an argument is formally valid is a generalization about both intensional interpretations and possible worlds; and a counterexample to this generalization is provided an intensional interpretation and possible world with the property that the actual argument that results from the intensional interpretation is divided by the possible world. But any intensional interpretation and possible world will determine an assignment of truth values to the unanalyzed components of the argument. In the example above the value **T** is assigned to the unanalyzed component A by associating the sentence Atlanta is in Georgia with A and considering the truth value of this sentence in the actual world. So any intensional interpretation and possible world will determine an extensional interpretation, and any counterexample to the formal validity of a symbolic argument will provide an extensional interpretation that divides its premises from its conclusion.

So we will have an extensional interpretation dividing the premises and conclusion of an argument if and only if we have a counterexample to its formal validity. That means we can take formal validity to be a generalization about extensional interpretations: an argument is formally valid if and only if its conclusion is true under every extensional interpretation that makes its premises all true. This means that an extensional interpretation that divides the premises of an argument from its conclusion amounts to a counterexample to formal validity.

We saw earlier that any dead-end open gap provides us with this sort of counterexample to formal validity. And that tells us that our system of derivations has enough rules, for it tells us that we are able to develop or close a gap whenever its proximate argument is valid. And, if the proximate argument is not valid, we would not expect to move further towards the completion of a proof. We will indicate this sort of completeness by saying that a system of derivations is *sufficient* when every dead-end open gap is divided by some extensional interpretation. Of course, in saying that system is sufficient, we do not say that every gap whose proximate is invalid has already reached a dead end. We would not expect this to be true since it would mean that we would never need to apply any rules at all in the case of an invalid argument.

Sufficiency is important, but there are further properties we might expect to hold of a good system of derivations. For example, we know that the proximate argument of a dead-end open gap is not valid; but that does not by itself show that the ultimate argument of a derivation with a dead-end gap will always be invalid, and testing the validity of the ultimate argument is the reason we construct the derivation. Moreover, sufficiency does not imply that we will even eventually reach a point where either all gaps close or there is a dead-end open gap; that is, a sufficient system might lead us to derivations that develop forever. In the next two subsections, we will see that our current system of derivations is well-behaved in both these respects.

2.3.2. Sound and safe rules

We have rules to close gaps only in cases where the argument associated with the gap is valid, and we have seen that the argument associated with a dead-end open gap is not (formally) valid. But what does this imply concerning the ultimate argument of a derivation, the one for which it was initially constructed? Ideally, the ultimate argument should be valid if all gaps eventually close and invalid if at least one gap reaches a dead end without closing. And, indeed, this is the case because of the connections between the rules for developing a derivations and the principles of entailment.

We will look in a little more detail at this connection and its consequences. In doing so, it will help to have some ways of talking about the relations between gaps at various stages of the development of a derivation. We can think of gaps as forming a tree that grows to the right and branches when a rule like Cnj leads us to develop a gap by dividing it into two or more new gaps. We will use the metaphor of a family tree and say that any gap that results from applying a rule is a *child* of the gap to which the rule is applied and that the latter gap is its *parent*. It will be convenient to apply the same terminology to gaps that continue unchanged while others develop: a gap at one stage that is open but unchanged at the next stage is understood to have a single child. Looking farther up or down a line of descent, we will say that some gaps are *ancestors* or *descendants* of others. In this terminology, the initial gap of a derivation is an ancestor of all gaps of all gaps at each later stage in its development; and they are all its descendants. Only open gaps will be part of these genealogies, so a gap that is closed at the next stage of its development has no children. Dead-end open gaps continue to have children if the derivation is continued at later stages (remember it need not be); they have reached a dead end in the sense that these children are always identical to their parents.

If we look at the relation between a gap to which the rule Cnj is applied and the children that result from applying it, we see that the law for conjunction as a conclusion tells that the proximate argument of the parent is valid if and only if the proximate arguments of both children are valid. And something analogous holds for the rule Ext and the law for conjunction as a premise. We can say something similar about rules that close gaps provided we understand a claim about each child of a gap that has no children to be true simply because there are no child to serve as a counterexample. That is, a gap to which ENV or EFQ applies has a valid proximate argument if and only if each of its children does because the gap to which the rule is applied has a valid proximate argument and it has no children. The same is true for QED when it is used to close a gap whose goal is among its active resources. We allow QED to be used also to close gaps whose goals are among their available but inactive resources, so a little more argument is needed in its case; but we will consider that later. For now, we will assume that QED is applied only in cases where the goal is among the active resources; and, in these cases, the law for premises tells us the proximate argument is valid. Finally, in the case of open gaps that remain unchanged as rules are applied elsewhere the proximate argument of the parent is the same as the proximate argument of the child so certainly one of these arguments is valid if and only if the other is.

