1.4. General principles of deductive reasoning

1.4.0. Overview

The properties and relations of sentence and propositions that are subject matter of deductive logic can be arranged in three groups.

1.4.1. A closer look at entailment
: Entailment will be at the heart of our study and we will begin by looking in some detail at a couple ways of formulating its definition.

1.4.2. Division
: It will be useful to have a special term for the kind of pattern of truth values that entailment rules out.

1.4.3. Relative exhaustiveness
: Although entailment does not encompass all the concepts of deductive logic, there is a similarly defined relation that does.

1.4.4. A general framework
: All the deductive properties and relations we will consider can be expressed in terms of relative exhaustiveness and expressed in a way that corresponds directly to definitions of them.

1.4.5. Reduction to entailment
: Although relative exhaustiveness provides a way of thinking about deductive properties and relations, entailment is way that they are most naturally established, and we need to consider how this can be done.

1.4.6. Laws for relative exhaustiveness
: The basic features of the system of relations among sets provided by relative exhaustiveness can be captured by three laws.

1.4.7. Laws for entailment
: Although laws governing relative exhaustiveness are in some ways more fundamental, it is laws governing entailment that we will use most.

1.4.8. Duality
: The specific principles concerning $\top$ and $\bot$ display a kind of symmetry that we will also find in principles for other logical forms.

Glen Helman 28 Aug 2008

1.4.1. A closer look at entailment

We have so far spoken of entailment somewhat informally as the relation between premises and a conclusion that merely extracts information from them and thus brings no risk of new error. Another way of putting the latter point is that a relation of entailment provides a conditional guarantee of the truth of the conclusion: it must be true if the premises are all true.

We will begin with a couple of ways of saying this more formally.

$\Gamma \Rightarrow \varphi$	if and only if	there is no logically possible world in which φ is false while all members of Γ are true
	if and only if	φ is true in every logically possible world in which all members of Γ are true

These are not two different concepts of entailment, for the two statements to the right of `if and only if` say the same thing. Still, they provide different perspectives on the concept. The second—which we will speak of as the *positive form* of the definition—is closely tied to the idea of a conditional guarantee of truth and to the reason why entailment is valuable. The first form—the *negative form*—makes the content of the concept especially clear, and this form of definition will generally be the more useful when we try to prove things concerning entailment. The other deductive properties and relations we have discussed or will go on to discuss can be given analogous pairs of definitions, a negative form ruling out certain patterns of truth values and another form stating a more positive generalization.

The equivalence of the two forms of the definition is a general feature of generalizations. When a generalization is false, it is because of *counterexample*, something of sort about which we generalize that does not have the property we have said all such things have. A counterexample to the claim that all birds fly is a bird that does not fly. In the positive definition of entailment, the generalization is about all possible worlds in which the premises are all true and such worlds are said to all have the property that the conclusion is true in them. A counterexample to such a generalization is then a world in which the premises are all true but the conclusion is not. The negative form of the definition then affirms the generalization by saying that there is no counterexample to it. As in this case, one good way to clarify a generalization is always to ask what sort of counterexample is being ruled out.

It is important to notice how little a claim of entailment says about the actual truth values of the premises and conclusion of an argument. We can distinguish four patterns of truth values that the premises and conclusion could exhibit. Of these, a claim that an argument is valid rules out only the one appearing at the far right of Figure 1.4.1-1.

	Patterns admitted			ruled out
Premises	all **T**	not all **T**	not all **T**	all **T**
Conclusion	**T**	**T**	**F**	**F**

Fig. 1.4.1-1. Patterns of truth values admitted and ruled out by entailment.

So, knowing that an argument is valid tells us about actual truth values only that we do not find the conclusion actually false when the premises are all actually true. The other three patterns all appear in the actual truth values of some valid arguments (though not all are possible for certain valid arguments because other deductive properties and relations of the sentences involved may rule them out).

To see examples of this, consider an argument of the simple sort we will focus on in the next chapter:

It's hot and sunny
It's humid but windy
———
It's hot and humid

This argument is clearly valid since its conclusion merely combines two items of information each of which is extracted from one of the premises. Depending on the state of the weather, the premises may be both true, both false, or one true and the other false; and, in any case where they are not both true the conclusion can be either true or false. In particular, if it's hot and humid but neither sunny nor windy, the conclusion will be true even though both premises are false. This should not be surprising: information can be extracted from a pair of sentences whether they are true or not, and the information extracted when they are not both true might be either true or false.

Of course, seeing one of these permitted patterns does not tell us that the argument is valid; no information that is limited to actual truth values can do that because validity concerns all possible worlds, not just the actual one. In particular, having true premises and a true conclusion does not make an argument valid. For example, the following argument is not valid:

Indianapolis is the capital of Indiana
———
Springfield is the capital of Illinois

For, although the single premise and the conclusion are both true, there is a logical possibility of the capital of Illinois being different while that of Indiana is as it actually is, so there is a possible world that provides a counterexample to the claim that the argument is valid.

