8.5. Proofs by choice and proofs of existence

8.5.0. Overview

Although formal proofs for disjunction involve some new ideas, these are mainly recombinations of ideas used for disjunction and universals.

8.5.1. The role of existentials in entailment

The role of existentials in entailment is analogous to the role of disjunctions in much the way the role of universals is analogous to that of conjunctions.

8.5.2. Derivations for existentials

Derivation rules for existentials then also exhibit an analogy with those for disjunction, with two basic rules supplemented by an often useful attachment rule.

8.5.3. First-order logic

This completes our account of entailment for first-order logic; higher-order logics concern quantifiers the generalize from predicates rather than individual terms but no complete system of derivations can be given for them.

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8.5.1. The role of existentials in entailment

As has been the case elsewhere in this chapter, we will be able to rely on our discussion of universals to simplify our discussion of principles of entailment for existentials. The differences between the principles governing universal and existential quantifiers will, in most cases, be analogous to differences between the principles for conjunction and disjunction. The laws of entailment for the universal quantifiers were modifications of laws for conjunction, and the rules for the existential quantifiers will be based in a similar way on rules for disjunction. Our planning rule for existential sentences as conclusions will take a different form from that for disjunctions, but even it is analogous to a rule that could have been used for that connective.

These analogies derive from the truth conditions for the unrestricted existential, which follow the conditions for disjunction in precisely the way the conditions for the universal follow those for conjunction. A sentence ∃x θx is true in a structure if and only if it has at least one true instance in a language expanded by the range **R** of that structure. In other words, an existential claim behaves like a disjunction of its instances when these instances are taken in a language that incorporates a term for each reference value. However, as was the case with the universal, the set of instances is not be the same for all structures, so we cannot employ any definite information about what the instances of an existential sentence are when stating general laws of entailment.

First, we will look at the role of an unrestricted existential as a premise. A disjunctive premise may be used to draw a conclusion by way of a proof by cases. In such a proof, we suppose in turn that each of the disjuncts is true and argue for the conclusion in each case. A comparable way of arguing from an existential would be to establish many arguments, each one considering an instance of the existential as one case. Since we cannot associate the existential with any definite set of instances, we cannot consider each of these arguments individually, so we must use adapt a device from our treatment of the universal: we need to set out the indefinitely many arguments by offering a general pattern. That is, to use an existential premise to draw a conclusion, we draw the conclusion from one instance of the existential in a way that sets a pattern for all other instances.

This sort of argument may be called a ***proof by choice***. To see how proofs by choice work, consider the two arguments below.

Anyone who worked late got overtime	*Anyone who worked late got overtime*
If anything broke down, Tom worked late	*If anything broke down, Tom worked late*
Something broke down	*X broke down*
Tom got overtime	*Tom got overtime*

The validity of the argument on the left can be traced to the validity of the one on the right. In the latter, we use the premise *X broke down* in place of the existential *Something broke down*, so we argue for the conclusion from an instance of the existential.

Of course, being able to draw a conclusion when using an instance of an existential does not, by itself, insure that we can draw the same conclusion using the existential. For example, given appropriate premises we can conclude *Larry will be happy* from *Larry will win a lottery*; but this does not insure that we can conclude *Larry will be happy* from *Someone will win a lottery*. So, to base the validity of the argument on the left on the validity of the one on the right, we will need to insure that whatever we can conclude using the instance *X broke down* also could be concluded using any other instance. That is, we need to insure that the argument on the right has a sort of generality. If we were employing proof by choice in a formal proof, we might signal this generality by saying, "Let *X* be anything that broke down." This would declare our intention to begin with the choice of an instance but without employing any special information about instance we have chosen.

It should be clear that there is some kinship between proofs by choice and the general arguments we have used to establish universal conclusions. Both the reasons for and the nature of this kinship can be brought out in another way by considering a second pair of arguments. In these arguments, the key premises of the earlier pair have been absorbed in the conclusion:

Anyone who worked late got overtime	*Anyone who worked late got overtime*
If anything broke down, Tom worked late	*If anything broke down, Tom worked late*
Something broke down → *Tom got overtime*	∀x (x *broke down* → *Tom got overtime*)

The validity of the argument on the left is tied to the validity of the left-hand argument of the earlier pair by the law for the conditional as a conclusion, and the validity of the right-hand arguments in both pairs are tied by that law and the law for the universal as a conclusion.

Consequently, the relation between the earlier two arguments can be understood by way of the relation between the new pair. And the new pair of arguments are clearly tied since their conclusions are equivalent by one of the confinement principles discussed in 8.1.4. The different forms taken by these conclusions show us that the inference ticket to *Tom got overtime* from *Something broke down*, can be based on a sort of general inference ticket to *Tom got overtime* from the instance *X broke down*. That is, to move from *Something broke down* to *Tom got overtime* we need a way of passing from *X broke down* to *Tom got overtime* that can be generalized to work for any instance.

Recalling the test we used for the generality of arguments in the case of the universal quantifiers, we can expect our analysis of the role of an existential as a premise to make reference to a term that is ***parametric*** in an appropriate sense. We will want a term that has no special connection to any elements of the argument—to any of its premises, its conclusion, or the predicate that the existential premise claims to be exemplified. So suppose the term a is unanalyzed term and does not appear in the set Γ, the sentence φ, or the existential ∃x θx, and consider the two arguments

$$\Gamma, \exists x\ \theta x\ /\ \varphi$$
$$\Gamma, \theta a\ /\ \varphi.$$

We can argue that each is valid if and only if the other is if we can show that each is divided by a structure if and only if the other is. If a structure S divides the premises and conclusion of the first, it will assign θ a non-empty extension, and we can form a structure S′ that divides the second argument by assigning a value in this extension to the term a. We can assign this extension to the term a without disturbing the interpretation of other vocabulary since, as a parameter, the term a stands apart from this vocabulary. So S′ will give θ the same extension as S does, and it will make θa true without changing the truth values of φ and the members of Γ. On the other hand, any structure dividing the second argument will give θ a non-empty extension (because the value of the term a will be in it) so this structure will make $\exists x\ \theta x$ true and also divide the first argument. Thus we will have a structure dividing one argument if and only if we have a structure dividing the other, and each argument is valid if and only if the other is. This gives us our ***law for the unrestricted existential as a premise***: if a is an unanalyzed term that does not appear in Γ, φ, or $\exists x\ \theta x$, then $\Gamma, \exists x\ \theta x \Rightarrow \varphi$ if and only if $\Gamma, \theta a \Rightarrow \varphi$.

