1.2. What is said: propositions

1.2.0. Overview

In 1.1.5, we noted the close relation between two properties of a deductive inference: it is a transition from premises to conclusion that is free of any risk of new error, and the information provided by its conclusion is already present in its premises. The relation between these properties points to a way of understanding the informational content of a sentence and provides the basis for a general picture of the function of language.

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1.2.1. Truth values and possible worlds

When an inference is deductive, its conclusion cannot be in error unless there is an error somewhere in its premises. The sort of error in question lies in a statement being false, so to know that an argument is valid is to know that its conclusion must be true unless at least one premise is false. Similarly, to know that a set of sentences is inconsistent —to know that it's members are deductively incompatible—is to know that these sentences cannot all be true. This means that the ideas of truth and falsity have a central place in deductive logic, and it will be useful to have some special vocabulary for them.

It is standard to speak of truth and falsity together as ***truth values*** and to abbreviate their names as **T** and **F**, respectively. So, to say that an argument is valid is to claim that there is no risk of the pattern of truth values for its premises and conclusion shown in Figure 1.2.1-1 occurring. That is (using some of the other terminology we have available), a conclusion is entailed by a set of assumptions if the truth value of the conclusion cannot be **F** when each of the assumptions has the truth value **T**.

$$\begin{array}{r|c} & \mathbf{T} \\ \textit{premises:} & \mathbf{T} \\ & \vdots \\ & \mathbf{T} \\ \hline \textit{conclusion:} & \mathbf{F} \end{array}$$

Fig. 1.2.1-1. The pattern of truth values that is not a risk when an argument is valid.

And a set is inconsistent if the truth values of its members cannot all be **T**.

Since to speak of no risk of error is to speak of no possibility of error, it is also useful to have some vocabulary for speaking of possibility and impossibility. The sort of possibility in question in deductive logic is very weak and the corresponding sort of impossibility is very strong. We will refer to this as ***logical*** possibility and impossibility. A description of a situation that runs counter to the laws of physics (for example, a locomotive floating 10 feet above the earth's surface without any abnormal forces acting on it) might be said to be physically impossible; but it need not be logically impossible, and we must consider many physical impossibilities when deciding whether a conclusion is deductively valid. For, otherwise, anything following from the laws of nature, including the laws themselves, would be a valid conclusion from any premises whatsoever, and these laws would not say anything more than mere descriptions of the facts they were designed to explain. In short, if there is any set of premises such that a sentence φ says something that they do not, then it is logically possible for φ to be false.

We can say that something is impossible by saying that "there is no possibility" of it being true. In saying this, we use a form of words analogous to one we might use to say that there is no photograph of Abraham Lincoln chopping wood. That is, in saying "there is no possibility," we speak of possibilities as if they were things like photographs. This way of speaking about possibilities is convenient, so it is worth spending a moment thinking about what sort of things possibilities might be. The sort of possibility of chief interest to us is a complete state of affairs or state of the world, where this is understood to include facts concerning the full course of history, both past and future. Since Leibniz, philosophers have used the phrase ***possible world*** as a particularly graphic way of referring to possibilities in this sense. For instance, Leibniz held that the goodness of God implied that the actual world must be the best of all possible worlds, and by this he meant that God made the entire course of history as good as it was logically possible for it to be.

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1.2.2. Ordering by content

When we judge the validity of an argument we are comparing the content of the conclusion to the information contained in the premises, and the ideas of truth values and possible worlds are designed to help us speak about the basis for that comparison. We can see more of what this sort of comparison involves and what similar comparisons are possible by focusing on comparisons of two sentences.

The term ***implies*** is a more common English synonym of entails, and we will use it often when considering an argument that has only one premise (i.e., an immediate inference in traditional terminology noted in 1.1.2). Thus φ implies (or entails) ψ when there is no risk that ψ will be in error without any error in φ—i.e., when there is no logically possible world in which ψ is false even though φ is true. When φ implies ψ, the content of ψ can be extracted from the content of φ, so to say that φ ⇒ ψ is to say that φ includes the content of ψ. Thus the relation of implication orders sentences according to their content.

If this relation holds in both directions—if both φ ⇒ ψ and φ ⇐ ψ—then each of the two sentences says everything the other does, so they provide exactly the same information, differing at most in their wording. For example, although one of the sentences Sam lives somewhere in northern Illinois or southern Wisconsin and Sam lives somewhere in southern Wisconsin or northern Illinois might be chosen over the other depending on the circumstances, they allow the same possibilities for Sam's residence and thus provide the same information about it. We will say that sentences that have the same informational content are ***(logically) equivalent*** (usually dropping the qualification logically since we will not be considering other sorts of equivalence). Our notation for logical equivalence—the sign ⇔ (***left right double arrow***)—reflects its tie to mutual implication.

The idea of logical equivalence can also be described directly in terms of truth values and possible worlds. When two sentences say the same thing there is no way for one to be in error when the other is not. That is to say, sentences are equivalent when there is no possible world in which they have different truth values. To put it another way, no what things are like, either both sentences will be accurate or both will be in error. So, when φ ⇔ ψ, we know that in every possible world φ and ψ will both have the same truth value.