Glen Helman 28 Aug 2008

1.4.2. Division

The pattern of truth values for premises and conclusion that is ruled out by entailment (i.e., true premises with a false conclusion) will recur often enough that it will be convenient to have special vocabulary for it. Let us say that a set Γ is *divided* from a set Δ whenever all members of Γ are true and all members of Δ are false. Whatever gives the sentences in Γ and Δ such values will be said to divide these sets.

Notice that this idea is asymmetric. When one set is divided from another it is the members of the first set that true and the members of the second that are false. You might think of sets being divided vertically, with the first set above the second. In this spatial metaphor, truth is thought of as higher than falsehood; and, although this is only a metaphor, it is a broadly useful one and is consistent with the appearance of Absurdity at the bottom of Figure 1.2.6-2 and Tautology at the top.

As with talk of sets of sentences as premises, it is really only the list of members of a set that we care about here, and we speak of sets only because the order of the list and the occurrence of repetitions in it do not matter. In particular, we will not distinguish between a sentence and a set that has only it as a member. So we can restate the negative definition of entailment as follows:

Γ ⇒ φ if and only if there is no possible world that divides Γ from φ.

We will also say that an argument is divided when its premises are divided from its conclusion, so we can say that an argument is valid when no possible world divides it.

Glen Helman 28 Aug 2008

1.4.3. Relative exhaustiveness

Clearly, we can use the idea of division in the way we can use it to define entailment to define a relation between sets rather than between a set and a sentence, and there is reason for doing this because the result constitutes a single fundamental idea that encompasses all the concepts of deductive reasoning. We have focused on entailment and will continue to do so, but it doesn't suffice by itself to capture all the ideas of deductive logic. We needed to add the idea of absurdity in 1.2.5 to capture the idea of inconsistency.

This more general relation is *relative exhaustiveness*. When it holds between a pair of sets, we will say that one set *renders* the other set *exhaustive*. Our notation for this idea will extend the use of the entailment arrow to allow a set to appear on the right. The negative and positive forms of its definition are as follows:

Γ ⇒ Δ	if and only if	there is no possible world in which all members of Δ are false while all members of Γ are true
	if and only if	in each possible world in which all members of Γ are true, at least one member of Δ is true

Or, in terms of division, Γ ⇒ Δ if and only if there is no possible world that divides Γ and Δ.

Entailment is the special case of this idea where the set Δ consists of a single sentence: to say that φ is entailed by Γ comes to the same thing as saying that φ is rendered exhaustive by Γ. In the other cases of relative exhaustiveness, it is either a set with several members or the empty set that is rendered exhaustive. In these cases, it does not make sense to speak of a conclusion. When the set on the right have several members, they need not be valid conclusions from the set that renders them exhaustive. Indeed, a jointly exhaustive pair of sentences will be rendered exhaustive by any set but often neither will be entailed by that set. This is particularly clear in the case of sentences like The glass is full and The glass is not full that are both jointly exhaustive and mutually exlcusive—i.e., that are contradictory. Although the set consisting of such pair is rendered exhaustive by any set, only an inconsistent set could entail both of these sentences.

Consequently, we need new terminology for sentences on the right of the arrow when they appear in groups. We will say that such sentences are *alternatives*. The conditional guarantee provided by a claim $\Gamma \Rightarrow \Delta$ of relative exhausitiveness is a guarantee that the alternatives Δ are not all false—i.e., that at least one is true—provided the premises Γ are all true. In particular, when $\Gamma \Rightarrow \varphi, \psi$, we have a guarantee that, if the members of Γ are all true, either φ or ψ is true.

Glen Helman 28 Aug 2008

1.4.4. A general framework

It is not surprising that the relative exhaustiveness should encompass deductive properties and relations if these are understood to consist in guarantees that certain parterns of truth values appear in no possible world, for to say that there is no world where certain sentences Γ are true and other sentences Δ are false is to say that $\Gamma \Rightarrow \Delta$. Of course, a given deductive property or relation may rule out a number of different patterns, and this means that it may consist of a number of different claims of relative exhaustiveness.

In the case of the properties and relations we will consider, no more than two claims of relative exhaustiveness are ever required, as can be seen in the following table. (When nothing appears to the left of the right of the arrow, the set in question is the empty set.)