We turn next to the role of existentials as conclusions. First, recall our account of the role of disjunction as conclusion: $\Gamma \Rightarrow \varphi \vee \psi$ if and only if $\Gamma, \overline{\varphi} \Rightarrow \psi$. We could have avoided the asymmetric treatment of the two components if we had resorted to an even heavier use of negation; applying the idea behind IP to the right side of the law, we get this: $\Gamma \Rightarrow \varphi \vee \psi$ if and only if $\Gamma, \overline{\varphi}, \overline{\psi} \Rightarrow \bot$. That is, a disjunction is a valid conclusion if and only if we can reduce to absurdity the supposition that its components are both false. We are often able to avoid this use of *reductio* arguments in the case of disjunction, but it would be awkward to do so in the case of the existential.

A strict analogue for the existential of this rule for disjunction would be to say that we can conclude an existential $\exists x\ \theta x$ from premises Γ if and only if we can reduce to absurdity the result of adding denials of all the instances of $\exists x\ \theta x$ to Γ. But there is no definite set of instances, so we cannot take this approach literally. We had a related problem in dealing with the universal as a premise, for the analogy with conjunction suggested that a universal premise might be replaced by the set of all its instances. And the problem there provides a solution here: we can say that an existential $\exists x\ \theta x$ follows from premises Γ if and only if we can reduce to absurdity the result of adding $\forall x\ \overline{\theta x}$ to Γ. This will be our ***law for the existential as a conclusion***.

$$\Gamma \Rightarrow \exists x\ \theta x \text{ if and only if } \Gamma, \forall x\ \overline{\theta x} \Rightarrow \bot$$

In it, we do not explain the role of the existential as a conclusion directly, but instead make a connection with the role of the universal as a premise. Like the awkwardness in handling disjunction, this can be traced to the fact that we maintain at most one goal. (A law for $\exists$ that makes no reference to $\forall$ is easier to state for relative exhaustiveness; see appendix B for the form it would take.)

This principle for the existential is closely related to the equivalence obversion, for (choosing one of the cases of obversion covered by the bar notation) we have

$$\neg\ \forall x\ \overline{\theta x} \Leftrightarrow \exists x\ \theta x.$$

This equivalence says that an existential is equivalent to the denial of a corresponding negative generalization. And the law for existential conclusions says that we can conclude a claim of exemplification if we can reduce a negative generalization to absurdity—that is, if we can do what would be needed to establish the denial of one.

This way of drawing an existential conclusion is called a ***non-constructive proof***. It enables us to establish a claim of exemplification without ever describing a particular example. (The use of the term *construction* here can be traced to geometry, where claims of exemplification are typically established by a geometric construction of the figure that is claimed to exist.) Non-constructive proofs of exemplification have been common in modern mathematics but have also been controversial. The doubts about them have not usually been doubts about their validity (though Brouwer, who was mentioned in 3.1.3, could be said to have doubted that). Instead these doubts have concerned the respect accorded such proofs, with some mathematicians feeling that the methods used in them render them undeserving of the respect that might be given to them due to the importance of their conclusions. The feature of non-constructive proofs that lies behind these doubts is a weakness that is granted even by those who accept such proofs happily: because they do not produce an example, they may provide little insight into the reasons why a claim of exemplification is true.

The deepest concerns about non-constructive proof are focused on arguments about abstract and, especially, infinite structures, and even Brouwer thought that non-constructive proofs were valid for reasoning about ordinary claims about the world of sense experience. Still, the indirection and uninformativeness of non-constructive arguments can be felt with ordinary reasoning and is often unnecessary, so it is worthwhile considering the alternative. A ***constructive proof*** of a claim of exemplification establishes the claim by first producing an example of the sort that is claimed to exist. The move from example to claim of exemplification appears formally as a step from an instance of an existential to the existential itself, and it is neatly captured in a principle of entailment commonly known as ***existential generalization***: $\theta\tau \Rightarrow \exists x\ \theta x$ for any term τ.

The conclusion of this entailment is not a generalization in the sense in which we have been using the term. But it may be said of someone who is making heavy use of words like *something* and *someone* that he is "speaking in generalities" and is not being specific. The principle of existential generalization is a license to move from a specific claim to a generality of an existential sort. We cannot rely on this principle alone—the issue of non-constructive arguments would never have arisen if we could—but it does provide a useful supplement in the way the principle of weakening supplements the law for disjunction as a conclusion. And, like weakening, we will count existential generalization as an attachment principle. (What is attached? In form, we could say it is the existential quantifier; in what is said, it is the other instances of the conclusion, the other ways in which it could be true.)

This completes our suite of principles for the unrestricted existential. Collected together, they are as follows:

Law for the unrestricted existential as a premise. For any unanalyzed term a appearing in neither Γ, Σ, nor $\exists x\ \theta x$, we have:

$$\Gamma, \exists x\ \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \theta a \Rightarrow \Sigma.$$

Law for the unrestricted existential as a conclusion.

$$\Gamma \Rightarrow \exists x\ \theta x \text{ if and only if } \Gamma, \forall x\ \overline{\theta x} \Rightarrow \bot.$$

Law of existential generalization.

$$\theta\tau \Rightarrow \exists x\ \theta x \text{ for any term } \tau.$$

The first of these is the principle underlying proofs by choice (in which we choose an example a of the sort claimed by the existential), the second underlies non-constructive proofs, and the third underlies constructive proofs.

