7.3.2. Generalizations as components

Sentences are analyzed into predicates and individual terms after we have completed all analysis by truth-functional connectives. On the other hand, we have already seen a number of cases where it is not possible to defer analysis as a generalization until all truth-functional connectives have been dealt with. The sentence *Everyone stood at the port or starboard rail* is not a disjunction, and *or* can be dealt with only after we have analyzed it as a universal. Still, analysis by truth-functional connectives will often precede the analysis of generalizations into quantifiers and predicates, and we have already seen the simplest case of this: the denial of a generalization. We will now go on to consider some other examples.

In some cases, this sort of analysis is a straightforward matter. Here is an example:

Everyone was contacted, but no one responded

Everyone was contacted $\land$ *no one responded*

($\forall$x: x *is a person*) x *was contacted* $\land$ ($\forall$x: x *is a person*) $\neg$ x *responded*

$$(\forall x\colon Px)\ Cx \land (\forall x\colon Px)\ \neg\ Rx$$

$$\forall x\ (Px \to Cx) \land \forall x\ (Px \to \neg\ Rx)$$

[C: λx (x *was contacted*); P: λx (x *is a person*); R: λx (x *responded*)]

The variable x is used in both generalizations here. This is quite legitimate since the pattern of binding can be understood as follows:

$$(\forall x\colon Px)\ Cx \land (\forall x\colon Px)\ \neg\ Rx$$

$$\forall x\ (Px \to Cx) \land \forall x\ (Px \to \neg\ Rx)$$

Since the lambda operators have been absorbed in the quantifier-plus-variable $\forall$x, variables are bound to this expression under the same conditions that would lead them to be bound to a lambda operator. The two occurrences of $\forall$x in each sentence apply to different expressions and can bind variables only in the expressions in their scopes, so neither can interfere with the operation of the other. Of course, this also means that the sentences could have been written just as well using different variables for the two quantifiers—e.g., as

$$(\forall x\text{: } Px)\ Cx \wedge (\forall y\text{: } Py)\ \neg\ Ry$$
$$\forall x\ (Px \rightarrow Cx) \wedge \forall y\ (Py \rightarrow \neg\ Ry)$$

but the independence of variables bound to different quantifiers will usually be used to economize on the number of letters that need to be devoted to variables, so it is the first way of writing the conjunctions that you will see most often.

When generalizations appear as components of conditionals or disjunctions, it usually will be obvious that the sentence as a whole is a truth-functional compound. However, there are cases where an analysis as a conjunction is possible even though the sentence does not so clearly have this form. In particular, it is often possible to understand a generalization whose class indicator or quantified predicate is logically complex as a conjunction of generalizations that share a domain or an attribute. For example, *Everything is fine and dandy* could be understood as a more compact equivalent of *Everything is fine and everything is dandy*. In making this restatement we have repeated a quantifier phrase and such a restatement does not always preserve meaning. However, in this case it does work and, in general, we can take a universal whose quantified predicate is formed by conjunction and restate it as a conjunction of universals. So we have the two alternative analyses:

$$\forall x\ (x\ \textit{is fine} \wedge x\ \textit{is dandy})$$
$$\forall x\ x\ \textit{is fine} \wedge \forall x\ x\ \textit{is dandy}$$

The first of these is preferable because it mirrors the form of the English sentence more closely, but the two are equivalent and we can claim a general equivalence between pairs of this sort:

$$\forall x\ (\rho x \wedge \theta x) \Leftrightarrow \forall x\ \rho x \wedge \forall x\ \theta x.$$

A similar principle holds for the restricted universal quantifier and principles hold for both often enough that we will employ special notation to indicate that. Let us write “(∀x …)” to indicate the *possibility* of a restriction so that a quantifier (∀x …) might take either of the forms ∀x or (∀x: ρx). Using this notation, we can write the more general principle as follows:

$$(\forall x \ldots)\ (\rho x \wedge \theta x) \Leftrightarrow (\forall x \ldots)\ \rho x \wedge (\forall x \ldots)\ \theta x.$$

In cases where the quantifier is restricted, it should be restricted in the same way in all three occurrences.