Concept	*in terms of relative exhaustiveness*
Γ entails φ	$\Gamma \Rightarrow \varphi$
φ is a tautology	$\Rightarrow \varphi$
φ and ψ are equivalent	both $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$
Γ excludes φ	$\Gamma, \varphi \Rightarrow$
Γ is inconsistent	$\Gamma \Rightarrow$
φ and ψ are mutually exclusive	$\varphi, \psi \Rightarrow$
φ is absurd	$\varphi \Rightarrow$
Γ is exhaustive	$\Rightarrow \Gamma$
φ and ψ are jointly exhaustive	$\Rightarrow \varphi, \psi$
φ and ψ are contradictory	both $\varphi, \psi \Rightarrow$ and $\Rightarrow \varphi, \psi$

This list adds only one concept to those already discussed, a generalization of the idea of a pair of jointly exhaustivess sentences to the exhaustiveness of a set.

The definition of this idea can be read off its description in terms of relative exhaustiveness. To say that $\Rightarrow \Gamma$ is to say that there is no possible world that divides the empty set and Γ. That is, there is no possible world that makes every member of the empty set true and every member of Γ false. But, since the empty set has no members, any possible world makes all its members true because there is no member to provide a counterexample to the claim that the world has made them all true. This means that the property of making all members of the empty set true adds nothing to the description of the sort of world ruled out by the claim that $\Rightarrow \Gamma$, and this claim can be stated more simply by saying that there is no possible world that makes all members of Γ false. That is, to state the definition in positive form, a set Γ is exhaustive when, in every possible world, at least one member of Γ is true. That is, if we take the sets of possible worlds left open by the various members of Γ and put them all together, they will all exhaust all possibilities.

In the same way, the definition of each of these properties and relations can be read off the right side of the table by applying the definition of relative exhaustiveness to the case or cases indicated. When the set on one side or the other of the arrow has 0, 1, or 2 members, a direction application of the definition can be simplified as we just saw in the case of exhaustiveness.

The ideas of division and relative exhaustiveness also provides ways of extending the idea of logical independence introduced in 1.2.3 to speak of the absence of any deductive property or relation. Let us say that a set Γ of sentences is *logically independent* when every way of assigning a truth value to each member of Γ is exhibited in at least one possible world. When the set has two members, this is the same as the earlier idea. When a set $\{\varphi\}$ containing a single sentence φ is logically independent in this sense, we can say that φ is *logically contingent* because there is at least one possible world in which it is true and at least one where it is false.

Relative exhaustiveness provides another way of looking at the same idea. When the sentences in a set are *not* independent, not every way of dividing them into a set of true sentences and a set of false sentences is logically possible. And when that is so, the set contains at least one pair of non-overlapping subsets Γ and Δ such that $\Gamma \Rightarrow \Delta$. So the members of a set are logically independent when the relation of relative exhaustiveness never holds between non-overlapping subsets. (It always holds when sets overlap because there is no way of dividing such sets.)

When a set is logically independent, each member is contingent and any two of its members are logically independent, but contingency of members and independence of pairs does not by itself imply that the set as a whole is logically independent. For example, assume that the sentences X is fast, X is strong, X has skill, and X has stamina form an independent set. Then the sentences

X is fast and strong	X has skill and stamina	X is fast and has stamina

are each contingent, and any two of them can be seen to be independent. However, the first two taken together entail the third, so these three more complex sentences do not form an independent set.

Glen Helman 28 Aug 2008

1.4.5. Reduction to entailment

Relative exhaustiveness generalizes relaxing the restriction to a single conclusion to allow several alternatives or none at all. To express the ideas captured by relative exhaustiveness in terms of entailment, we need add ways of expressing both of these ideas.

We have already seen a way of expressing the idea of rendering exhaustive the empty set of alternatives. In 1.2.5, we characterized inconsistency in terms of entailment and absurdity by what was called the Basic Law for Inconsistency. If we restate that law expressing inconsistency in terms of relative exhaustiveness, it says

$$\Gamma \Rightarrow \text{ if and only if } \Gamma \Rightarrow \bot$$

so a set renders exhaustive an empty set of alternatives if and only if it entails the absurdity $\bot$. Both of these are conditional guarantees of something that cannot happen, so they have the effect of ruling out the possibility that the conditions of the guarantee (i.e., the truth of all members of Γ) can ever be met.

To express the idea of rendering exhaustive multiple alternatives using entailment we need help from the concept of contradictoriness. When sentences φ and ψ are contradictory (i.e., when $\varphi \otimes \psi$), they always have opposite truth values. so making one true comes to the same thing as making the other false. Since the difference between having a sentence as an assumption and having it as an alternative lies in the truth value assigned to it in the pattern that is being ruled out. This means that having a sentence as an alternative comes to the same thing as having a sentence contradictory to it as an assumption; that is,

$$\text{if } \overline{\varphi} \otimes \varphi\text{, then } \Gamma \Rightarrow \varphi, \Delta \text{ if and only if } \Gamma, \overline{\varphi} \Rightarrow \Delta$$

If we apply this idea repeatedly (perhaps infinitely many times), we can move any set of alternatives to the left of the arrow. To make it easier to state the result of doing this, we will use $\overline{\Gamma}$ for the result of replacing each member of Γ by a sentence contradictory to it.