Putting this all together, we see that the ultimate argument of a derivation is valid if and only if, at every stage of its development, every one of its descendants has a valid proximate argument. And two things follow from this. If there is any stage when an argument has no descendants—that is, any stage when all gaps have closed—we can say for sure that each of its descendants has a valid argument—because there is none that does not. So, if all gaps of a derivation close, we can be sure that the ultimate argument of the derivation is valid. On the other hand, if a dead-end open gap appears, the initial gap has a descendant whose proximate argument is not valid, and its own proximate argument is therefore invalid. So, if a dead-end open gap appears, the ultimate argument of a derivation is invalid. That is, we have shown both that the ultimate argument of a derivation is valid if all gaps close and that it is invalid if there is at least one open gap.

Now, an argument is valid (and formal validity is what is in question here) if and only if there is no extensional interpretation that divides its premises from its conclusion. So principles that tie the validity of proximate arguments at some stages in the development of a derivation to the validity of proximate arguments at other stages at the same time tie the existence of dividing interpretations at different stages. In fact, we can state stronger principles that say not merely that the existence and non-existence of dividing interpretations is preserved as we develop a derivation but indeed that any dividing interpretations are themselves preserved.

R is *(utterly) sound*	*when*	an (extensional) interpretation divides a gap to which the rule R is applied *only if* it divides some child of the gap
R is *safe*	*when*	an (extensional) interpretation divides a gap to which the rule R is applied *if* it divides some child of the gap

When a rule is utterly sound we never lose any open-gap-dividing interpretations as we apply the rule and, when it is safe, we never gain any. The reason for the qualification utterly will be discussed later, and we will suppress its use in the meantime.

These two properties do not have the same significance. If any rule were unsound, all gaps of a derivation might close even though the original argument was invalid. This would undermine the central function of proofs: to establish validity. An unsafe rule would analogously undermine the use of derivations to establish invalidity because it would introduce the possibility that a derivation for a valid argument could produce a dead-end open gap. But the role of derivations in establishing invalidity is less central, and its full use depends also on a property (discussed in 2.3.4) that will fail for the systems of the last two chapters. So soundness is more fundamental than safety.

Moreover, moves corresponding to unsafe rules are an important part of explicit deductive reasoning. For example, a natural approach when we seek a way to prove a mathematical result is to introduce a lemma (in the sense is discussed in 1.4.7) as a stepping stone to a final result. If the lemma represents a significant step beyond the premises, it may be no more obviously a valid conclusion from the premises than is the final conclusion we hope to establish. The introduction of such a lemma can be described as a conjecture, and this conjecture may be wrong: the lemma may not be a valid conclusion from our premises even when the final conclusion is valid. In short, by seeking to reach our conclusion by way of this lemma, we may be entering a blind alley. This is just the sort of thing that would appear in the context of derivations as a dead-end open gap in a derivation whose initial argument is valid. Conjecturing a lemma can be thought of as a step in discovering a proof that is valuable but unsafe.

Our interest in deductive reasoning is somewhat different from a mathematicians'. We are not aiming not at new and surprising conclusions but instead at fuller understanding of the steps by which deductive conclusions are reached. Consequently, we will not be considering the large deductive steps for which conjecturing lemmas is the only practical approach. We will make use of lemmas—and we will look at rules for doing so in 2.4—but the chief value of lemmas for us lies in a restricted range of cases where we can be sure that they are safe.