There is a corresponding set of principles for the restricted existential. These can be reached by way of restatements of a restricted existential in unrestricted form and thus by way of principles for conjunction. The process is more straightforward than in the case of the restricted universal, so we will only consider the results, which are the following:

Law for the restricted existential as a premise. For any unanalyzed term a appearing in neither Γ, φ, nor $(\exists x\!: \rho x)\ \theta x$, we have:

$$\Gamma, (\exists x\!: \rho x)\ \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \rho a, \theta a \Rightarrow \Sigma.$$

Law for the restricted existential as a conclusion.

$$\Gamma \Rightarrow (\exists x\!: \rho x)\ \theta x \text{ if and only if } \Gamma, (\forall x\!: \rho x)\ \overline{\theta x} \Rightarrow \bot.$$

Law of restricted existential generalization.

$$\rho\tau, \theta\tau \Rightarrow (\exists x\!: \rho x)\ \theta x \text{ for any term } \tau.$$

Again these provide the basis for proofs by choice and for both non-constructive and constructive proofs of exemplification. The last says that we can establish a claim of existence if we can show of the value of τ both that it is in the domain of the existential and that it has the attribute. The first law says we can draw a conclusion from an existential if we conclude it from arbitrary choice of a value that is in the domain and has the attribute.

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8.5.2. Derivations for existentials

To implement the laws we have just been considering, we will again use ideas introduced in connection with universals. In particular, a proof by choice will be marked by a veil of ignorance flagged by a parameter, and it will have a supposition that sets out the example chosen. However, the complications that appeared with the rules for exploiting universals may be left with those rules, since we manage planning for an existential conclusion simply by passing the buck on to universals.

The two basic rules for the unrestricted existential are ***Proof by Choice*** (PCh) and ***Non-constructive Proof*** (NcP):

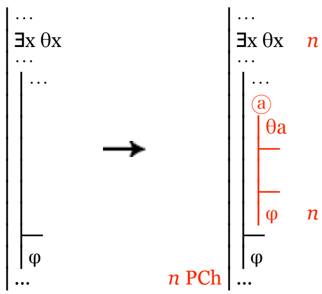


Fig. 8.5.2-1. Developing a derivation at stage *n* by exploiting an unrestricted existential; the parameter a is new to the derivation.

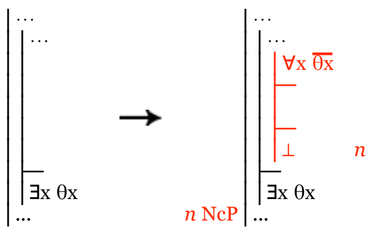


Fig. 8.5.2-2. Developing a derivation at stage *n* by planning for an unrestricted existential.

Notice that the existential is rendered inactive in the first rule. Also remember that the parameter that is used in this rule should be new to the derivation; that will insure that the supposition that is introduced represents the only information about this parameter that may be used in closing the gap. The second rule will often be a very indirect way of reaching an existential goal, and the following attachment rule, ***Existential Generalization*** (EG), will often simplify derivations considerably:

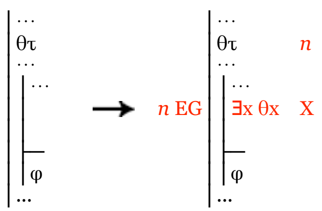
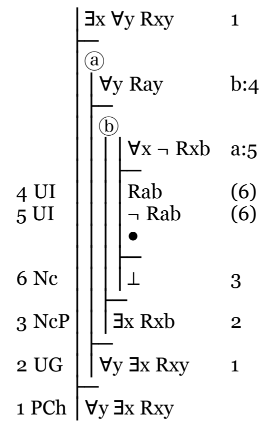
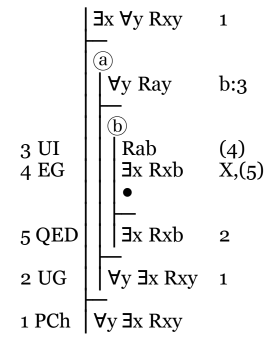


Fig. 8.5.2-3. Developing a derivation at stage *n* by adding an unrestricted existential that has an instance among the active resources.

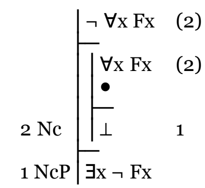
Although this is an attachment rule and therefore not part of the basic system, you should be as ready to use it as the two above.

Here are two derivations that illustrate these rules. Each shows that a claim of uniformly general exemplification implies the corresponding claim of general exemplification without a claim of uniformity. The derivation on the left uses a non-constructive proof of the existential that is set as the goal in stage 2 while the one on the right uses EG to give a constructive proof of this existential. Both derivations begin by exploiting the existential premise, but derivations for the same entailment could have been developed by planning for the initial conclusion first; and, when NcP is used, it would be possible to postpone the exploitation of the initial premise until after NcP is applied. (It would be a good exercise at this point to write down these other derivations for this argument.)





The savings here in length and complexity by using EG are typical of cases where it can be used. Since it can be used only when an existential is entailed by the resources, it will often be unavailable in derivations that fail, and NcP is required also in some derivations for valid arguments. A derivation showing the obversion principle ¬ ∀x Fx ⇒ ∃x ¬ Fx is simple example of this; EG cannot be applied because the premise does not entail any sentence ¬ Fτ from which we could generalize.



The rules for restricted quantifiers take on the same general forms. The two basic rules are ***Proof by Restricted Choice*** (PRCh) and ***Restricted Non-constructive Proof*** (RNcP):

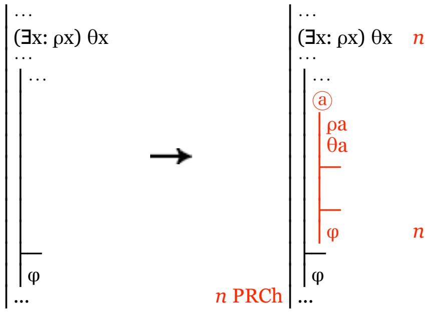


Fig. 8.5.2-4. Developing a derivation at stage *n* by exploiting a restricted existential; the parameter a is new to the derivation.

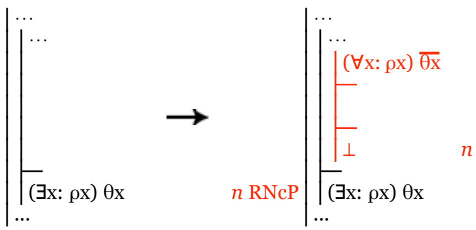


Fig. 8.5.2-5. Developing a derivation at stage *n* by planning for a restricted existential.

The exploitation rule introduces two suppositions; they stipulate of the example chosen not only that it have the attribute of the claim of exemplification but also that it be in the domain of this claim. An analogous move in an English proof would be to say, "Let *r* be a real number, and suppose it is between 0 and 1" as a way of exploiting the fact *Some real number is between 0 and 1*.