BASIC LAW FOR RELATIVE EXHAUSTIVENESS. $\Gamma \Rightarrow \Delta, \Sigma$ if and only if $\Gamma, \overline{\Delta} \Rightarrow \Sigma$.

That is, extra alternatives can be removed if we put sentences contradictory to them among the assumptions. This gives us two ways of restating claims of relative exhaustiveness as entailments: (i) we may replace all but one alternative by contradictory sentences among the assumptions or (ii) we may replace all alternatives by contradictory sentences and replace the resulting empty set of alternatives by $\bot$.

The following table summarizes the application of these ideas to state all the deductive properties we have considered using entailment, absurdity, and contradictoriness:

Concept	*in terms of entailment and other ideas*
Γ entails φ	$\Gamma \Rightarrow \varphi$
φ is a tautology	$\Rightarrow \varphi$
φ and ψ are equivalent	both $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$
Γ excludes φ (i.e., $\Gamma, \varphi \Rightarrow$)	$\Gamma, \varphi \Rightarrow \bot$
Γ is inconsistent (i.e., $\Gamma \Rightarrow$)	$\Gamma \Rightarrow \bot$
φ and ψ are mutually exclusive (i.e., $\varphi, \psi \Rightarrow$)	$\varphi, \psi \Rightarrow \bot$
φ is absurd (i.e., $\varphi \Rightarrow$)	$\varphi \Rightarrow \bot$
Γ is exhaustive (i.e., $\Rightarrow \Gamma$)	$\overline{\Gamma} \Rightarrow \bot$
φ and ψ are jointly exhaustive (i.e., $\Rightarrow \varphi, \psi$)	$\overline{\varphi}, \overline{\psi} \Rightarrow \bot$ (or $\overline{\varphi} \Rightarrow \psi$ or $\overline{\psi} \Rightarrow \varphi$)
φ and ψ are contradictory (i.e., both $\varphi, \psi \Rightarrow$ and $\Rightarrow \varphi, \psi$)	both $\varphi, \psi \Rightarrow \bot$ and $\overline{\varphi}, \overline{\psi} \Rightarrow \bot$

Here $\overline{\varphi}$ is any sentence contradictory to φ, and $\overline{\Gamma}$ is the result of replacing each member of Γ by a sentence contradictory to it

Of course, either of the two further ways of stating exhaustiveness could be used instead of the second entailment required for two sentences to be contradictory. And, when a non-empty set Γ is said to be exhaustive, we could leave one member behind as a conclusion rather than adding $\bot$; that is, $\Rightarrow \Gamma, \varphi$ when $\overline{\Gamma} \Rightarrow \varphi$.

Glen Helman 28 Aug 2008

1.4.6. Laws for relative exhaustiveness

Most of our concern with deductive reasoning will not be with particular examples, but instead with general laws. Most of these will be generalizations about specific logical forms that will be introduced chapter by chapter, but some very general ones can be stated now. We will look first at relative exhaustiveness, the content of whose laws is the clearest, and then turn to entailment.

We will consider three basic principles for relative exhaustiveness, two of which are related to the laws of reflexivity and transitivity for implication that we considered in 1.2.2. For any sentence φ and any sets Γ, Δ, Σ, and Θ of sentences:

Repetition. $\Gamma, \varphi \Rightarrow \varphi, \Delta$ (for any sentence φ and any sets Γ and Δ).

Cut. If $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma, \varphi \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ (for any sentence φ and any sets Γ and Δ).

Monotonicity. If $\Gamma \Rightarrow \Delta$, then $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ (for any sets Γ, Δ, Σ, and Θ).

The repetition law tells that relative exhaustiveness holds whenever an assumption appears also among the alternatives. When this is so, the truth of the assumptions certainly guarantees the truth of at least one alternative. The reflexivity law for implication is the special case of this where the sets Γ and Δ are both empty, where the only alternative is also the sole assumption. Relative exhaustiveness itself is not reflexive in general, but there is only one counterexample. The empty set does not render itself exhaustive, but the cases of the repetition law where Γ and Δ are the same set tell us that all non-empty sets render themselves exhaustive.

The name of the cut law reflects the disappearance of the φ in the conclusion that is drawn. This is a very fundamental law, and instances and consequences of it (the transitivity of implication is one) are clearer in their import than the law itself. But, to see the import of this law in its full generality, notice that the relation $\Gamma \Rightarrow \varphi, \Delta$ implies that Γ guarantees that either φ or a member of Δ is true. But if φ is true, we know that a member of Δ will be true also (because $\Gamma, \varphi \Rightarrow \Delta$). And this means that, given Γ, at least one member of Δ is bound to be true, which is what $\Gamma \Rightarrow \Delta$ says.