Earlier, we set aside uses of QED in which the goal of the gap we close is among the available resources of the gap but not among the active ones. To discuss such uses of QED, we need to consider the property of soundness more closely. The reason for the qualification utter used earlier lies in the difference between the property stated above and the following property:

R is *(minimally) sound*	*when*	an (extensional) interpretation divides a gap to which the rule R is applied and all ancestors of this gap *only if* it divides some child of the gap

The difference lies in the added phrase and all ancestors of this gap. The addition makes minimal soundness apparently weaker than utter soundness because, for minimal soundness, we do not ask that an interpretation divide a child gap unless it divides not only the parent gap but also all ancestors. One reason for parenthesizing the qualifications utterly and minimally in the names of the two properties is that, when all rules are safe, a rule that is minimally sound is also utterly sound. For, when all rules are safe, an interpretation that divides a gap will also divide all its ancestors. When there is a difference between the two sorts of soundness, it lies in their handling of the spurious dividing interpretations introduced by unsafe rules: with an utterly sound rule, such interpretations will continue to divide descendants while, with a minimally sound rule, they might not.

And the reason for calling the second property *minimal* soundness is that, even when not all rules are safe, minimal soundness is enough to insure that the ultimate argument of a derivation is valid whenever all gaps close. For if all rules are minimally sound, we can be sure that any interpretation that divides a gap and all its ancestors will divide some child and all ancestors of this child (since these are just the parent and its ancestors). But any interpretation that divides the ultimate argument of a derivation also divides any ancestor (since it has none), so, if all rules are minimally sound, this interpretation will also divide some child and all its ancestors—and so on. That is, as with utter soundness, when all rules are minimally sound, an interpretation that divides the ultimate argument must divide some descendant at each stage; therefore, if all gaps close, there can be no interpretation dividing the ultimate argument.

Now, for a rule that closes gaps to be minimally sound, it is enough that is closes a gap only when there is no extensional interpretation that makes the goal of the gap false while making its active resources and the active resources of all its ancestors true. That is, for a gap-closing rule to be minimally sound, it is enough that there be no interpretation that makes the goal of the gap false while making all active resources of the gap *and all active resources of its ancestors* true. This means that it is enough that goal of the gap being closed to be entailed by its active resources together the active resources of its ancestors. With the rules we have so far, all available resources are included among the active resources of a gap and its ancestors, so it is enough goal is among its available resources. But we can be even more generous since, by the law for lemmas, adding to a collection of resources something that is entailed by them will not change what they entail. In short, we can state rules for closing gaps and have them minimally sound if the conclusion of the gap is among its active resources, is among the active resources of its ancestors, or is a further resource entailed by these resources. The available resources of a gap always include its active resources and the active resources of its ancestors, but in 2.4.3 we will consider rules which add to the available resources conclusions that they entail. We have just seen that this sort of addition will not undermine the minimal soundness of QED.

Although we will sometimes need to distinguish soundness and safety (or even utter and minimal soundness) in later discussions, most often we will not. We will say that a system is *conservative* when its rules are all safe and minimally sound (which comes to the same thing as being all safe and utterly sound). As we develop a derivation in a conservative system, open-gap-dividing interpretations are neither gained nor lost though they may be spread out among an increasing number of descendant gaps.

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2.3.3. Presenting counterexamples

A dead-end open gap is always divided by an interpretation that also divides the ultimate argument of the derivation, and we will complete derivations that uncover invalidity by displaying this division. We will do that by exhibiting the interpretation that divides a dead-end open gap and calculating the truth values of the original premises and conclusions on that interpretation. In the example discussed in 2.3.1, this calculation is shown in the following table:

A B C	(A∧⊤) ∧ B / B ∧ (⊤∧C)
T T F	T T Ⓣ Ⓕ T F

Here the values of unanalyzed components have not been repeated on the right, but they are used to calculate the values of compounds containing them, with the order of calculation being guided by parentheses. In performing this calculation we are confirming that the interpretation dividing the gap really does constitute a counterexample to the ultimate argument; and we will say that, in constructing the table, we are *presenting a counterexample*. It will be our standard way of concluding the treatment of an argument whose derivation fails.

It is not always the case that the unanalyzed components of the ultimate argument all appear among the resources and goal of a dead-end gap. When unanalyzed components do not appear there, values must still be assigned to them in order for a truth value to be defined for each sentence in the ultimate argument; but it will not matter what value we assign to these further unanalyzed components. If an interpretation divides the gap, any way we choose to extend it to unanalyzed components not appearing in the gap's proximate argument will still divide that gap and therefore divide the ultimate argument.