Notice that ⇒ is related to ⇔ in much the way that ≥ is related to =. That is, when φ ⇒ ψ, either φ says everything that ψ does as well as something more or the two sentences are equivalent. When φ does say something more than ψ, it will rule out some possibilities that ψ leaves open. To see an example of this, consider the following series of successively more specific statements, each implied by the one below it:

The package will arrive sometime

⇑

The package will arrive next week

⇑

The package will arrive next Wednesday

Each of the first two sentences leaves open some possibilities that are ruled out by the sentence below it. And in general, as we add information, we reduce the range of possibilities left open and increase the range that are ruled out. We will often speak of a sentence that rules out more and leaves open less as making a ***stronger*** claim and of one that rules out less and leaves open more as making a ***weaker*** claim. So, in the list above, the sentences closer to the bottom make the stronger claims and those closer to the top make the weaker ones.

We have been employing analogies between implication and numerical ordering and the related sorts of comparison that are associated with terms like stronger and weaker. These analogies rest on properties of implication that can be made explicit in two basic laws:

REFLEXIVITY OF IMPLICATION. φ ⇒ φ (for any sentence φ).

TRANSITIVITY OF IMPLICATION. If φ ⇒ ψ and ψ ⇒ χ, then φ ⇒ χ (for any sentences φ, ψ, and χ).

The first says implication is ***reflexive*** in the sense that any sentence φ implies itself, and second says it is ***transitive*** in the sense that implication by a premise φ carries over from a valid conclusion ψ to any sentence χ implied by that conclusion. That is, we do not count steps in a chain of related items (as is done with parent of, grandparent of, etc.) but simply report the existence of a chain (as is done with ancestor of).

Equivalence inherits the laws governing implication and obeys one further one:

REFLEXIVITY OF EQUIVALENCE. φ ⇔ φ (for any sentence φ).

SYMMETRY OF EQUIVALENCE. If φ ⇔ ψ then ψ ⇔ φ (for any sentences φ and ψ).

TRANSITIVITY OF EQUIVALENCE. If φ ⇔ ψ and ψ ⇔ χ, then φ ⇔ χ (for any sentences φ, ψ, and χ).

To say that a relation is ***symmetric*** is to say that it is reversable, and equivalence is reversable because it amounts to implication in both directions. For the same reason, the reflexivity of equivalence simply states the reflexivity of implication twice over and the transitivity of implication carries through in both directions needed to have φ ⇔ χ given φ ⇔ ψ and ψ ⇔ χ.

However, not everything that can be said about implication holds also for equivalence. For example, the principle of transitivity for implication tells us that implication is transferable by implication though in different directions for premises and conclusions. That is, anything implied by a valid conclusion from φ is itself a valid conclusion from φ, and anything that implies a premise implying χ is itself a premise implying χ. Or, in other words, weakening a conclusion preserves implication, as does strengthening a premise. However, equivalence is not transferable in either direction by implication (though both implication and equivalence are transferable by equivalence). Seen from this point of view, this failure of transferability is probably not surprising; we wouldn't expect the fact that implication is transferable by itself to imply that another relation is transferable by implication. However, the reason this property fails for equivalence an important feature of the connection between equivalence and implication: when we say that φ ⇔ ψ, we are looking at each of φ and ψ as both a premise and a conclusion, so we can replace one by another sentence only if that sentence is both at least as strong and at least as weak as the one it replaces.

1.2.3. Contrasting content

We arrived at the relation of implication by considering entailment by a single premise. If we do the same with exclusion, we arrive at another relation between sentences. If φ excludes ψ, then the set $\{\varphi, \psi\}$ formed of the two is inconsistent. Since the members of a set have no order, it will be equally true that ψ excludes φ; and this reversability is reflected in the usual terminology for this relation. When there is no possible world in which φ and ψ are together true, φ and ψ are ***mutually exclusive***. There is no standard notation for this relation; but, when it is convenient to have a symbol for it, we will write $\varphi \times \psi$ to say that φ and ψ are mutually exclusive. This use of the ***multiplication sign*** is intended to suggest crossing out of the possibility that sentences are both true.

Mutually exclusiveness is neither reflexive nor transitive like implication, but it does obey the following two laws:

Symmetry of $\times$. If $\varphi \times \psi$ then $\psi \times \varphi$ for any sentences φ and ψ.
Contravariance of $\times$. If $\varphi \Rightarrow \psi$, then whenever $\psi \times \chi$ we also have $\varphi \times \chi$ for any sentences φ, ψ, and χ.

The first of these notes the reversability of $\times$. The term contravariance in the second alludes to the fact that it tells us that exclusion is transferable by implication but in a direction opposite to the direction of the implication arrow. That is, if ψ excludes χ then anything φ is *implied by* also excludes χ. This law indicates that strength in sentences is important for mutual exclusiveness: indeed, to say that sentences are mutually exclusive is to say any case of weakness in one—any possibility left open—is made up for by the other.

Mutually exclusive sentences are opposed to one another, and they can be thought of as opposites. But there are different sorts of opposites. Some, like The glass is full and The glass is empty are extremes that may both fail in intermediate cases. Others, like The glass is full and The glass is not full cover all the ground between them and do not leave room for a third alternative.

The difference between these sorts of opposition is tied to another relation between sentences that we haven't discussed yet. Sentences φ and ψ are ***jointly exhaustive*** when there is no possible world in which both are false, when there is no possible world that both rule out. If we put together the possibilities left open by such sentences, the result will include all possibilities because any possibility ruled out by one must be left open by the other; and, in this sense, these sentences jointly exhaust all possibilities. Such sentences certainly differ in meaning—as we will see later, they can be said to have no common content—but they are not opposites since they need not be incompatible. They might be thought of instead as complementary since, in regard to possibilities left open, each picks up where the other leaves off. We will use a simple ***large circle*** $\bigcirc$ as our notation for this relation, with $\varphi \bigcirc \psi$ intended to suggest that φ and ψ between them leave open the full range of possibilities.