The idea behind monotonicity is that the truth of an instance of relative exhaustiveness can never be damaged by adding assumptions or alternatives. (The law mentions added sets of both assumptions Σ and alternatives Θ, but either of these might be the empty set.) If we add assumptions, we are narrowing the range of possibilities left open for the alternatives to exhaust; and, if we add alternatives, we are adding further ways of covering these possibilities. Either way, we are making it harder to find a counterxample to the claim of relative exhaustiveness. The term monotonic is applied to trends that never change direction. More specifically, it is applied to a quantity that does not both increase and decrease in response to changes in another quantity. In this case, it reflects the fact that adding assumptions will never lead to a decrease in the sets of alternatives rendered exhaustive and adding alternatives will never lead to a decrease in the sets of assumptions rendering them exhaustive.

The cut law and monotonicity combine to yield the transitivity of implication. For, if $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \chi$, then both $\varphi \Rightarrow \psi, \chi$ and $\varphi, \psi \Rightarrow \chi$ by monotonicity, and we can cut ψ from these two to get $\varphi \Rightarrow \chi$. However, relative exhaustiveness itself is not transitive. It is true that, if $\Gamma \Rightarrow \psi$ and $\psi \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$. But knowing that $\Gamma \Rightarrow \Sigma$ and $\Sigma \Rightarrow \Delta$ for a larger set Σ is not enough to insure that $\Gamma \Rightarrow \Delta$. With $\Gamma \Rightarrow \Sigma$ we have a guarantee, given Γ, only that at least one member of Σ is true while $\Sigma \Rightarrow \Delta$ guarantees the truth of at least one member of Δ only given the truth of *all* members of Σ.

There is a sense in which cut and monotonicity are inverse principles since cut allows us to eliminate assumptions and alternatives while monotonicity allows us to add them. The nature of the inversion can be seen more clearly by considering a generalization of cut:

Generalized cut. If $\Gamma, \Sigma \Rightarrow \Delta, \Theta$ and moreover $\Gamma, \Sigma' \Rightarrow \Delta, \Theta'$ for all other non-overlapping sets Σ' and Θ' such that $\Sigma' \cup \Theta' = \Sigma \cup \Theta$, then $\Gamma \Rightarrow \Delta$ (for any sets Γ, Δ, Σ, and Θ).

This principle says that we can drop a group of premises and alternatives provided the relation holds no matter how they are distributed between assumptions and alternatives. The intuitive idea is that, if it does not matter what specific role these sentences play, they need not appear at all. The basic cut law applies this idea to a single sentence, and its application to any finite set follows from that law. For example, consider the case of two sentences. Putting together the following instances of cut (which cut ψ in the first two cases and φ in the third)

$$\text{if } \Gamma \Rightarrow \psi, \varphi, \Delta \text{ and } \Gamma, \psi \Rightarrow \varphi, \Delta, \text{ then } \Gamma \Rightarrow \varphi, \Delta$$
$$\text{if } \Gamma, \varphi \Rightarrow \psi, \Delta \text{ and } \Gamma, \varphi, \psi \Rightarrow \Delta, \text{ then } \Gamma, \varphi \Rightarrow \Delta$$
$$\text{if } \Gamma \Rightarrow \varphi, \Delta \text{ and } \Gamma, \varphi \Rightarrow \Delta, \text{ then } \Gamma \Rightarrow \Delta$$

we can say

$$\text{if } \Gamma \Rightarrow \psi, \varphi, \Delta \text{ and } \Gamma, \psi \Rightarrow \varphi, \Delta \text{ and } \Gamma, \varphi \Rightarrow \psi, \Delta \text{ and } \Gamma, \varphi, \psi \Rightarrow \Delta, \text{ then } \Gamma \Rightarrow \Delta.$$

That is, if relative exhaustiveness holds no matter how φ and ψ are added to assumptions Γ and alternatives Δ, then it holds without any addition. One reason for considering the generalized principle is that any relation that satisfies it together with repetition and monotonicity will be the relation of relative exhaustiveness corresponding to some set of possibilities, so these three principles encompass all there is to be said in general about relative exhaustiveness.