The analogue of EG for restricted existentials is the rule ***Restricted Existential Generalization*** (REG):

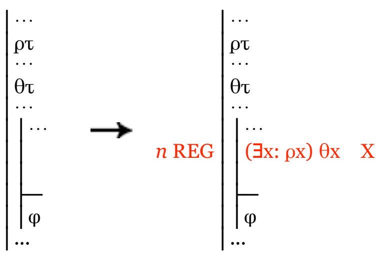


Fig. 8.5.2-6. Developing a derivation at stage *n* by adding a restricted existential whose domain and attribute predicates are found among the active resources applying to the same term.

As an example of REG, let us construct a derivation to show that *Every horse is a mammal* implies *Any head of a horse is a head of a mammal* (an entailment mentioned in 7.1.1 as being beyond the scope of Aristotle's syllogistic logic).

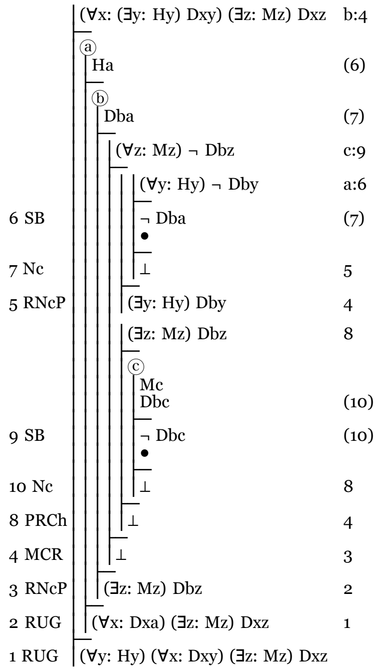
	(∀x: Hx) Mx	b:3
	ⓐ	
	(∃y:Hy) Day	2
	ⓑ	
	Hb	(3)
	Dab	4
3 SB	Mb	4
4 REG	(∃z:Mz) Daz	X,(5)
	•	
5 QED	(∃z:Mz) Daz	2
2 PRCh	(∃z:Mz) Daz	1
1 RUG	(∀x: (∃y: Hy) Dxy) (∃z: Mz) Dxz	

[H: λx (x *is a horse*); M: λx (x *is a mammal*); D: λxy (x *is a head of* y)]

Existential generalization is used at stage 4 and saves us having to enter (∀z: Mz) ¬ Daz as a supposition to be reduced to absurdity. That is a small simplification in this case; but REG can provide a more substantial simplification when it is used to provide an auxiliary resource for a detachment rule, as in the following derivation of an alternative analysis of *Any head of a horse is a head of a mammal* that treats *a horse* as marking a generalization with wide scope.

	(∀x: (∃y: Hy) Dxy) (∃z: Mz) Dxz	b:4
	ⓐ	
	Ha	(3)
	ⓑ	
	Dba	(3)
3 REG	(∃y: Hy) Dby	X,(4)
4 SB	(∃z: Mz) Dbz	(5)
	•	
5 QED	(∃z: Mz) Dbz	2
2 RUG	(∀x: Dxa) (∃z: Mz) Dxz	1
1 RUG	(∀y: Hy) (∀x: Dxy) (∃z: Mz) Dxz	

Here the alternative to REG is a use of MCR to exploit the premise for b after planning for (∃z: Mz) Dxz by NcP, and each of the two gaps opened by MCR would require some work to complete. Here is what a completed derivation along those lines would look like:



Although this is not a very natural argument for the entailment, it provides a good illustration of the basic rules for both restricted universals and restricted existentials; and this sort of approach could be unavoidable in the case of an argument that was not valid.

As was the case with universals, it is possible to capture the logical properties of restricted existentials by way of their restatement using unrestricted quantifiers. The rules for doing this are ***Restricted Existential Premise*** (REP) and ***Restricted Existential Conclusion*** (REC). The latter takes two forms since it needs to be applied to resources as well as goals in order to use EG in place of REG. The second form of REC is counted as an attachment rule since it and REP could not both be counted as progressive.

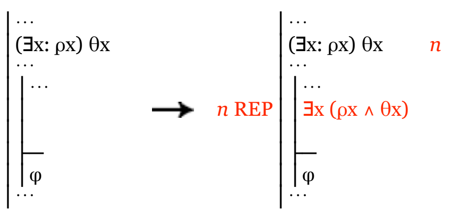


Fig. 8.5.2-7. Developing a derivation at stage *n* by restating a restricted existential resource.

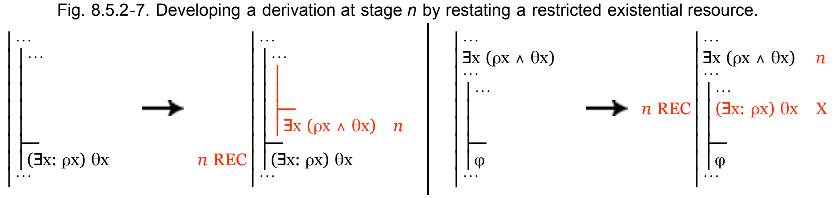


Fig. 8.5.2-8. Developing a derivation at stage *n* by restating a restricted existential goal or restating a resource as a restricted existential.

Each of PRCh, RNcP, and REG could be replaced by a use of one of these along with other rules—PCh followed by Ext, NcP followed by uses of UI together with MPT or CR, and EG preceded by Adj, respectively.

Arguments for the soundness and completeness of this system carry over from 7.7 without any new wrinkles. We solved all the key problems there, and a number are not even repeated here.

However, we cannot avoid the consequences of the failure of decisiveness. To find finite counterexamples whenever they exist, we would need to modify the rules for exploiting existential resources in the way the rule for planning for a universal goal was modified in 7.8.1. Without such rules, we will not reach dead-end open gap in any derivation whose resources contain a weak, though unrestricted, claim of general exemplification (e.g., the sentence of the form ∀x ∃y Rxy). We will label the modified rules ***Supplemented Proof by Choice*** (PCh+) and ***Supplemented Proof by Restricted Choice*** (PRCh+).