Like exclusiveness, exhaustiveness is neither reflexive nor transitive but is symmetric. However, its difference from exclusiveness is reflected in the fact that it is not contravariant but is transferable by implication in a different way.

Symmetry of $\bigcirc$. If $\varphi \bigcirc \psi$ then $\psi \bigcirc \varphi$ (for any sentences φ and ψ).
Covariance of $\bigcirc$. If $\varphi \Rightarrow \psi$, then whenever $\varphi \bigcirc \chi$ we also have $\psi \bigcirc \chi$ (for any sentences φ, ψ, and χ).

The second law claims a transferability of $\bigcirc$ by implication that, unlike that for $\times$, follows the direction of the implication arrow (which is what the term covariance points to). That is, weakness in sentences is what is important for joint exhaustiveness: any point of strength in one of a pair of jointly exhaustive sentences—any possibility ruled out—is matched by a corresponding weakness in the other.

Although neither $\times$ nor $\bigcirc$ is transitive, linking sentences by the two relations in either order does tell us something about the logical relations about the sentences at each end.

Alternation law for $\times$ and $\bigcirc$. If $\varphi \times \psi$ and $\chi \bigcirc \psi$, then $\varphi \Rightarrow \chi$ (for any sentences φ, ψ, and χ).

Notice that, because of the symmetry of $\times$ and $\bigcirc$, saying that $\varphi \times \psi$ and $\chi \bigcirc \psi$ comes to the same thing as saying either that $\varphi \times \psi$ and $\psi \bigcirc \chi$ or that $\chi \bigcirc \psi$ and $\psi \times \varphi$. The reason for the law is that, if φ is true and χ is false, then either φ and ψ are both true or χ and ψ are both false. In other words, if $\varphi \Rightarrow \chi$ fails to be true, then so must one of $\varphi \times \psi$ and $\chi \bigcirc \psi$.

When sentences are not only mutually exclusive but also jointly exhaustive, they are opposed in the second way described above: since they cannot both be false, one or the other is bound to hold and there is no room for a third alternative. We will say that two sentences for which this is so are ***contradictory***. Contradictory sentences—like The glass is full and The glass is not full—are bound to have opposite truth values. We will combine the notation for the two relations that make up this idea and write $\varphi \otimes \psi$ to say that φ and ψ are contradictory (using the symbol ***circled times***). Although our use of the term contradictory is the standard one in discussions of deductive logic, in ordinary speech, it is often applied to sentences that are only mutually exclusive. In particular, when a claim is said to be "self-contradictory," what is meant is that part of what it says excludes something else it says. Such a sentence will not contradict itself in the sense in which we will use the term because that would require that it be both true and false in each possible world. (Being true and false in each possible world is a problem only if there are possible worlds, but that's an assumption we will make.)

Contradictoriness inherits the symmetry of exhaustivness and exclusion. We have just seen that reflexivity fails for it in an even stronger way than for them, and $\otimes$ also has a special property that implies a similarly strong failure of transitivity.

Symmetry of $\otimes$. If $\varphi \otimes \psi$ then $\psi \otimes \varphi$ (for any sentences φ and ψ).

Doubling law for $\otimes$. If $\varphi \otimes \psi$ and $\psi \otimes \chi$, then $\varphi \Leftrightarrow \chi$ (for any sentences φ, ψ, and χ).

The second law follows from the alternation law for $\times$ and $\bigcirc$. It is alo easy to understand directly: contradictory sentences have opposite truth values and taking the opposite (in this sense) twice over returns you to where you started. Although this sort of property can be found for other relations (mirror image of is one), there is no standard name for the particular form it takes here. But an operation which is undone when repeated a second time (like the operation of reversing course) is known as an ***involution*** (in one sense of the term), and the doubling law tells us that the operation of moving from the content of a sentence to the content of a sentence contradictory to it is an involution.

The four basic deductive relations between two sentences that we have considered are shown in the following table:

Relation	*holds when there is no possible world in which sentences have these values:*	
φ implies ψ ($\varphi \Rightarrow \psi$)	φ is **T**	ψ is **F**
φ is implied by ψ ($\varphi \Leftarrow \psi$)	φ is **F**	ψ is **T**
φ and ψ are mutually exclusive ($\varphi \times \psi$)	φ is **T**	ψ is **T**
φ and ψ are jointly exhaustive ($\varphi \bigcirc \psi$)	φ is **F**	ψ is **F**

These are the only relations that can be defined by ruling out a specific pattern of truth values for two sentences because there are only four such patterns. Ruling out more than one pattern does not give us many more relations. If we rule out the first two patterns, we say $\varphi \Leftrightarrow \psi$, and if we rule out the last two patterns, we say $\varphi \otimes \psi$; but, if we were to rule out any other pair of patterns, we would simply rule out a truth value for one of the sentences in all possible worlds, and ruling out three patterns would leave just one pattern and would specify the truth values of both sentences. While the idea of a sentence that cannot be false or cannot be true is an important one, it is a property of the single sentence rather than a relation between two. So, in one sense, the six relations for which we have terminology are the only ones possible.

When none of these relations hold between a pair of sentences φ and ψ—that is, when each of four patterns of truth values for the two appears in some possible world—we will say that φ and ψ are ***logically independent***. Not only are logically independent sentences unordered by implication, they are not tied by any deductive relation.

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1.2.4. Truth conditions and propositions

In making the various comparisons we have been considering, what have needed to know about a sentence in order to compare it to others is its truth values in various possible worlds. We will describe this aspect of a sentence's meaning as its ***truth conditions***. That is, when we know, for any given possible world, whether or not a sentence is true, we know the conditions under which the sentence is true; and, when we know those conditions, we can tell whether or not it is true in a given possible world.