Glen Helman 28 Aug 2008

1.4.7. Laws for entailment

Entailment holds in those cases of relative exhaustiveness where there is a single alternative, so a natural place to look for its laws is in the instances of the laws of repetition, cut, and monotonicity for single alternatives. With a minor exception in the case of cut, that is the source of the following laws:

LAW FOR PREMISES. $\Gamma, \varphi \Rightarrow \varphi$ (for any sentence φ and any sets Γ and Δ);
LAW FOR LEMMAS. If $\Gamma, \varphi \Rightarrow \psi$ and $\Gamma \Rightarrow \varphi$, then $\Gamma \Rightarrow \psi$ (for any sentence φ and set Γ);
MONOTONICITY. If $\Gamma \Rightarrow \varphi$, then $\Gamma, \Delta \Rightarrow \varphi$ (for any sentence φ and any sets Γ and Δ).

The law for premises and the law of monotonicity for entailment are simply instances of the laws of repetition and monotonicity for relative exhaustiveness where certain sets have been chosen to be empty or have a single member. No such simple restrictions will convert the cut law into a law for entailment; but, if $\Gamma, \varphi \Rightarrow \psi$ and $\Gamma \Rightarrow \varphi$ then we have $\Gamma, \varphi \Rightarrow \psi$ and $\Gamma \Rightarrow \varphi, \psi$ by applying monotonicity to the second, and an instance of cut will give us $\Gamma \Rightarrow \psi$.

The first law is renamed to reflect the role it will usually play, to justify concluding a premise. The name of the second also refers to its function. The term lemma can be used for a conclusion that is drawn not because it is of interest in its own right but because it helps us to draw further conclusions. This law tells us that if we add to our premises Γ a lemma φ that we can conclude from them, anything ψ we can conclude using the enlarged set of premises can be concluded from the original set Γ. Or, to put it in a way that suggest its relation to monotonicity, we can drop from a set a premises and sentence that is entailed by the rest.

A more direct inverse to monotonicity would be a principle that allowed us to drop any set of premises provided each of its members was entailed by the premises that remain. This is a legitimate principle, and it follows from the law for lemmas in cases where a finite set of premises is dropped. However, rather than stating this generalized form of the law of lemmas, we will consider a related principle that has a slightly different function.

CHAIN LAW. If $\Gamma \Rightarrow \psi$ for each assumption ψ in Δ and $\Delta \Rightarrow \varphi$, then $\Gamma \Rightarrow \varphi$ (for any sentence φ and any sets Γ and Δ).

This follows from the generalized form of the law for lemmas using monotonicity and, in combination with the law for premises, it implies both of them. We refer to this principle as the chain law since it enables us to link valid arguments together to get new valid arguments. If premises Γ enable us to conclude each of the premises Δ of a second argument, then its conclusion follows from Γ directly.

This idea is similar to the idea behind the transitivity of implication (and has that as a special case), and the law for premises is similarly related to the reflexivity of implication. However, these laws for entailment are not directly principles of reflexivity and transitivity since those ideas only make sense for relations between the same sorts of things. Let us define a relation of *set entailment* by saying that a set Γ entails a set Δ if Γ entails every member of Δ. Set entailment comes to the same thing as relative exhaustiveness when Δ has only one member, but otherwise the two are different. The law for premises tells us that set entailment is reflexive, and the chain law tells us that it is transitive. And the reflexivity and transitivity of set entailment can be shown to give a complete account of the general laws of entailment.

Finally, let us look briefly at the law of monotonicity for entailment. Although it will play only an auxiliary role in our discussion of deductive reasoning, it is a distinguishing characteristic of deductive reasoning that such a principle holds. For, when reasoning is not risk free, additional data can show that a initially well-supported conclusion is false without undermining the original premises on which the conclusion was based. If such further data were added to the original premises, the result would no longer support the conclusion.

Indeed, the risk in good but risky inference can be thought of as a risk that further information will undermine the quality of the inference, so risky inference (or, more precisely, the way the quality of such inference is assessed) is, in general, *non-monotonic* in the sense that additions to the premises can reduce the set of conclusions that are justified. This is true of inductive generalization and of inference to the best explanation of available data, but the term non-monotonic is most often applied to inferences that are based on features of typical or normal cases. One standard example is the argument from the premise Tweety is a bird to the conclusion Tweety flies. This conclusion is reasonable when the premise exhausts our knowledge of Tweety; but the inference is not free of risk, and the conclusion would no longer be reasonable if we were to add the premise that Tweety is a penguin.

Glen Helman 28 Aug 2008

1.4.8. Duality

The properties of $\top$ and $\bot$ take a particularly symmetric form when stated in the context of relative exhaustiveness.

	as a premise	*as an alternative*
Tautology	if $\Gamma, \top \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$	$\Rightarrow \top$
Absurdity	$\bot \Rightarrow$	if $\Gamma \Rightarrow \bot, \Delta$, then $\Gamma \Rightarrow \Delta$

That is, while $\top$ contributes nothing as a premise and may be dropped, it is sufficient by itself as the only alternative (no matter how small our set of premises). And while $\bot$ is sufficient by itself as a premise (no matter how small the set of alternatives is), it contributes nothing as an alternative and may be dropped.

