8.5.1. The role of existentials in entailment

As has been the case elsewhere in this chapter, we will be able to rely on our discussion of universals to simplify our discussion of principles of entailment for existentials. The differences between the principles governing universal and existential quantifiers will, in most cases, be analogous to differences between the principles for conjunction and disjunction. The laws of entailment for the universal quantifiers were modifications of laws for conjunction, and the rules for the existential quantifiers will be based in a similar way on rules for disjunction. Our planning rule for existential sentences as conclusions will take a different form from that for disjunctions, but even it is analogous to a rule that could have been used for that connective.

These analogies derive from the truth conditions for the unrestricted existential, which follow the conditions for disjunction in precisely the way the conditions for the universal follow those for conjunction. A sentence $\exists x\ \theta x$ is true in a structure if and only if it has at least one true instance in a language expanded by the range **R** of that structure. In other words, an existential claim behaves like a disjunction of its instances when these instances are taken in a language that incorporates a term for each reference value. However, as was the case with the universal, the set of instances is not be the same for all structures, so we cannot employ any definite information about what the instances of an existential sentence are when stating general laws of entailment.

First, we will look at the role of an unrestricted existential as a premise. A disjunctive premise may be used to draw a conclusion by way of a proof by cases. In such a proof, we suppose in turn that each of the disjuncts is true and argue for the conclusion in each case. A comparable way of arguing from an existential would be to establish many arguments, each one considering an instance of the existential as one case. Since we cannot associate the existential with any definite set of instances, we cannot consider each of these arguments individually, so we must use adapt a device from our treatment of the universal: we need to set out the indefinitely many arguments by offering a general pattern. That is, to use an existential premise to draw a conclusion, we draw the conclusion from one instance of the existential in a way that sets a pattern for all other instances.

This sort of argument may be called a ***proof by choice***. To see how proofs by choice work, consider the two arguments below.

Anyone who worked late got overtime	*Anyone who worked late got overtime*
If anything broke down, Tom worked late	*If anything broke down, Tom worked late*
Something broke down	*X broke down*
Tom got overtime	*Tom got overtime*

The validity of the argument on the left can be traced to the validity of the one on the right. In the latter, we use the premise *X broke down* in place of the existential *Something broke down*, so we argue for the conclusion from an instance of the existential.

Of course, being able to draw a conclusion when using an instance of an existential does not, by itself, insure that we can draw the same conclusion using the existential. For example, given appropriate premises we can conclude *Larry will be happy* from *Larry will win a lottery*; but this does not insure that we can conclude *Larry will be happy* from *Someone will win a lottery*. So, to base the validity of the argument on the left on the validity of the one on the right, we will need to insure that whatever we can conclude using the instance *X broke down* also could be concluded using any other instance. That is, we need to insure that the argument on the right has a sort of generality. If we were employing proof by choice in a formal proof, we might signal this generality by saying, "Let *X* be anything that broke down." This would declare our intention to begin with the choice of an instance but without employing any special information about instance we have chosen.

It should be clear that there is some kinship between proofs by choice and the general arguments we have used to establish universal conclusions. Both the reasons for and the nature of this kinship can be brought out in another way by considering a second pair of arguments. In these arguments, the key premises of the earlier pair have been absorbed in the conclusion:

Anyone who worked late got overtime *If anything broke down, Tom worked late*	*Anyone who worked late got overtime* *If anything broke down, Tom worked late*
Something broke down → *Tom got overtime*	∀x (x *broke down* → *Tom got overtime*)

The validity of the argument on the left is tied to the validity of the left-hand argument of the earlier pair by the law for the conditional as a conclusion, and the validity of the right-hand arguments in both pairs are tied by that law and the law for the universal as a conclusion.

Consequently, the relation between the earlier two arguments can be understood by way of the relation between the new pair. And the new pair of arguments are clearly tied since their conclusions are equivalent by one of the confinement principles discussed in 8.1.4. The different forms taken by these conclusions show us that the inference ticket to *Tom got overtime* from *Something broke down*, can be based on a sort of general inference ticket to *Tom got overtime* from the instance *X broke down*. That is, to move from *Something broke down* to *Tom got overtime* we need a way of passing from *X broke down* to *Tom got overtime* that can be generalized to work for any instance.