It will also be convenient to be able to speak of this kind of meaning or aspect of meaning as an entity in its own right. We will do this by speaking of the truth conditions of a sentence as encapsulated in the ***proposition*** expressed by the sentence. This proposition can be thought of as a way of dividing the full range of possible worlds into those in which the sentence is true and those in which it is false—i.e., into the possibilities it rules out and the ones it leaves open. And a proposition can be pictured as a division of an area representing the full range of possibilities into two regions.

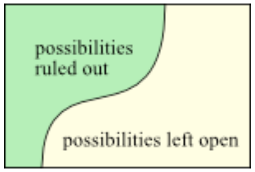

Fig. 1.2.4-1. A proposition dividing the full range of possible worlds into possibilities ruled out and possibilities left open.

Since a sentence that rules out more possibilities makes a stronger claim, the size of the region occupied by the possibilities it rules out can be thought to correspond to the strength of the claim it makes.

Relations between the propositions expressed by a pair of sentences can also be depicted in this way. The regions ruled out are shown shaded in the top row in Figure 1.2.4-2, and the regions left open are shown hatched in the bottom row.

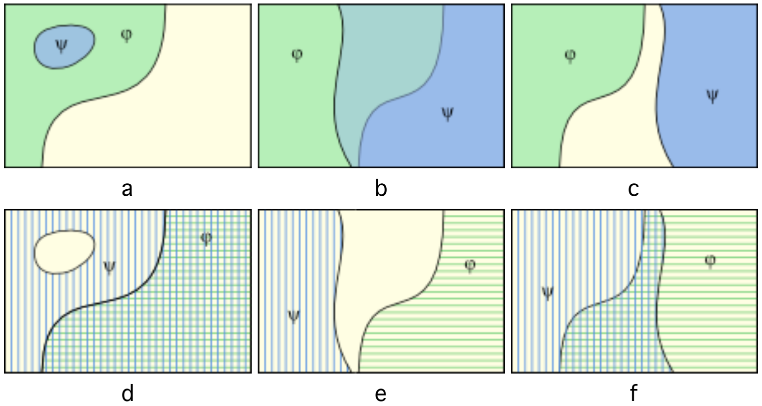

Fig. 1.2.4-2. Three relations between sentences φ and ψ. (a, d) φ implies ψ. (b, e) φ and ψ are mutually exclusive. (c, f) φ and ψ are jointly exhaustive. Regions ruled out by sentences are shaded in the top row—in green for φ and in blue for ψ. The regions left open are hatched in the bottom row—hatched horizontally for φ and vertically for ψ.

When $\varphi \Rightarrow \psi$ (see a and d above), the implied sentence ψ cannot rule out any possibility not already ruled out by the implying sentence φ, so the region ruled out by φ must include the region ruled out by ψ (and the region left open by φ must therefore be included in the region left open by ψ). If φ and ψ are mutually exclusive (see b and e above), there can be no overlap in the regions they leave open so the regions ruled out by the two must together cover the full range of possibilities. Here φ rules out all worlds at the left of the rectangle and ψ rules out all worlds at the right, with both ruling out a swath of worlds in the middle. Finally, when φ and ψ are jointly exhaustive, the situation is reversed (see c and f above): the regions left open by the two must together cover all possibilities so the regions they rule out cannot overlap. In the diagram a swath of worlds through the middle is left open by both.

1.2.5. Tautologies and absurdities

There are two extreme examples of truth conditions or propositions. A sentence that is true in all possible worlds says nothing. It has no informational content because it leaves open all possibilities and rules nothing out. For example, the weather "forecast" Either it will rain or it won't has no chance of being wrong and is, therefore, completely worthless as a prediction. We will say that such a sentence is a ***tautology***. Although there are many (indeed, infinitely many) tautologies, all express the same proposition; and the words that they use to express it are beside the point since they all say nothing in the end. In short, any two tautologies are logically equivalent. It will be convenient to establish a particular tautology and mark it by special notation. We will call this sentence ***Tautology*** and use the sign $\top$ (***down tack***) as our notation for it.

Notice that $\top$ is entailed by any set of premises because it will not add information to any set of sentences; and, for the same reason, its presence among the premises contributes nothing to the validity of an argument.

LAW FOR $\top$ AS A CONCLUSION. $\Gamma \Rightarrow \top$ (for any set Γ).

LAW FOR $\top$ AS A PREMISE. $\Gamma, \top \Rightarrow \varphi$ if and only if $\Gamma \Rightarrow \varphi$ (for any set Γ and sentence φ).

Although they are stated for $\top$, these laws will hold for all tautologies since they hold simply in virtue of the proposition expressed by $\top$.

These laws are different in character from the ones we have been considering since they concern the logical properties of a specific sort of sentence rather than the general principles governing logical relations. They are also a first sample of a common pattern in the laws of deductive reasoning that we will consider. Entailment is so central to deductive reasoning that an account of the role of a kind of sentence in entailment as a conclusion and as a premise will usually tell us all we need to know about it.

At the other extreme of truth conditions from tautologies are sentences that rule out all possibilities. The fact that such a sentence is the opposite of a tautology might suggest that it is maximally informative, but it sets an upper bound on informativeness in a different way: any genuinely informative sentence must say less. The ultimate aim of providing information is to narrow down possibilities until a single one remains, for this would provide a complete description of the history of the universe. To go beyond this would leave us with nothing because there is no way to distinguish among possibilities if all are ruled out. For example, the forecast It will rain, but it won't is far from non-committal since it stands no chance of being right, but it is no more helpful than a tautologous one.