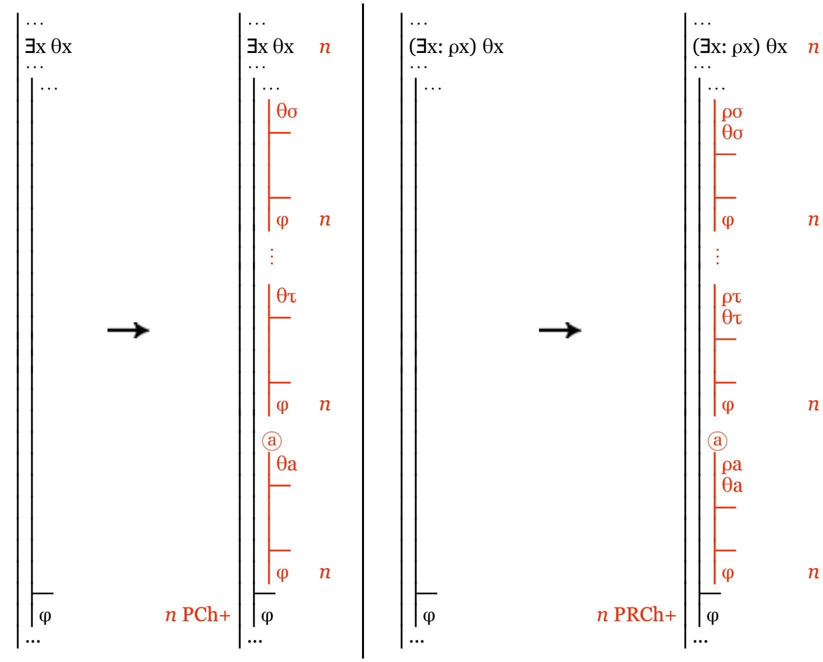
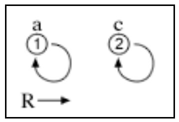


Fig. 8.5.2-9. Developing a derivation at stage *n* by exploiting an unrestricted or a restricted existential; the parameter a is new to the derivation and the terms σ, ..., τ include at least one from each current alias set for the gap

The following derivation illustrates these rules. It shows that a claim of general exemplification need not imply uniformity by finding a counterexample to the entailment ∀x ∃y Rxy ⇒ ∃y ∀x Rxy.

	∀x ∃y Rxy	a:2, c:9
	∀y ¬ ∀x Rxy	a:3, c:10
2 UI	∃y Ray	5
3 UI	¬ ∀x Rxa	4
	Raa	(7)
	•	
7 QED	Raa	6
	ⓒ	
	¬ Rca	
9 UI	∃y Rcy	12
10 UI	¬ ∀x Rxc	11
	Rca	
	(unfinished but will close)	
	∀x Rxc	12
	Rcc	
	¬ Rac	
	○	Raa, ¬ Rca, Rcc, ¬ Rac ⇏ ⊥
	⊥	14
14 IP	Rac	13
	•	15
15 QED	Rcc	13
	ⓔ	
	(unfinished)	
	Rec	13
13 UG+	∀x Rxc	12
	ⓓ	
	Rcd	
	(unfinished)	
	∀x Rxc	12
12 PCh+	∀x Rxc	11
11 CR	⊥	8
8 IP	Rca	6
6 UG+	∀x Rxa	5
	ⓑ	
	Rab	
	(unfinished)	
	∀x Rxa	5
5 PCh+	∀x Rxa	4
4 CR	⊥	1
1 NcP	∃y ∀x Rxy	



Although this is long and cumbersome, the development of the dead-end gap goes through precisely the steps you would need to go through in your own thinking to arrive the same counterexample:

The premise says that everything stands in relation R to something or other. So let's suppose we have an object a such that Raa. But we if we stop there, everything will stand in R to a and the conclusion will be true. So let's suppose we have a second object c that doesn't stand in R to a. Now c must stand in R to something if the premise is to be true and it can't stand in R to a, so let's suppose it stands in R to itself. Now, to make the conclusion false we must be sure that not everything stands in R to c, so we better suppose that a does not. So we've described a possible world containing objects a and c where Raa, ¬ Rca, Rcc, ¬ Rac; and that's enough to make the premise true and the conclusion false.

Developing the unfinished gaps would lead to other counterexamples. For example, the last open gap in this derivation explores the possibility of making the premise true by having a stand in R to another object b and it would, among other things, lead us to a counterexample in which each of a and b is stands in R to the other but neither stands in R to itself.

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8.5.3. First-order logic

Although we will go on to give some consideration to derivations for the description operator, our system of derivations is now essentially complete. It is intended to capture entailments that derive from truth-functional logic and the logical properties of identity, predication, and the quantifiers. This range of logical forms is the concern of ***first-order logic***. (Usage varies a little, and sometimes identity is not included; in that case, our subject is "first-order logic with identity.") The qualification *first-order* derives from the fact that we analyze quantification only over individuals and not over properties and relations. Thus we cannot analyze the sentence *Objects a and b are identical if and only if every property of one is a property of the other* and we cannot ask whether this sentence is a tautology. The representation of such ***higher-order*** quantification symbolically would present few new problems. We would need bindable variables that functioned syntactically as predicates, notation for complex predicates of predicates (with our quantifiers serving as simple predicates of predicates), and quantifiers applying to such predicates of predicates. This would give us ***second-order logic***. To go further, we might introduce quantification for predicates of predicates—and so on. If this process is continued to all (finite) orders, we end up with what is known as ***higher-order logic*** or ***(simple) type theory***.

While higher-order logic introduces nothing really new in its syntax, the account of entailment for it is a completely different game, and the new problems appear already with second-order logic. In particular, there can be no sound system for settling questions of validity for second-order logic that is even complete, much less decisive. Indeed, a full understanding of validity for second-order logic would provide a full understanding of all truths concerning positive integers. But it was shown by Kurt Gödel in the early 1930s that these truths cannot be captured by anything like a system of derivations. (This is the result mentioned in 7.7.1 as the basis on which Church showed that there could be no system of derivations for first-order logic that was decisive as well as sound and complete.)

So there is a reason for distinguishing the theory of first-order quantification, from higher-order logic. Frege's work did not make this distinction. The subject matter he addressed included the whole of what is now known as type theory because he was interested in connections with arithmetic, whose truths he wished to explain as logical tautologies. Although he provided what was essentially a complete account of validity for first-order logic, his treatment of other areas introduced inconsistencies. These were repaired shortly after (in the first decade of the 20th century) by Bertrand Russell, whose work led to the current conception of type theory. First-order logic came to be distinguished within type theory and was permanently set in its present form by Gödel when he showed that Frege's initial ideas provided a complete account of validity for this part of logic.