Recalling the test we used for the generality of arguments in the case of the universal quantifiers, we can expect our analysis of the role of an existential as a premise to make reference to a term that is ***parametric*** in an appropriate sense. We will want a term that has no special connection to any elements of the argument—to any of its premises, its conclusion, or the predicate that the existential premise claims to be exemplified. So suppose the term a is unanalyzed term and does not appear in the set Γ, the sentence φ, or the existential ∃x θx, and consider the two arguments

$$\Gamma,\ \exists x\ \theta x\ /\ \varphi$$
$$\Gamma,\ \theta a\ /\ \varphi.$$

We can argue that each is valid if and only if the other is if we can show that each is divided by a structure if and only if the other is. If a structure S divides the premises and conclusion of the first, it will assign θ a non-empty extension, and we can form a structure S′ that divides the second argument by assigning a value in this extension to the term a. We can assign this extension to the term a without disturbing the interpretation of other vocabulary since, as a parameter, the term a stands apart from this vocabulary. So S′ will give θ the same extension as S does, and it will make θa true without changing the truth values of φ and the members of Γ. On the other hand, any structure dividing the second argument will give θ a non-empty extension (because the value of the term a will be in it) so this structure will make ∃x θx true and also divide the first argument. Thus we will have a structure dividing one argument if and only if we have a structure dividing the other, and each argument is valid if and only if the other is. This gives us our ***law for the unrestricted existential as a premise***: if a is an unanalyzed term that does not appear in Γ, φ, or ∃x θx, then Γ, ∃x θx ⇒ φ if and only if Γ, θa ⇒ φ.

We turn next to the role of existentials as conclusions. First, recall our account of the role of disjunction as conclusion: Γ ⇒ φ ∨ ψ if and only if Γ, $\overline{\varphi}$ ⇒ ψ. We could have avoided the asymmetric treatment of the two components if we had resorted to an even heavier use of negation; applying the idea behind IP to the right side of the law, we get this: Γ ⇒ φ ∨ ψ if and only if Γ, $\overline{\varphi}$, $\overline{\psi}$ ⇒ ⊥. That is, a disjunction is a valid conclusion if and only if we can reduce to absurdity the supposition that its components are both false. We are often able to avoid this use of *reductio* arguments in the case of disjunction, but it would be awkward to do so in the case of the existential.

A strict analogue for the existential of this rule for disjunction would be to say that we can conclude an existential ∃x θx from premises Γ if and only if we can reduce to absurdity the result of adding denials of all the instances of ∃x θx to Γ. But there is no definite set of instances, so we cannot take this approach literally. We had a related problem in dealing with the universal as a premise, for the analogy with conjunction suggested that a universal premise might be replaced by the set of all its instances. And the problem there provides a solution here: we can say that an existential ∃x θx follows from premises Γ if and only if we can reduce to absurdity the result of adding ∀x $\overline{\theta x}$ to Γ. This will be our ***law for the existential as a conclusion***.

$$\Gamma \Rightarrow \exists x\, \theta x \text{ if and only if } \Gamma, \forall x\, \overline{\theta x} \Rightarrow \bot$$

In it, we do not explain the role of the existential as a conclusion directly, but instead make a connection with the role of the universal as a premise. Like the awkwardness in handling disjunction, this can be traced to the fact that we maintain at most one goal. (A law for ∃ that makes no reference to ∀ is easier to state for relative exhaustiveness; see appendix B for the form it would take.)

This principle for the existential is closely related to the equivalence obversion, for (choosing one of the cases of obversion covered by the bar notation) we have

$$\neg\ \forall x\ \overline{\theta x} \Leftrightarrow \exists x\ \theta x.$$

This equivalence says that an existential is equivalent to the denial of a corresponding negative generalization. And the law for existential conclusions says that we can conclude a claim of exemplification if we can reduce a negative generalization to absurdity—that is, if we can do what would be needed to establish the denial of one.