The example below is designed to illustrate this. Of the three interpretations shown, the first divides only the first dead-end gap (since it assigns the value **T** to the goal of the second dead-end gap), and the last divides only the second open gap (for a similar reason); but the middle one divides both open gaps. With 4 unanalyzed components, there are 2×2×2×2 = 2^4 = 16 possible interpretations, so there are 13 interpretations that do not divide either gap. The soundness and safety of our rules insures that the 3 interpretations shown above constitute counterexamples to the ultimate argument and that the other 13 do not.

	│A ∧ B	1
1 Ext	│A	
1 Ext	│B	(4)
	││○	A, B ⇏ C
	││C	2
	│││●	
4 QED	│││B	3
	│││○	A, B ⇏ D
	│││D	3
3 Cnj	││B ∧ D	2
2 Cnj	│C ∧ (B ∧ D)	

A B C D	A ∧ B / C ∧ (B ∧ D)	
T T F T	Ⓣ Ⓕ T	divides first dead-end gap
T T F F	Ⓣ Ⓕ F	divides both dead-end gaps
T T T F	Ⓣ Ⓕ F	divides second dead-end gap

Any one of these three interpretations is enough to provide a counterexample. Beginning with chapter 6, it will prove to be most convenient to assign **F** to an unanalyzed component whenever we have a choice, and here that would lead us to the middle interpretation in the case of both gaps. But, for now, when an unanalyzed component does not appear in the proximate argument of a dead-end gap, the choice of the value to assign to it is entirely arbitrary.

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2.3.4. Reaching decisions

We know that, if a system of derivations has individual rules that are both sound and safe and is, as a whole, sufficient, it will never give us an incorrect answer regarding the validity of an argument. But it is entirely possible that such a system will give us no answer at all. If we ever run out of rules to apply, we will have an answer. For, if this happens without all gaps closing, we will have at least one open gap that has reached a dead-end. However, without some guarantee that we will eventually run out of rules, we have no guarantee that we will eventually have an answer. And such a guarantee is not trivial because, once we get to the last two chapters, we will be working in a system some of whose derivations do go on forever.

We will say that a system is *decisive* when we always reach a point where either all gaps are closed or there is a dead-end open gap. It should be clear that our system so far is decisive. The rules Ext and Cnj replace conjunctions among the resources and goals of gap by simpler sentences and must therefore eventually eliminate all conjunctions. At that point the only rules that might apply are QED, ENV, and EFQ, but each of these closes a gap and there will be only a limited number of gaps to close. We will say that a rule is *direct* when it is like one of these —that is, when it closes a gap, replaces a resource by one or more simpler resources, or replaces a goal by one or more simpler goals. All of the rules we have considered so far are direct in this sense.

More broadly, we will say that a rule is *progressive* when it, in some sense, brings us closer to a point where no more rules can be applied. The qualification in some sense is important because many different measures of distance could be used. We might measure distance from the end first of all by the complexity of sentences appearing as resources and goals and, once all resources and goals are of minimum complexity, by the number of open gaps. If we use a measure of this sort, direct rules are progressive.

But there are many measures of this sort, differing in the way they measure complexity; and this is not the only way measuring distance from the end. We would always want direct rules to count as progressive on any measure of distance we use, but some measures will count more rules as progressive. For example, a rule that introduces a sentence more complex than any previously in the derivation will not be direct, but it might still count as progressive if there is a limit on the number of such sentences that can be introduced in this way. For then a rule that introduces such a sentence brings us closer to the end by reducing the number that can be introduced later. We do need to require that, whatever measure of distance is used, there is some minimum reduction of distance that makes a rule progressive; for we must insure that we cannot squeeze in an infinite series of steps by, for example, going halfway to the end, halfway from the point to the end, and so on.

As we saw in the case of our current rules, a system whose rules are each progressive will be decisive because, if applying a rule always reduces our distance from the end (by at least some minimum amount), then we will eventually reach a point where the distance has been reduced so much that no more rules can be applied. At that point, any gap that is left open will have reached a dead end, and the derivation will have provided an answer about the validity of the original system. We have seen also that if a such system is sufficient and conservative, the answer provided is always the correct one. A system that always eventually provides an answer and a correct one, can be said to provide a *decision procedure* for validity.

Our current system is sufficient, conservative, and decisive, and it therefore provides a decision procedure. But we can cut up its properties in another way. Because it is decisive as well as accurate in its answers, we can say both of the following about any derivation:

> The ultimate argument of a derivation is valid if and only if eventually all gaps close.
>
> The ultimate argument of a derivation is invalid if and only if eventually we reach a dead-end open gap.