Sentences that rule out all possibilities make logically impossible claims and we will refer to them as ***absurd***. As was the case with tautologies, any two absurd sentences are logically equivalent. Also, as with tautologies, we will introduce a particular example of an absurdity, named ***Absurdity***, and we use the special notation $\bot$ (the perpendicular sign, or ***up tack***) for it.

A simple law describes the role of absurdities as premises. We state it for the specific absurdity $\bot$.

LAW FOR $\bot$ AS A PREMISE. $\Gamma, \bot \Rightarrow \varphi$ (for any set Γ and sentence φ).

An argument with an absurdity among its premises is valid by default. Since its premises cannot all be true, there is no risk of *new* error no matter what the conclusion is.

We will eventually have a law for $\bot$ on the right of the sign $\Rightarrow$, but that will come only once we have assigned a broader meaning to that arrow. The idea of an entailment with an absurd conclusion is a fundamental one and cannot be restated in any simpler way using entailment. Since $\bot$ is bound to be false, we can have $\Gamma \Rightarrow \bot$ only it is not possible for the premises Γ to all be true. That is, we have the following:

BASIC LAW FOR INCONSISTENCY. A set Γ is inconsistent if and only if $\Gamma \Rightarrow \bot$.

This characterization of inconsistency will help us to concentrate on entailment. Laws governing inconsistency—and, by way of it, principles governing exclusion and mutual exclusiveness—will appear as principles governing valid arguments with the conclusion $\bot$. In fact, we are not really dispensing with the idea of inconsistency since an absurdity amounts to a sentence that forms an inconsistent set all by itself. The role of entailment will be to enable us to study the full range of inconsistent sets by way of this simple example.

1.2.6. Logical space and the algebra of propositions

Logic is concerned with propositions in the way mathematics is concerned with numbers, but propositions are not numbers. While numbers can be ordered in a linear way, the collection of propositions has a more complex structure. The series of examples of increasing strength we looked at in 1.2.2 did form a single chain, and we might have extended this chain to begin with ⊤ and end with ⊥. But it should be clear that we could have gone in many different directions to add content to these propositions—with ⊥ the only exception.

This metaphor of many directions suggests a space of more than one dimension; and, although the structure of a collection of propositions differs not only from the 1-dimensional number line but also from the structure of ordinary 2- or 3-dimensional space, spatial metaphors and diagrams can help in thinking its structure. These metaphors and can be associated with the term ***logical space*** that was introduced by the philosopher Ludwig Wittgenstein (1889-1951).

We will actually use two different sorts of spatial metaphor. One is the metaphor used in 1.2.4 to depict propositions. In it, possible worlds are the points of logical space, and propositions determine regions in the space by drawing a boundary between the possibilities they rule out and the ones they leave open. But we use a different metaphor when we speak of increasing strength in many different directions. According to this second metaphor, propositions are points in space rather than regions and possible worlds function behind the scenes as something like the dimensions of the space. If we were to apply this idea in any very realistic way, the space would have too many dimensions to be visualized, but in artificially simple cases this sort of space can be depicted by a figure in ordinary 2- or 3-dimensional space.

Let's begin to look further at these ideas by considering an very simple example of the first sort of logical space. Suppose there were only 4 possible worlds. A proposition will either rule out or leave open each of these possibilities. Figure 1.2.6-1 illustrates two such propositions.

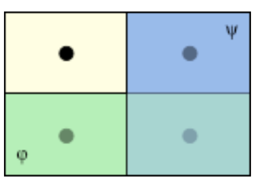

Fig. 1.2.6-1. The possibilities (the shaded bottom and right halves) that are ruled out by two propositions.

Each of these propositions rules out two of the four possibilities (in the shaded areas) and leaves open two others. The proposition expressed by the sentence φ rules out the two possibilities at the bottom of the diagram and the one expressed by ψ rules out the ones at the right. As a result both rule out the possible world in the lower right of the diagram and neither rules out the one in the upper left.

Of course, these are not the only propositions that can be expressed given this range of possibilities. A proposition has two options for each possible world: it may rule it out or leave it open. With 4 possible worlds this means that there are $2 \times 2 \times 2 \times 2 = 16$ propositions in all, and 6 of these rule out two possible worlds.

We can illustrate all 16 of these propositions by using a logical space of the second sort. Figure 1.2.6-2 depicts (in two dimensions) a 3-dimensional figure that is one possible representation of a 4-dimensional cube. It is labeled to suggest what sorts of sentences might express these propositions.

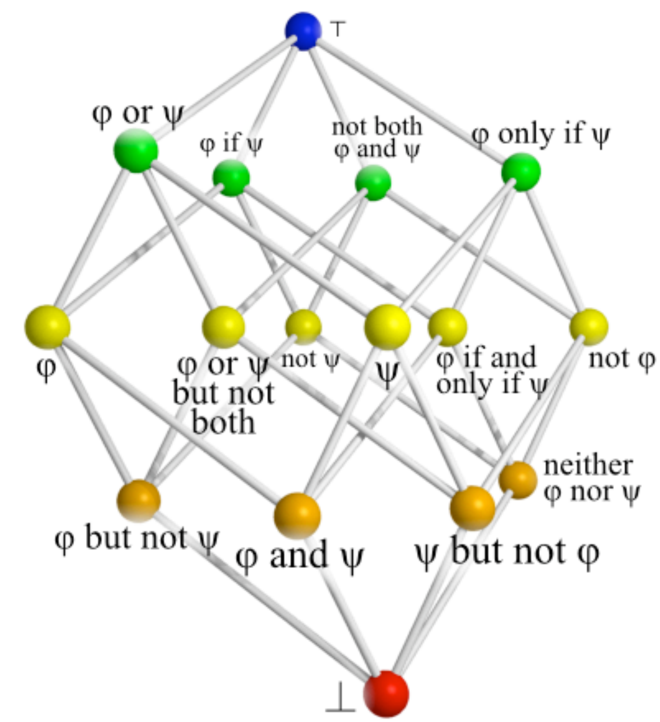

Fig. 1.2.6-2. The sixteen propositions when there are 4 possible worlds.