The symmetry here might be traced to the symmetry of relative exhaustiveness: since $\top$ and $\bot$ are contradictory, having one as an assumption comes to the same thing as having the other as an alternative according to the basic law of relative exhaustiveness discussed in 1.4.5. However, there is a more general idea behind this symmetry that will apply also to cases where sentences are not contradictory.

We will spend a moment looking more closely at the pattern that contradictoriness provides for here in order to make it easier to recognize in other cases. To take the simplest example of symmetry in the table above, we might state the lower left and upper right as follows:

$$\bot \Rightarrow$$

$$\top \Leftarrow$$

(where an arrow running right to left is understood to have its alternatives on its left and its premises on its right). That is, the difference lies in interchanging Absurdity and Tautology and reversing the direction of the arrow—or, what comes to the same thing, interchanging premises and alternatives. If we apply the same transition to the principle at the upper left we get

$$\text{if } \Gamma, \bot \Leftarrow \Delta, \text{ then } \Gamma \Leftarrow \Delta$$

or, rewriting so the arrows run left to right (without change of premises and alternatives),

$$\text{if } \Delta \Rightarrow \bot, \Gamma, \text{ then } \Delta \Rightarrow \Gamma$$

which differs from the principle for Absurdity as an alternative on the lower right above only in the interchange of Γ and Δ; and, since each could be any set, exchanging them does not alter the content of the principle. The possibility of this sort of transformation can be expressed by saying that $\top$ and $\bot$ on the one hand and premise (or assumption) and alternative on the other constitute pairs of *dual* terms. We will run into other pairs of dual terms later.

Glen Helman 28 Aug 2008

1.4.s. Summary

1 Entailment may be defined in two equivalent ways, negatively as the relation that holds when the conclusion is false in no possible world in which all the premises are true or positively as the relation which holds when the conclusion is true in all such worlds. The negative form has the advantage of focusing attention on the sort of possible world that serves as a counterexample to a claim of entailment. The positive form characterizes a relation of entailment as a conditional guarantee of the truth of the conclusion, a guarantee conditional on the truth of the premises.

2 The requirements for a world to serve as a counterexample to entailment suggest the general idea of dividing a pair of sets by making all members of the first true and all members of the second false. A world will be said to divide an argument when it divides the premises and conclusion.

3 The idea of division enables us to define a relation of relative exhaustiveness between sets: one set renders another exhaustive when there is no possible world that divides the two sets. We will extend the notation for entailment to express this relation between sets Γ and Δ as Γ ⇒ Δ. Entailment is the special case of this where Δ has only one member. When Δ has more than one member, its members will be referred to as alternatives because a relation of relative exhaustive provides a conditional guarantee only that at least one member of the second set it true.

4 Since a set of alternatives can have more than one member or be empty, relative exhaustiveness encompasses all the deductive properties and relations we have considered (as well as an extension of the idea of joint exhaustiveness to any set of sentences). We way a property or relation is expressed using relative exhaustiveness is tied directly to the negative form of its definition. When no relation of relative exhaustiveness holds no matter how a set is divided into two parts, all patterns of truth values for its members are possible and the set is logically independent. A single sentence that forms a logically independent set is logically contingent.

5 Definitions in terms of relative exhaustiveness can be converted into definitions in terms of entailment by replacing empty sets of alternative with ⊥ and reducing the size of multiple sets by replacing members with contradictory sentences among the assumptions (using the basic law for relative exhaustiveness).

6 Relative exhaustiveness satisfies three basic principles: repetition (the relation holds whenever an assumption is repeated as an alternative), cut (if the relation holds whether a sentence appears as an assumption or as an alternative, the sentence need not appear as either), and monotonicity (when the relation holds, it will continue to hold with added assumptions or alternatives). The term monotonic reflects the fact that the number of cases of relative exhaustiveness never decreases when the set of assumptions or set of alternatives increases. The cut law may be generalized to say that, if the relation holds no matter how sentences from some set are distributed among assumptions and alternatives, it holds when these sentences do not appear as either. Generalized cut, repetition, and monotonicity together suffice to imply all the principles governing relative exhaustiveness.

7 The basic principles governing entailment are closely related to those governing relative exhaustiveness. Two of these—the law for premises (any premise is a valid conclusion), monotonicity (adding premises never damages validity)—are special cases of laws for relative exhaustiveness. A third is a slight variation on an instance of the cut law: the law for lemmas says that a premise may be dropped if it is entailed by the other premises. The latter licenses the use of lemmas, valid conclusions that are of interest only as premises in further arguments. A more general law, called the chain law, says that anything entailed by a set of valid conclusions from given premises is itself a valid conclusion. This, together with a law for premises, yields all laws of entailment, and these two principles amount to principles of reflexivity and transitivity for the relation between sets that holds when one set entails each member of the other. Although this places monotonicity in the background, it is significant in distinguishing entailment from other forms of good inference, whose riskiness means that they are non-monotonic because adding information that the risk has not paid off will undermine their quality.