The if parts of these together say that the system is accurate, and we have seen that they follow from its conservativeness (along with sufficiency in the case of the second statement). The only if parts follow from the if parts given decisiveness.

For example, we can show the only if part of the first by showing that, if gaps do not eventually all close, the derivation's ultimate argument is not valid. Suppose that the gaps never all close; we want to show that in this case the ultimate argument is not valid. Since the system is decisive, if gaps never all close, we must eventually reach a dead-end open gap; and the if part of the second statement then tells us that the argument is invalid. In a similar way, if we suppose that we never eventually reach a dead-end gap, we can show that the argument is not invalid, and this establishes the only if part of the second statement.

Moreover, the only if parts of the two claims above together imply decisiveness since, because an argument will always be either valid or invalid, they imply that eventually either all gaps close or we reach a dead-end gap.

But these two claims, like the properties of soundness and safety, are not of equal importance. The first is closely tied to the use of derivations to establish validity while the second is similarly related to their use to find counterexamples and establish invalidity. The first is of special interest also because it can be established in some cases where decisiveness fails, and we will take it as the key property of our system of derivations in chapters 7 and 8 when we must abandon decisiveness.

It is standard to give different names to the two parts of the first statement:

> The ultimate argument of a derivation is valid *if* eventually all gaps close
>
> The ultimate argument of a derivation is valid *only if* eventually all gaps close

When we can be sure that the *if*-statement is true, we say that the system is *sound*. We have seen that a system will be sound if all its rules are at least minimally sound. When we can be sure that the *only-if*-statement is true, we say the system is *complete* because such a system provides a proof for each valid argument.

We can show that a system is complete if we know that its rules are safe and the system as whole is sufficient and we know also that any derivation whose ultimate argument is valid eventually reaches an end. The latter is not full decisiveness since it applies only to derivations whose ultimate argument is valid, this sort of partial decisiveness is something we will be able to establish for the indecisive systems of chapters 7 and 8. Consequently, all systems that we will study in the course are both sound and complete.

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2.3.s. Summary

1 When a derivation is constructed for an invalid argument, we eventually reach a point where an open gap has reached a dead end without closing. We mark such a gap with a white circle ○ and write its active resources and goal with the sign ⇏ between to indicate that they do not form a valid argument. We call this argument the proximate argument of the gap to distinguish it from the ultimate argument for which the derivation is constructed. The invalidity of the proximate argument may only be formal is the sense that some intensional interpretation of its unanalyzed components—some way of associating actual sentences with them—yields an invalid argument (though others may yield valid ones). A test of formal validity is whether there is an extensional interpretation of unanalyzed components, an assignment of truth values to them, that makes premises true and conclusion false. We will often be concerned with formal validity, so we extend to assignments of truth values the ideas of dividing premises from a conclusion and of constituting a counterexample to an argument. And we speak of a gap being divided when its proximate argument is. The fact that any dead-end open is divided—that its proximate argument has a counterexample—indicates that our system is sufficient in the sense of having enough rules to close all dead-end gaps whose proximate arguments are valid.

2 We can be sure that a counterexample to a proximate argument is a counterexample to the derivation's ultimate argument provided all our rules are safe in the sense of never leading us to try to prove a valid argument by completing a proof of an invalid one. When the converse is true, when we our rules never lead us to develop a gap that can be divided by considering only gaps that cannot be divided, they are utterly sound. Since our real interest is in the ultimate argument of derivation, it is really enough to preserve the division of gaps only when all ancestors of the gap have also been divided; rules that do this are minimally sound; when all rules are safe, minimally sound rules are also utterly sound. The idea of minimal soundness enables us to justify the use of available but inactive resources (to, for example, close gaps) even when not all rules are safe. A system whose rules are all safe and minimally sound is conservative.

3 Since a dead-end open gap is divided by an interpretation is this interpretation is also a counterexample to the ultimate argument of

the derivation, we will present such a counterexample as a way of finishing off a derivation that fails.

4 A system will be decisive (in the sense that a derivation will always come to an end) provided its rules are all progressive (in the sense of always leading us closer to a point where no more can be done). Many rules are progressive because they are direct (in the sense of either closing a gap or replacing a goal or active resource by one or more simpler sentences). A decisive system which is sufficient and conservative (and is therefore correct in the answers it gives) provides a decision procedure for (formal) validity. Not all systems we consider will provide decision procedures but all will be sound in the sense of providing proofs only for valid arguments and complete in the sense of leading us to a proof whenever an argument is formally valid.