You can imagine that the propositions φ (which appears at the left) and ψ (near the center) are the two propositions depicted in Figure 1.2.6-1.

The levels in the structure correspond to grades of strength, with Absurdity at the bottom ruling out all possible worlds and Tautology at the top ruling out none. A line connects propositions that differ only with respect to one possible world. The proposition lower in the diagram rules out this world and the one above it leaves the world open, so the lower proposition implies the one above it. Each of the four propositions immediately above Absurdity then leaves open just one possible world. Lines connecting propositions that differ with respect to a given world are parallel (in the 3-dimensional figure, not in its 2-dimensional projection); and, in this sense, the worlds can be thought of as the dimensions on which the content of propositions can vary.

The other comparisons of content we have considered are depicted here, too, but a little less clearly. Diametrically opposite propositions are contradictory. φ and not φ are examples, and so are φ or ψ and neither φ nor ψ. Any pair of propostions that imply the corresponding members of a contradictory pair are mutually exclusive, so the mutually exclusive propositions are the ones that lie on or below a diagonal in this sense. Similarly, the jointly exhaustive pairs lie on or above a diagonal.

The relation between the two sorts of diagram can be seen by replacing each proposition in Figure 1.2.6-2 by its representation using a diagram of the sort illustrated in Figure 1.2.6-1. Putting the two sorts of illustration together in this way gives us the following picture of the same 16 propositions.

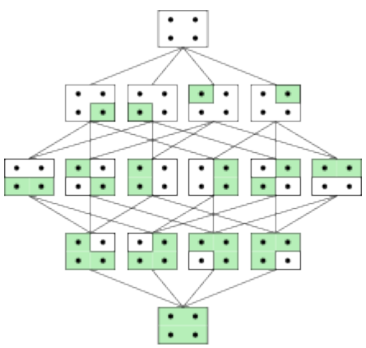

Fig. 1.2.6-3. The propositions generated by 4 possible worlds depicted as regions in one logical space (the repeated rectangle) and as points in another (the overall diagram).

The spacing of the nodes differs between Figures 1.2.6-2 and 1.2.6-3 but the left-to-right order at each level is the same and the regions associated with φ and ψ are the same as those depicted in Figure 1.2.6-1.

The whole structure of Figure 1.2.6-2 can be seen as a complex diamond formed of four diamonds whose corresponding vertices are linked. A simple diamond is the structure of the 2 × 2 = 4 propositions we would have with only 2 possible worlds. The structure in Figure 1.2.6-2 doubles the number of possible worlds and squares the number of propositions. If we were to double the number of possible worlds again to 8, we would square the number of propositions to get 256. The structure they would form could be obtained by replacing each node in the structure of Figure 1.2.6-2 by a small structure of the same form and replacing each line by a bundle of 16 lines connecting the corresponding nodes.

To get a sense of the structure of the set of propositions for a realistically large set of possible worlds, imagine carrying out this process over and over again. The result will always have an upper and lower limit (⊤ and ⊥) and many different nodes on each of its intermediate levels. As the number of possible worlds increases the distribution of possible worlds among the various degrees of strength (which is 1, 4, 6, 4, 1 in Figure 1.2.6-2) will more and more closely approximate a bell curve. But the bell shape of the curve will also narrow significantly, and bulk of the propositions will be found in intermediate degrees of strength. In short, the shape of the space of propositions departs further and further from a single line with ⊤ and at the bottom ⊥ as this space gets closer to a realistic degree of complexity.

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1.2.s. Summary

1 The relation of entailment concerns the possibilities of truth and falsity for premises and conclusions; that is, it concerns the truth values of these sentences in various possible worlds. The possibilities in question are logical possibilities, which may be understood as the situations whose description is permitted by the semantic rules of the language.

2 Entailment by a single premise, or implication, is a relation between sentences that orders them by their content. More precisely, $\varphi \Rightarrow \psi$ when φ says everything that is said by ψ. When sentences imply each other, they say the some thing, and we say they are equivalent, a relation for which we use the sign $\Leftrightarrow$. When $\varphi \Rightarrow \psi$ but these sentences are not equivalent, φ says more than ψ and we will often say that φ makes a stronger claim and ψ a weaker one.

Implication is a reflexive and transitive relation. The latter property tells us that implication can be transferred by implication: implication is preserved if we weaken the conclusion to something it implies or strengthen the premise to something that it is implied by. Equivalence is reflexive and transitive and also symmetric, but it is not preserved when sentences are strengthened or weakened.

3 Sentences can also be compared to describe differences in what they say. Sentences that cannot both be true are mutually exclusive (a relation for which we use the sign $\times$). The claims made by such sentences are opposite but opposite in a way that permits a third alternative. Sentences which are complementary in the sense that each must be true if the other is false are jointly exhaustive (for which our notation is $\bigcirc$). When these two relations both hold, sentences are contradictory (and we use the combined sign $\otimes$). Contradictory sentences always have opposite truth values make claims that are opposite in a way that permits no third alternative. Sentences that are neither mutually exclusive nor jointly exhaustive and neither or which implies the other are logically independent.