8.5.s. Summary

Existentials bear the kind of analogy to disjunctions that universals bear to conjunctions, and their role in entailment reflects this. Our principle for the unrestricted existential as premise says that the existential will support a proof by choice. This is a sort of proof by cases in which cases for each instance of the existential are handled not one by one but by using a parameter to consider a single instance that sets the pattern for all the rest. The pattern-setting instance can thus be thought of as an example, chosen in ignorance of its specific identity, of the sort that the existential claims to exist. There are two approaches to establishing an existential conclusion. Our general principle for the unrestricted existential as a conclusion uses the idea of non-constructive proof, in which a claim of exemplification is based on the reduction to absurdity of a corresponding negative universal. In a constructive proof, an existential conclusion is based on the proof of an instance, which thus "constructs" an example of the sort the existential claims to exist. Constructive proofs are supported by the attachment principle of existential generalization. There are analogous principles for the restricted existential as a premise and as a conclusion and of restricted existential generalization.

The laws for existential premises and conclusions are implemented in exploitation and planning rules using some ideas from the rules for universals. The principles for unrestricted existentials are implemented in the rules Proof by Choice (PCh), Non-constructive Proof (NcP), and Existential Generalization (EG); and, for the restricted existential, we have analogous rules of Proof by Restricted Choice (PRCh), Restricted Non-constructive Proof (RNcP), and Restricted Existential Generalization (REG). An alternative approach to the deductive properties of restricted existentials uses rules Restricted Existential Premise (REP) and Restricted Existential Conclusion (REC) to restate them using unrestricted quantifiers or conclude them from such restatements. Also as was the case with the universal quantifier, to uncover counterexamples to invalid arguments using finite ranges (when such counterexamples exist), we need supplemented forms of proof by choice and restricted choice, PCh+ and PRCh+.

The arguments for soundness and completeness also contain no new twists. The system we have now completed accounts for the entailments of what is known as first-order logic. That is, we consider quantification only over individuals and not over properties, properties of those properties, or any other second-order or higher-order entities. Although higher-order logic, or type theory, has attracted interest since Frege, it cannot be given a complete system of derivations.

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8.5.x. Exercise questions

1. Use the system of derivations to establish each of the following:

- **a.** ∃x Fx, ∀x (Fx → Gx) ⇒ ∃x Gx
- **b.** (∃x: Fx) Gx, (∀x: Gx) Hx ⇒ (∃x: Fx) Hx
- **c.** ∀x (Fx → Ga) ⇔ ∃x Fx → Ga
- **d.** Fa ⇔ (∃x: x = a) Fx
- **e.** (∃x: Fx) ∀y Rxy ⇒ ∀x (∃y: Fy) Ryx
- **f.** (∃x: Gx) Fx, ¬ Fa ⇒ (∃x: ¬ x = a) Gx
- **g.** ∀x (Fx → Ga),∀x (Ga → Fx), ∃x Fx ⇒ ∀x Fx
- **h.** *Everyone loves everyone who loves anyone*, *Someone loves someone* ⇒ *Everyone loves everyone*
- **i.** *Something is such that nothing other than it is done* ⇔ *At most one thing is done*

2. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, describe a structure that divides an open gap. You need not worry about infinite derivations.

- **a.** ∃x Fx, ∃x Gx ⇒ ∃x (Fx ∧ Gx)
- **b.** (∃x: Fx) Gx, (∃x: Fx) Hx, (∀x: Fx) (∀y: Fy) x = y ⇒ ∃x (Gx ∧ Hx)

3. In the following, choose one of each bracketed pair of premises and one each bracketed pair of words or phrases in the conclusion so as to make a valid argument; then analyze the premises and conclusion and construct a derivation to show that the argument is valid.

a. *Some road sign was colored*
[*Every stop sign was a road sign* | *Every road sign was a traffic marker***]**
[*If anything was red, it was colored* | *If anything was colored, it was painted***]**
Some **[***stop sign* | *traffic marker***]** *was* **[***red* | *painted***]**

b. *Someone who owns a snake was pleased*
[*Every cobra is a snake* | *Every snake is a reptile***]**
Someone who owns a **[***cobra* | *reptile***]** *was pleased*

8.5.xa. Exercise answers

1. Some of the derivations below are given in two forms, one that does not use EG and REG and another that does. It is also possible to use the rules REP and REC along with the rules for unrestricted existentials. Such answers are not shown but the they can constructed from the answers that are given by using the substitutions of rules shown in the following table:

rule	*alternative approach using REP and REC*
PRCh	REP, PCh, Ext
RNcP	REC, NcP (with later uses SB, SC, and MCR replaced by UI and either MPT or CR)
REG	Adj, EG, REC

a.

	∃x Fx	1
	∀x (Fx → Gx)	a:2
	ⓐ	
	Fa	(3)
2 UI	Fa → Ga	3
3 MPP	Ga	
	∀x ¬ Gx	a:5
5 UI	¬ Ga	(6)
	•	
6 Nc	⊥	
4 NcP	∃x Gx	1
1 PCh	∃x Gx	

	∃x Fx	1
	∀x (Fx → Gx)	a:2
	ⓐ	
	Fa	(3)
2 UI	Fa → Ga	3
3 MPP	Ga	(4)
4 EG	∃x Gx	X, (5)
	•	
5 QED	∃x Gx	1
1 PCh	∃x Gx	

b.

