8.3.3. *Exactly n*

It is also possible to give a somewhat simpler symbolic representations of the quantifier phrase *exactly n* C*s* than we get by way of truth-functional compounds of *at least*-*m* forms. Here are a couple of approaches for the case of *exactly 1*:

I forgot just one thing
Something is such that (I forgot it and nothing else)
∃x *I forgot* x *and nothing else*
∃x (<u>*I*</u> *forgot* <u>x</u> ∧ *I forgot nothing other than* x)
∃x (Fix ∧ *nothing other than* x *is such that (I forgot it)*)
∃x (Fix ∧ (∀y: y *is other than* x) ¬ *I forgot* y)

$$\exists x\,(Fix \land (\forall y{:}\ \neg\, y = x)\ \neg\, Fiy)$$
$$\exists x\,(Fix \land \forall y\,(\neg\, y = x \rightarrow \neg\, Fiy))$$

I forgot just one thing
Something is such that (I forgot it and it was all I forgot)
∃x *I forgot* x *and* x *was all I forgot*
∃x (<u>*I*</u> *forgot* <u>x</u> ∧ x *was all I forgot*)
∃x (Fix ∧ *everything I forgot is such that (* x *was it)*)
∃x (Fix ∧ (∀y: *I forgot* y) x *was* y)

$$\exists x\,(Fix \land (\forall y{:}\ Fiy)\ x = y)$$
$$\exists x\,(Fix \land \forall y\,(Fiy \rightarrow x = y))$$

[F: λxy (x *forgot* y); i: *me*]

And, in general, *Exactly one thing is such that (… it …)* can be analyzed as any of the following (where θx abbreviates … x …):

$\exists x\,(\theta x \land (\forall y{:}\ \neg\, y = x)\ \neg\, \theta y)$	$\exists x\,(\theta x \land \forall y\,(\neg\, y = x \rightarrow \neg\, \theta y))$
$\exists x\,(\theta x \land (\forall y{:}\ \theta y)\ x = y)$	$\exists x\,(\theta x \land \forall y\,(\theta y \rightarrow x = y))$

The forms in columns are equivalent by the symmetry of identity and the following equivalences:

$$(\forall x{:}\ \rho x)\ \theta x \Leftrightarrow (\forall x{:}\ \overline{\theta x})\ \overline{\rho x}$$
$$\varphi \rightarrow \psi \Leftrightarrow \overline{\psi} \rightarrow \overline{\varphi}$$

The first of these is traditionally called ***contraposition*** and that name is sometimes used for the second also. The first licenses the restatement of *Only dogs barked* by *Everything that barked was a dog*. The second would apply to the same pair of sentences when they are represented using unrestricted quantifiers and also to the restatement of *The match burned only if oxygen was present* by *If the match burned, then oxygen was present*.

The initial unrestricted quantifier in the above analyses of *exactly 1 thing* can also be replaced by a restricted quantifier. The following analysis of a slightly more complex example uses this sort of variation on the second pattern above:

I forgot just one number
Some number I forgot is such that (it was all the numbers I forgot)
(∃x: x *is a number I forgot*) x *was all the numbers I forgot*
(∃x: x *is a number* ∧ *I forgot* x) *every number I forgot is such that (*x *was it)*
(∃x: x *is a number* ∧ *I forgot* x) (∀y: y *is a number I forgot*) x *was* y
(∃x: Nx ∧ Fix) (∀y: y *is a number* ∧ *I forgot* y) x *was* y

(∃x: Nx ∧ Fix) (∀y: Ny ∧ Fiy) x = y

And, in general, *Exactly 1* C *is such that (… it …)* can be analyzed as

(∃x: x *is a* C ∧ … x …) (∀y: y *is a* C ∧ … y …) x = y

The analogous variation on the first pattern would be

(∃x: x *is a* C ∧ … x …) (∀y: y *is a* C ∧ ¬ y = x) ¬ … y …

In the case of, *I forgot just one number*, this pattern would amount to saying *Some number that I forgot is such that I forgot no other number*.

The sentence *There is exactly 1* C can be understood as *Exactly 1* C *is such that (it is)* and the dummy predicate λx (x *is*) can be dropped to yield the analysis

(∃x: x *is a* C) (∀y: y *is a* C) x = y

which can be understood to say *Some* C *is such that (it is all the* C*s there are)*.

This sort of pattern will be important for the analysis of definite descriptions in 8.4.1, but the first approach (i.e., by way of *nothing else*) is probably the more natural way of extending the analysis to claims of *exactly n* for numbers *n* > 1—as in the following example:

Exactly 2 things are in the room
2 things are such that (they are in the room but and nothing else is)
∃x (∃y: ¬ y = x) x *and* y *are in the room but and nothing else is*
∃x (∃y: ¬ y = x) ((x *is in the room* ∧ y *is in the room*) ∧ *nothing other than* x *and* y *is in the room*)
∃x (∃y: ¬ y = x) ((Nxr ∧ Nyr) ∧ (∀z: z *is other than* x *and* y) ¬ z *is in the room*)
∃x (∃y: ¬ y = x) ((Nxr ∧ Nyr) ∧ (∀z: z *is other than* x ∧ z *is other than* y) ¬ Nzr)

∃x (∃y: ¬ y = x) ((Nxr ∧ Nyr) ∧ (∀z: ¬ z = x ∧ ¬ z = y) ¬ Nzr)

[N: λxy (x *is in* y); r: *the room*]

The general forms for *exactly 2 things are such that (… they …)* and *exactly 2* C*s are such that (… they …)* along these lines are the following (using θ for λx (… x …) and ρ for λx (x *is a* C)):

∃x (∃y: ¬ y = x) ((θx ∧ θy) ∧ (∀z: ¬ z = x ∧ ¬ z = y) ¬ θz)
(∃x: ρx) (∃y: ρy ∧ ¬ y = x) ((θx ∧ θy) ∧ (∀z: ρz ∧ ¬ z = x ∧ ¬ z = y) ¬ θz)

Notice that the restricting predicate ρ is added to each of the three quantifiers in the second. In particular, *Exactly 2 boxes are in the room* means *2 boxes are such that (they are in the room and no other boxes are)* rather than *2 boxes are such that (they are in the room and nothing else is)*, which says that two boxes are the only things in the room.