The relations $\times$, $\bigcirc$, and $\otimes$ are all symmetric. Mutual exclusion is preserved when sentences are strengthened; since this change from conclusion to premise is in a direction opposite to the entailment arrow, we will say that mutual exclusion is contravariant. On the other hand, joint exhaustiveness is preserved when sentences are weakened, and we will say it is covariant. Contradictoriness is not preserved when sentences are either strengthened or weakened, nor is it transitive; but it satisfies a doubling law to the effect that anything contradictory to something contradictory to φ is equivalent to φ.

4 The deductive relations a sentence stands in depend on its truth values in various possible worlds. That is, they depend on its truth conditions. These truth conditions are encapsulated in the proposition it expresses, which can be thought of as a way of dividing all possibilities into those it rules out and those it leaves open. This means that a proposition can be depicted as a division of space into two regions.

5 At one extreme are tautologies, which rule out no possibilities and thus have no content. All tautologies are equivalent and we will distinguish one, Tautology, for which we use the notation $\top$. At the other extreme are sentences that rule out all possibilities. Such sentences are absurd and all are equivalent to the single representative Absurdity, for which we use the notation $\bot$.

An argument with a tautology as a conclusion, is always valid; but a tautologous premise contributes nothing to the validity of an argument. An argument with an absurd premise is always valid but by default since its premises cannot all be true. An argument with an absurd conclusion is valid when and only when its premises form an inconsistent set, and this will enable us to study inconsistency by way of entailment.

6 Although certain groups of sentences can be ordered linearly between $\bot$ and $\top$ as a series of claims with steadily increasing content, the full range of propositions expressed by sentences are better thought of as inhabiting a much more complex logical space. This space might be a space of possibilities with propositions appearing as ways of dividing the space into regions, or it might be a space that has as its points propositions themselves. Logical space in this second sense has a bottom in the proposition expressed by $\bot$ and a top provided by $\top$. When there are a significant number of possible worlds, there will be many more propositions with intermediate content than there are strong propositions near $\bot$ or weak ones near $\top$.

1.2.x. Exercise questions

1. Each of the following claims that a deductive relation holds between a pair of sentences. In each case, judge whether the claim is true and, if not, the describe a sort of possibility that shows it is not true. Briefly explain your answers. For example, The package will arrive sometime does not entail The package will arrive next week because the possibility that it will arrive before or after next week is ruled out by the conclusion but not by the premise. In answering, it is safe to understand these sentences all in the most straightforward way; you will miss the point of the exercise if you try to look for subtle or obscure possibilities.

- **a.** The package will arrive next Tueday $\Rightarrow$ The package will arrive next week
- **b.** The package will arrive next week $\Rightarrow$ The package will arrive next Tuesday
- **c.** The package will arrive next Tueday $\times$ The package will arrive next week
- **d.** The package will arrive next Tuesday $\times$ The package will arrive next Wednesday
- **e.** The package will arrive before next Tueday $\bigcirc$ The package will arrive after next Tuesday
- **f.** The package will arrive next Tuesday or before $\bigcirc$ The package will not arrive before next Wednesday
- **g.** The package will arrive after next Tuesday $\Leftrightarrow$ The package will arrive next Wednesday or later
- **h.** The bridge will open at the end of May $\Leftrightarrow$ The bridge will open before June
- **i.** The package will arrive before next Wednesday $\otimes$ The package will arrive after next Wednesday
- **j.** The bridge will open before June $\otimes$ The bridge will open in June or later or never at all

2. Some of the following claims about deductive relations hold for any sentence φ, some for no sentence φ, and others hold only if φ is a tautology or only if it is absurd. In each case, say which is so and explain your answer.

a. $\varphi \Rightarrow \varphi$	**b.** $\varphi \Rightarrow \top$	**c.** $\varphi \Rightarrow \bot$
d. $\varphi \Leftarrow \varphi$	**e.** $\varphi \Leftarrow \top$	**f.** $\varphi \Leftarrow \bot$
g. $\varphi \bigcirc \varphi$	**h.** $\varphi \bigcirc \top$	**i.** $\varphi \bigcirc \bot$
j. $\varphi \times \varphi$	**k.** $\varphi \times \top$	**l.** $\varphi \times \bot$
m. $\varphi \Leftrightarrow \varphi$	**n.** $\varphi \Leftrightarrow \top$	**o.** $\varphi \Leftrightarrow \bot$
p. $\varphi \otimes \varphi$	**q.** $\varphi \otimes \top$	**r.** $\varphi \otimes \bot$

3. The headings at the left of the table give information about the relation of φ and ψ and those at the top give information about the relation of ψ and χ. Fill in cells of the table by indicating what, if anything, you can conclude in each case about the relation of φ and χ. Nearly all the cases where you can conclude something are covered by laws stated in this section; however, this exercise does not ask you to remember or look up the laws but instead to think through each case afresh. Since there are a number of symmetries, you will find that some questions are really being asked several times over.