8 The laws describing the behavior of ⊤ and ⊥ in the context of relative exhaustiveness exhibit a kind of symmetry that we will see in other laws later. The sentences ⊤ and ⊥ are dual as are the terms premise and alternative (or the left and right of an arrow) in the sense that replacing each such term in a law by the one dual to it will produce another law.

Glen Helman 28 Aug 2008

1.4.x. Exercise questions

1. Any claim that a deductive relation holds can be stated as one or more claims that one set of sentences cannot be divided from another. (i) Restate each of the following claims in that way, and (ii) explicitly describe the sort of possibility that would divide the sets in question and is thus ruled out by claiming that the deductive relation holds. Nonsense words have been used to help you think to think how a possibility would be described without worrying whether that possibility could really occur.

For example, the claim that The widget plonked is equivalent to The widget plinked can be restated by saying that (i) the set consisting of the first sentence cannot be divided from the set consisting of the second sentence and vice versa. That is, (ii) it rules out any possibility in which the widget plonked but did not plink and any possibility in which the widget plinked but did not plonk.

- **a.** The gizmo is a widget and The gizmo is a gadget are mutually exclusive
- **b.** The gizmo is a widget and The gizmo is a gadget are jointly exhaustive
- **c.** The widget plinked is a tautology
- **d.** The widget plonked is absurd
- **e.** The widget was a gadget renders exhaustive the alternatives The widget plinked and The widget plonked
- **f.** The widget was a gizmo, The widget plinked, and The widget plonked are inconsistent

2. The basic law for relative exhaustiveness can be used not only to replace alternatives by assumptions but also to replace assumptions by alternatives. For example, The widget is blue entails The widget is colored can be restated to say (i) The widget is blue and The widget is not colored are inconsistent, (ii) The widget is not blue and The widget is colored form an exhaustive set, and (iii) The widget is not colored entails The widget is not blue.

In the following, you will be asked to restate some statements of deductive relations by replacing alternatives with assumptions or assumptions with alternatives. You may add or remove ordinary negation to state the contradictories of sentences.

- **a.** Restate the following as a claim of entailment: The gadget is red and The gadget is green are mutually exclusive
- **b.** Restate the following as a claim of entailment: Someone is in the auditorium and There are empty seats in the auditorium are jointly exhaustive
- **c.** Restate the following as a claim of absurdity: A widget is a widget is a tautology
- **d.** Restate the following as a claim of tautologousness: A widget is a gadget is absurd
- **e.** Restate the following as a claim of inconsistency: The widget is a gadget or gizmo and The widget is not a gadget entail The widget is a gizmo
- **f.** Restate the following so that each assumption is replaced by an alternative and each alternative by an assumption: The widget has advanced and The widget has plonked render exhaustive the alternatives The widget has finished the task and The widget has broken

Glen Helman 28 Aug 2008

1.4.xa. Exercise answers

1. **a.** (i) The set consisting of The gizmo is a widget and The gizmo is a gadget cannot be divided from the empty set; that is, (ii) there is no possibility of the gizmo being both a widget and a gadget.

b. (i) The empty set cannot be divided from the set consisting of The gizmo is a widget and The gizmo is a gadget; that is, (ii) there is no possibility of the gizmo being neither a widget nor a gadget

c. (i) The empty set cannot be divided from the set consisting of only The widget plinked; that is, (ii) there is no possibility that the widget did not plink

d. (i) The set consisting of only The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget plonked

e. (i) The set consisting of only The widget was a gadget cannot be divided from the set consisting of The widget plinked and The widget plonked; that is, (ii) there is no possbility that the widget was a gadget while not either plinking or plonking.

f. (i) The set consisting of The widget was a gizmo, The widget plinked, and The widget plonked cannot be divided from the empty set; that is, (ii) there is no possibility that the widget was a gizmo and both plinked and plonked

2. **a.** The gadget is red entails The gadget is not green (*or:* The gadget is green entails The gadget is not red)

b. The auditorium is empty entails There are empty seats in the auditorium (*or:* There are no empty seats in the auditorium entails The auditorium is not empty)

c. A widget is a not widget is absurd

d. A widget is a not gadget is a tautology

e. The widget is a gadget or gizmo, The widget is not a gadget, and The widget is not a gizmo are inconsistent

f. The widget has not finished the task and The widget has not broken render exhaustive The widget has not advanced and The widget has not plonked

Glen Helman 28 Aug 2008