	(∃x: Fx) Gx	1
	(∀x: Gx) Hx	a:2
	ⓐ	
	Fa	(4)
	Ga	(2)
2 SB	Ha	(5)
	(∀x: Fx) ¬ Hx	a:4
4 SB	¬ Ha	(5)
	•	
5 Nc	⊥	3
3 RNcP	(∃x: Fx) Hx	1
1 PRCh	(∃x: Fx) Hx	

	(∃x: Fx) Gx	1
	(∀x: Gx) Hx	a:2
	ⓐ	
	Fa	(3)
	Ga	(2)
2 SB	Ha	(3)
3 REG	(∃x: Fx) Hx	X, (4)
	•	
4 QED	(∃x: Fx) Hx	1
1 PRCh	(∃x: Fx) Hx	

c.

```
         │ ∀x (Fx → Ga)        b:3
         ├─
         │ │ ∃x Fx             2
         │ ├─
         │ │ ⓑ
         │ │ │ Fb              (4)
         │ │ ├─
3 UI     │ │ │ Fb → Ga         4
4 MPP    │ │ │ Ga              (5)
         │ │ │ ●
         │ │ ├─
5 QED    │ │ │ Ga              2
         │ ├─
2 PCh    │ │ Ga                1
         ├─
1 CP     │ ∃x Fx → Ga
```

```
         │ ∃x Fx → Ga                    4
         ├─
         │ ⓑ
         │ │ │ Fb                        (8)
         │ │ ├─
         │ │ │ │ ¬ Ga                    (4)
         │ │ │ ├─
4 MTT    │ │ │ │ ¬ ∃x Fx                 5
         │ │ │ │ │ │ ∀x ¬ Fx           b:7
         │ │ │ │ │ ├─
7 UI     │ │ │ │ │ │ ¬ Fb              (8)
         │ │ │ │ │ │ ●
         │ │ │ │ │ ├─
8 Nc     │ │ │ │ │ │ ⊥                 6
         │ │ │ │ │ ├─
6 NcP    │ │ │ │ │ ∃x Fx               5
         │ │ │ │ ├─
5 CR     │ │ │ │ ⊥                     3
         │ │ │ ├─
3 IP     │ │ │ Ga                        2
         │ │ ├─
2 CP     │ │ Fb→Ga                       1
         ├─
1 UG     │ ∀x (Fx → Ga)
```

```
         │ ∃x Fx → Ga          4
         ├─
         │ ⓑ
         │ │ │ Fb              (3)
         │ │ ├─
3 EG     │ │ │ ∃x Fx           X, (4)
4 MPP    │ │ │ Ga              (5)
         │ │ │ ●
         │ │ ├─
5 QED    │ │ │ Ga              2
         │ ├─
2 CP     │ │ Fb → Ga         1
         ├─
1 UG     │ ∀x (Fx → Ga)
```

d.

```
         │ Fa                          (2)
         ├─
         │ │ (∀x: x = a) ¬ Fx          a:2
         │ ├─
2 SC     │ │ ¬ a = a                   (3)
         │ │ ●
         │ ├─
3 DC     │ │ ⊥                         1
         ├─
1 RNcP   │ (∃x: x = a) Fx
```

```
         │ Fa                  (2)
         ├─
1 EC     │ a = a               (2)
2 REG    │ (∃x: x = a) Fx      (3)
         │ ●
         ├─
3 QED    │ (∃x: x = a) Fx
```

```
         │ (∃x: x = a) Fx      1
         ├─
         │ ⓑ
         │ │ b = a             a—b
         │ │ Fb                (2)
         │ │ ●
         │ ├─
2 QED=   │ │ Fa                1
         ├─
1 PRCh   │ Fa
```

e.

```
         │ (∃x: Fx) ∀y Rxy               2
         ├─
         │ ⓐ
         │ │ ⓑ
         │ │ │ Fb                        (4)
         │ │ │ ∀y Rby                    b:5
         │ │ ├─
         │ │ │ │ (∀y: Fy) ¬ Rya          b:4
         │ │ │ ├─
4 SB     │ │ │ │ ¬ Rba                   (6)
5 UI     │ │ │ │ Rba                     (6)
         │ │ │ │ ●
         │ │ │ ├─
6 Nc     │ │ │ │ ⊥                       3
         │ │ ├─
3 RNcP   │ │ │ (∃y: Fy) Rya              2
         │ ├─
2 PRCh   │ │ (∃y: Fy) Rya                1
         ├─
1 UG     │ ∀x (∃y: Fy) Ryx
```

```
         │ (∃x: Fx) ∀y Rxy               2
         ├─
         │ ⓐ
         │ │ ⓑ
         │ │ │ Fb                        (4)
         │ │ │ ∀y Rby                    b:3
         │ │ ├─
3 UI     │ │ │ Rba                       (4)
4 REG    │ │ │ (∃y: Fy) Rya              X, (5)
         │ │ │ ●
         │ │ ├─
5 QED    │ │ │ (∃y: Fy) Rya              2
         │ ├─
2 PRCh   │ │ (∃y: Fy) Rya                1
         ├─
1 UG     │ ∀x (∃y: Fy) Ryx
```

f.

```
         │ (∃x: Gx) Fx                   1
         │ ¬ Fa                          (4)
         ├─
         │ ⓑ
         │ │ Gb                          (3)
         │ │ Fb                          (4)
         │ ├─
         │ │ │ (∀x: ¬ x = a) ¬ Gx        b:3
         │ │ ├─
3 SC     │ │ │ b = a                     a—b
         │ │ │ ●
         │ │ ├─
4 Nc=    │ │ │ ⊥                         2
         │ ├─
2 RNcP   │ │ (∃x: ¬ x = a) Gx            1
         ├─
1 PRCh   │ (∃x: ¬ x = a) Gx
```

g.

	∀x (Fx → Ga)	c:3
	∀x (Ga → Fx)	b:5
	∃x Fx	2
	ⓑ	
	ⓒ	
	Fc	(4)
3 UI	Fc → Ga	4
4 MPP	Ga	(6)
5 UI	Ga → Fb	6
6 MPP	Fb	(7)
	●	
7 QED	Fb	2
2 PCh	Fb	1
1 UG	∀x Fx	

h. *Everyone loves everyone who loves someone*
Someone loves someone

	(∀x: Px) (∀y: Py ∧ (∃z: Pz) Lyz) Lxy	b:5, a:9
	(∃x: Px) (∃y: Py) Lxy	3
	ⓐ	
	Pa	(9)
	ⓑ	
	Pb	(5), (11)
	ⓒ	
	Pc	(7), (10)
	(∃y: Py) Lcy	4
	ⓓ	
	Pd	(6)
	Lcd	(6)
5 SB	(∀y: Py ∧ (∃z: Pz) Lyz) Lby	c:8
6 REG	(∃z: Pz) Lcz	X, (7)
7 Adj	Pc ∧ (∃z: Pz) Lcz	X, (8)
8 SB	Lbc	(10)
9 SB	(∀y: Py ∧ (∃z: Pz) Lyz) Lay	b:12
10 REG	(∃z: Pz) Lbz	X, (11)
11 Adj	Pb ∧ (∃z: Pz) Lbz	X, (12)
12 SB	Lab	(13)
	●	
13 QED	Lab	4
4 PRCh	Lab	3
3 PRCh	Lab	2
2 RUG	(∀y: Py) Lay	1
1 RUG	(∀x: Px) (∀y: Py) Lxy	