	$\psi \Rightarrow \chi$	$\psi \Leftarrow \chi$	$\psi \Leftrightarrow \chi$	$\psi \times \chi$	$\psi \bigcirc \chi$	$\psi \otimes \chi$
$\varphi \Rightarrow \psi$						
$\varphi \Leftarrow \psi$						
$\varphi \Leftrightarrow \psi$						
$\varphi \times \psi$						
$\varphi \bigcirc \psi$						
$\varphi \otimes \psi$						

1.2.xa. Exercise answers

1. a. The package will arrive next Tueday entails The package will arrive next week because the package arriving next Tuesday is one of ways for it to be true that it arrives next week
   b. The package will arrive next week does not entail The package will arrive next Tuesday because the premise would still be true if it arrived another day next week
   c. The package will arrive next Tuesday and The package will arrive next week are not mutually exclusive because both will be true if it does arrive next Tuesday
   d. The package will arrive next Tuesday and The package will arrive next Wednesday are mutually exclusive since the package cannot arrive both days
   e. The package will arrive before next Tueday and The package will arrive after next Tuesday are not jointly exhaustive since both will be false if it arrives on next Tuesday
   f. The package will arrive next Tuesday or before and The package will not arrive before next Wednesday are jointly exhaustive because, if the second is false—i.e., if it does arrive before next Wednesday—then the first must be true
   g. The package will arrive after next Tuesday is equivalent to The package will arrive next Wednesday or later because arriving next Wednesday or later than that are the two ways in which a package could arrive after next Tuesday
   h. The bridge will open at the end of May is not equivalent to The bridge will open before June since it is not now the end of May so the bridge could open before June by opening even earlier than the end of May
   i. The package will arrive before next Wednesday and The package will arrive after next Wednesday are not contradictory because both will be false if it arrives on next Wednesday
   j. The bridge will open before June and The bridge will open in June or later or never at all are contradictory because opening before June, opening in June, opening later than June, and not opening at all exhaust all possibilities and are mutually incompatible
2. a. $\varphi \Rightarrow \varphi$ holds always because φ cannot fail to be true if it is true
   b. $\varphi \Rightarrow \top$ holds always because $\top$ cannot fail to be true no matter what φ is like
   c. $\varphi \Rightarrow \bot$ holds only when φ is absurd because, if there is any possibility of φ being true, there is a possibility of $\bot$ being false when φ is true
   d. $\varphi \Leftarrow \varphi$ holds always because the truth of φ is guaranteed by its own truth
   e. $\varphi \Leftarrow \top$ holds only when φ is a tautology because if there is any possibility of φ being false, there is a possibility of it being false when $\top$ is true
   f. $\varphi \Leftarrow \bot$ holds always because there is no possibility of $\bot$ being true so no possibility of φ being false when $\bot$ is true
   g. $\varphi \bigcirc \varphi$ holds only when φ is a tautology because if there is any possibility of φ being false, it does not, together with itself exhaust all possibilities
   h. $\varphi \bigcirc \top$ holds always becuase $\top$ covers all possibilities by itself, so it certainly exhausts them when taken together with φ
   i. $\varphi \bigcirc \bot$ holds only when φ is a tautology becuase, since $\bot$ leaves open no possibilities, it contributes nothing to exhausting them all and φ must do that all by itself
   j. $\varphi \times \varphi$ holds only when φ is absurd because, unless φ rules out all possibilities, there will be a possibility of it being true along with itself
   k. $\varphi \times \top$ holds only when φ is absurd because, since $\top$ is bound to be true, any possibility of φ being true will be a possibility of both being true
   l. $\varphi \times \bot$ holds always because, since $\bot$ cannot be true, it cannot be true together with any sentence (even itself)
   m. $\varphi \Leftrightarrow \varphi$ holds always since a sentence must have the same truth value as itself
   n. $\varphi \Leftrightarrow \top$ holds only when φ is a tautology because, if φ is bound to have the same truth value as a tautology, it must be one
   o. $\varphi \Leftrightarrow \bot$ holds only when φ is absurd because, if φ is bound to have the same truth value as an absurd sentence, it must be one
   p. $\varphi \otimes \varphi$ never holds because no sentence can be both true and false at the same time
   q. $\varphi \otimes \top$ holds only when φ is absurd because φ is bound to be false if its value is opposite that of a sentence that is bound to be true
   r. $\varphi \otimes \bot$ holds only when φ is a tautology because φ is bound to be true if its value is opposite that of a sentence that is bound

to be false

3. The appearance of "—" in a cell in the table below indicates that nothing can be concluded in general about the relation between φ and χ.

	ψ ⇒ χ	ψ ⇐ χ	ψ ⇔ χ	ψ × χ	ψ ◯ χ	ψ ⊗ χ
φ ⇒ ψ	φ ⇒ χ	—†	φ ⇒ χ	φ × χ	—†	φ × χ
φ ⇐ ψ	—*	φ ⇐ χ	φ ⇐ χ	—*	φ ◯ χ	φ ◯ χ
φ ⇔ ψ	φ ⇒ χ	φ ⇐ χ	φ ⇔ χ	φ × χ	φ ◯ χ	φ ⊗ χ
φ × ψ	—*	φ × χ	φ × χ	—*	φ ⇒ χ	φ ⇒ χ
φ ◯ ψ	φ ◯ χ	—†	φ ◯ χ	φ ⇐ χ	—†	φ ⇐ χ
φ ⊗ ψ	φ ◯ χ	φ × χ	φ ⊗ χ	φ ⇐ χ	φ ⇒ χ	φ ⇔ χ

In cells marked with †, the fact that no relations hold in general can be seen by noting that if ψ is a tautology, the relations between it and φ and χ will hold no matter what sentences they are. And, similarly, in the cells marked with *, the relations between ψ and each of the other sentences will hold no matter what they are if ψ is absurd. There are various considerations which can be used to show that nothing more can be said in other cases, but it is probably easiest just to confirm for yourself that no further possibilities for the truth values of φ and χ are ruled out by the given information.

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