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2.3.x. Exercise questions

Use the basic system of derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample (that is, give an interpretation that divides an open gap and calculate truth values for the premises and conclusion from it—as is done in the example in 2.3.3):

1. $A \Rightarrow A \land B$
2. $A \land B \Rightarrow A \land (B \land A)$
3. $B \land E, C \land \top \Rightarrow (A \land B) \land (C \land D)$
4. $A \land B, B \land C, C \land D \Rightarrow A \land D$
5. $A, B \land A, D \Rightarrow B \land ((C \land A) \land D)$

For more exercises, use the exercise machine.

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2.3.xa. Exercise answers

1.

```
        │A           (2)
        ├─
        ││•
        │├─
2 QED   ││A          1
        ││○          A ⇏ B
        │├─
        ││B          1
        ├─
1 Cnj   │A ∧ B
```

A	B	A	/	A ∧ B
T	F	Ⓣ		Ⓕ

2.

```
        │A ∧ B              1
        ├─
1 Ext   │A                  (4),(6)
1 Ext   │B                  (5)
        ││•
        │├─
4 QED   ││A                 2
        │││•
        ││├─
5 QED   │││B                3
        │││•
        ││├─
6 QED   │││A                3
        │├─
3 Cnj   ││B ∧ A             2
        ├─
2 Cnj   │A ∧ (B ∧ A)
```

3.

```
        │B ∧ E                      1
        │C ∧ ⊤                      2
        ├─
1 Ext   │B                          (5)
1 Ext   │E
2 Ext   │C                          (7)
2 Ext   │⊤
        │││○                        B, C, E, ⊤ ⇏ A
        ││├─
        │││A                        4
        │││•
        ││├─
5 QED   │││B                        4
        │├─
4 Cnj   ││A ∧ B                     3
        │││•
        ││├─
7 QED   │││C                        6
        │││○                        B, C, E, ⊤ ⇏ D
        ││├─
        │││D                        6
        │├─
6 Cnj   ││C ∧ D                     3
        ├─
3 Cnj   │(A ∧ B) ∧ (C ∧ D)
```

A	B	C	D	E	B ∧ E,	C ∧ ⊤	/	(A ∧ B)	∧	(C ∧ D)
F	T	T	F	T	Ⓣ	Ⓣ T		F	Ⓕ	F

This derivation could have been ended after stage 4 when the first open gap has reached a dead end. Often answers will show a derivation continued further than necessary in order to show how the further steps would have worked out. The counterexample presented here divides both dead-end gaps; there are others that divide one of the two. Notice that ⊤ is not assigned a value at the left of the table. Since its value is fixed by the stipulation that it is a tautology, a value need not and cannot be assigned to it as part of an extensional interpretation.

4.

	$A \wedge B$	1
	$B \wedge C$	2
	$B \wedge D$	3
1 Ext	A	(5)
1 Ext	B	
2 Ext	B	
2 Ext	C	
3 Ext	B	
3 Ext	D	(6)
	•	
5 QED	A	4
	•	
6 QED	D	4
4 Cnj	$A \wedge D$	

Clearly, there is redundancy in the active resources of the gaps after stage 3. Since both gaps close, the exploitation of the second premise at stage 2 is not necessary (though it would be necessary before any gap could reach a dead end). It would be possible to state rules so that the resource B was not repeated at stages 2 and 3, but such repetition does not ordinarily enlarge derivations significantly and makes it easier to check whether rules have been applied fully and correctly.

5.

	A	(6)
	$B \wedge A$	1
	D	(7)
1 Ext	B	(5)
1 Ext	A	
	•	
5 QED	B	2
	○	$A, B, D \nRightarrow C$
	C	4
	•	
6 QED	A	4
4 Cnj	$C \wedge A$	3
	•	
7 QED	D	3
3 Cnj	$(C \wedge A) \wedge D$	2
2 Cnj	$B \wedge ((C \wedge A) \wedge D)$	

A	B	C	D	A	$B \wedge A$	D	/	$B \wedge ((C \wedge A) \wedge D)$
T	T	F	T	T	T	T		F F F

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