Everyone loves everyone

Note that stages 4 and 6 serve only to move us from (∃y: Py) Lcy to (∃z: Pz) Lcz—i.e., to change a bound variable. If sentences that differ only in the choice of a letter for a bound variable are regarded as the same, (∃y: Py) Lcy could be used as a premise for Adj at stage 7 and the use of PRCh at stage 4 would not be needed.

i. *Something is such that nothing other than it is done*
[When *nothing* is analyzed using a negative generalization, a derivation like that below but without stages 6 and 7 could be used.]

	∃x ¬ (∃y: ¬ y = x) Dy	2
	∃x (∃y: ¬ y = x) (Dx ∧ Dy)	3
	ⓐ	
	¬ (∃y: ¬ y = a) Dy	6
	ⓑ	
	(∃y: ¬ y = b) (Db ∧ Dy)	4
	ⓒ	
	¬ c = b	(10)
	Db ∧ Dc	5
5 Ext	Db	(8)
5 Ext	Dc	(9)
	(∀y: ¬ y = a) ¬ Dy	b:8, c:9
8 SC	b = a	a—b, c
9 SC	c = a	a—b—c, (10)
	●	
10 Nc=	⊥	7
7 RNcP	(∃y: ¬ y = a) Dy	6
6 CR	⊥	4
4 PRCh	⊥	3
3 PCh	⊥	2
2 PCh	⊥	1
1 RAA	¬ ∃x (∃y: ¬ y = x) (Dx ∧ Dy)	

At most one thing is done

At stage 10, the conclusion ⊥ could also be justified as coming by DC from ¬ c = b alone since c = a serves to make b and c co-aliases.

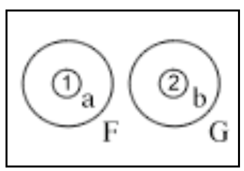
At most one thing is done

	¬ ∃x (∃y: ¬ y = x) (Dx ∧ Dy)	(9)
	∀x (∃y: ¬ y = x) Dy	a:2, b:4
2 UI	(∃y: ¬ y = a) Dy	3
	ⓑ	
	¬ b = a	
	Db	(6)
4 UI	(∃y: ¬ y = b) Dy	5
	ⓒ	
	¬ c = b	(7)
	Dc	(6)
6 Adj	Db ∧ Dc	X, (7)
7 REG	(∃y: ¬ y = b) (Db ∧ Dy)	X, (8)
8 EG	∃x (∃y: ¬ y = x) (Dx ∧ Dy)	X, (9)
	•	
9 Nc	⊥	5
5 PRCh	⊥	3
3 PRCh	⊥	1
1 NcP	∃x ¬ (∃y: ¬ y = x) Dy	

Something is such that nothing other than it is done

2. a.

	∃x Fx	1
	∃x Gx	2
	ⓐ	
	Fa	(5)
	ⓑ	
	Gb	(7)
	∀x ¬ (Fx ∧ Gx)	a:4, b:6
4 UI	¬ (Fa ∧ Ga)	5
5 MPT	¬ Ga	
6 UI	¬ (Fb ∧ Gb)	7
7 MPT	¬ Fb	
	○	Fa, ¬ Fb, ¬ Ga, Gb ⇏ ⊥
	⊥	3
3 NcP	∃x (Fx ∧ Gx)	2
2 PCh	∃x (Fx ∧ Gx)	1
1 PCh	∃x (Fx ∧ Gx)	



b.

	(∃x: Fx) Gx	1
	(∃x: Fx) Hx	2
	(∀x: Fx) (∀y: Fy) x = y	a:3
	ⓐ	
	Fa	
	Ga	(7)
	ⓑ	
	Fb	
	Hb	(8)
3 SB	(∀y: Fy) a = y	b:4
4 SB	a = b	a—b
	∀x ¬ (Gx ∧ Hx)	a:6
6 UI	¬ (Ga ∧ Ha)	7
7 MPT	¬ Ha	(8)
	•	
8 Nc=	⊥	5
5 NcP	∃x (Gx ∧ Hx)	2
2 PRCh	∃x (Gx ∧ Hx)	1
1 PRCh	∃x (Gx ∧ Hx)	

3. a. *Some road sign was colored*
Every road sign was a traffic marker
If anything was colored, it was painted

	(∃x: Sx) Cx	1
	(∀x: Sx) Tx	a:2
	∀x (Cx → Px)	a:3
	ⓐ	
	Sa	(2)
	Ca	(4)
2 SB	Ta	(5)
3 UI	Ca → Pa	4
4 MPP	Pa	(5)
5 REG	(∃x: Tx) Px	X, (6)
	•	
6 QED	(∃x: Tx) Px	1
1 PRCh	(∃x: Tx) Px	

Some traffic marker was painted

b. *Someone who owns a snake was pleased*
Every snake is a reptile

	(∃x: Px ∧ (∃y: Sy) Oxy) Dx	1
	(∀x: Sx) Rx	b:4
	ⓐ	
	Pa ∧ (∃y: Sy) Oay	2
	Da	(7)
2 Ext	Pa	(6)
2 Ext	(∃y: Sy) Oay	3
	ⓑ	
	Sb	(4)
	Oab	(5)
4 SB	Rb	(5)
5 REG	(∃y: Ry) Oay	X, (6)
6 Adj	Pa ∧ (∃y: Ry) Oay	X, (7)
7 REG	(∃x: Px ∧ (∃y: Ry) Oxy) Dx	X, (8)
	•	
8 QED	(∃x: Px ∧ (∃y: Ry) Oxy) Dx	3
3 PRCh	(∃x: Px ∧ (∃y: Ry) Oxy) Dx	1
1 PRCh	(∃x: Px ∧ (∃y: Ry) Oxy) Dx	

Someone who owns a reptile was pleased