This way of drawing an existential conclusion is called a ***non-constructive proof***. It enables us to establish a claim of exemplification without ever describing a particular example. (The use of the term *construction* here can be traced to geometry, where claims of exemplification are typically established by a geometric construction of the figure that is claimed to exist.) Non-constructive proofs of exemplification have been common in modern mathematics but have also been controversial. The doubts about them have not usually been doubts about their validity (though Brouwer, who was mentioned in 3.1.3, could be said to have doubted that). Instead these doubts have concerned the respect accorded such proofs, with some mathematicians feeling that the methods used in them render them undeserving of the respect that might be given to them due to the importance of their conclusions. The feature of non-constructive proofs that lies behind these doubts is a weakness that is granted even by those who accept such proofs happily: because they do not produce an example, they may provide little insight into the reasons why a claim of exemplification is true.

The deepest concerns about non-constructive proof are focused on arguments about abstract and, especially, infinite structures, and even Brouwer thought that non-constructive proofs were valid for reasoning about ordinary claims about the world of sense experience. Still, the indirection and uninformativeness of non-constructive arguments can be felt with ordinary reasoning and is often unnecessary, so it is worthwhile considering the alternative. A ***constructive proof*** of a claim of exemplification establishes the claim by first producing an example of the sort that is claimed to exist. The move from example to claim of exemplification appears formally as a step from an instance of an existential to the existential itself, and it is neatly captured in a principle of entailment commonly known as ***existential generalization***: $\theta\tau \Rightarrow \exists x\ \theta x$ for any term τ.

The conclusion of this entailment is not a generalization in the sense in which we have been using the term. But it may be said of someone who is making heavy use of words like *something* and *someone* that he is "speaking in generalities" and is not being specific. The principle of existential generalization is a license to move from a specific claim to a generality of an existential sort. We cannot rely on this principle alone—the issue of non-constructive arguments would never have arisen if we could—but it does provide a useful supplement in the way the principle of weakening supplements the law for disjunction as a conclusion. And, like weakening, we will count existential generalization as an attachment principle. (What is attached? In form, we could say it is the existential quantifier; in what is said, it is the other instances of the conclusion, the other ways in which it could be true.)

This completes our suite of principles for the unrestricted existential. Collected together, they are as follows:

Law for the unrestricted existential as a premise. For any unanalyzed term a appearing in neither Γ, Σ, nor $\exists x\ \theta x$, we have:

$$\Gamma, \exists x\ \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \theta a \Rightarrow \Sigma.$$

Law for the unrestricted existential as a conclusion.

$$\Gamma \Rightarrow \exists x\ \theta x \text{ if and only if } \Gamma, \forall x\ \overline{\theta x} \Rightarrow \bot.$$

Law of existential generalization.

$$\theta\tau \Rightarrow \exists x\ \theta x \text{ for any term } \tau.$$

The first of these is the principle underlying proofs by choice (in which we choose an example a of the sort claimed by the existential), the second underlies non-constructive proofs, and the third underlies constructive proofs.

There is a corresponding set of principles for the restricted existential. These can be reached by way of restatements of a restricted existential in unrestricted form and thus by way of principles for conjunction. The process is more straightforward than in the case of the restricted universal, so we will only consider the results, which are the following:

Law for the restricted existential as a premise. For any unanalyzed term a appearing in neither Γ, φ, nor $(\exists x\colon \rho x)\ \theta x$, we have:

$$\Gamma, (\exists x\colon \rho x)\ \theta x \Rightarrow \Sigma \text{ if and only if } \Gamma, \rho a, \theta a \Rightarrow \Sigma.$$

Law for the restricted existential as a conclusion.

$$\Gamma \Rightarrow (\exists x\colon \rho x)\ \theta x \text{ if and only if } \Gamma, (\forall x\colon \rho x)\ \overline{\theta x} \Rightarrow \bot.$$

Law of restricted existential generalization.

$$\rho\tau, \theta\tau \Rightarrow (\exists x\colon \rho x)\ \theta x \text{ for any term } \tau.$$

Again these provide the basis for proofs by choice and for both non-constructive and constructive proofs of exemplification. The last says that we can establish a claim of existence if we can show of the value of τ both that it is in the domain of the existential and that it has the attribute. The first law says we can draw a conclusion from an existential if we conclude it from arbitrary choice of a value that is in the domain and has the attribute.

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