Although the analysis as conjoined generalizations was not the most natural one in the case of *Everything is fine and dandy*, there is another sort of case where it is more natural. Consider the sentence *All boys and girls are invited*. This claims that the attribute of being invited holds universally for boys and also for girls. That is, it could be stated as a conjunction of two generalizations: *All boys were invited* ∧ *all girls were invited*. The sentence *All boys and girls are invited* can be analyzed also as a single generalization, but care must be taken in stating the restricting predicate. It must express membership in the class consisting of all boys and all girls; that is, we need a predicate that is true of any child of either sex. Of course, λx (x *is a child*) would do; but if we are to employ the vocabulary of the original sentence, the best we can do is λx (x *is a boy* ∨ x *is a girl*). Thus, we have the following pair of equivalent analyses:

$$(\forall x\colon Bx \lor Gx)\ Ix$$
$$(\forall x\colon Bx)\ Ix \land (\forall x\colon Gx)\ Ix$$

[B: λx (x *is a boy*); G: λx (x *is a girl*); I: λx (x *is invited*)]

Here the second has the advantage of reflecting the use of *and* in the English sentence by a use of conjunction. The pair is an instance of a general equivalence:

$$(\forall x\colon \rho x \lor \pi x)\ \theta x \Leftrightarrow (\forall x\colon \rho x)\ \theta x \land (\forall x\colon \pi x)\ \theta x$$

It is enlightening to state this using unrestricted universal quantifiers

$$\forall x\ (\ (\rho x \lor \pi x) \to \theta x) \Leftrightarrow \forall x\ (\rho x \to \theta x) \land \forall x\ (\pi x \to \theta x)$$

because we can then justify it by the following general equivalence for the conditional (which is closely associated with the idea behind proofs by cases):

$$(\varphi \lor \psi) \to \chi \Leftrightarrow (\varphi \to \chi) \land (\psi \to \chi)$$

Together with the equivalence for the universal and conjunction noted above, this allows us to argue as follows:

$$\begin{aligned}\forall x\ (\ (\rho x \lor \pi x) \to \theta x) &\Leftrightarrow \forall x\ (\ (\rho x \to \theta x) \land (\pi x \to \theta x)\ )\\ &\Leftrightarrow \forall x\ (\rho x \to \theta x) \land \forall x\ (\pi x \to \theta x)\end{aligned}$$

This locates the source of the change from disjunction to conjunction when the single generalization is restated as two in the features of restricted universal generalizations that make them analogous to conditionals.

While it is possible to analyze *All boys and girls are invited* so that the word *and* in the class indicator turns out to mark the overall form of the sentence, things do not always work out like this—as the next few examples will show. Consider first a direct negative generalization with the same domain as the generalization above. Suppose, for example, we wish to say the property of having been forgotten fails for all boys and girls. We can state this as a conjunction of generalizations (e.g., *No boy was forgotten and no girl was either*)—or with a conjunctive class indicator if we make an affirmative generalization whose predicate incorporates negation (e.g., *All boys and girls were unforgotten*). But if we want a compound quantifier phrase using the quantifier word *no*, we will be forced to employ *or*—as in *No boy or girl was forgotten*. The closest we could come to this while using *and* with a negative quantifier word would be something like *None of the boys and girls was forgotten*. (The sentence *No boys and girls were forgotten* may *sound* fine, but its meaning is elusive.) An analysis of the negative generalization *No boy or girl was forgotten* as a universal quantification whose restricting predicate contains disjunction is probably the most natural one in this case because it preserves the connective appearing in the original sentence.

The conjoined noun phrase *boys and girls* can be used also in stating a complementary negative generalization—e.g., *Only boys and girls are invited*. The domain of this generalization is the class of everything that is not a boy or girl. This suggests the analysis

$$(\forall x\!: \neg\, (x \textit{ is a boy} \vee x \textit{ is a girl}))\ \neg\, x \textit{ is invited}$$
$$(\forall x\!: \neg\, (Bx \vee Gx))\ \neg\, Ix$$
$$\forall x\, (\neg\, (Bx \vee Gx) \rightarrow \neg\, Ix)$$

$$[B\!: \lambda x\, (x \textit{ is a boy});\ G\!: \lambda x\, (x \textit{ is a girl});\ I\!: \lambda x\, (x \textit{ is invited})]$$

Of course, we could restate $\neg\, (Bx \vee Gx)$ as $\neg\, Bx \wedge \neg\, Gx$ by one of De Morgan's laws and in this way eliminate disjunction in favor of conjunction. The form we would get would be expressed more directly in English by *Nothing that is not a boy and not a girl is invited*. But the claim these sentences make cannot be analyzed as a conjoined pair of generalizations. In particular, the conjunction *Only boys are invited* ∧ *only girls are invited* is quite different in its implications: you could reasonably conclude from it that no one at all is invited.

Somewhat similar (and related) problems concern the quantified predicates of negative generalizations. Compounding with *and* cannot be captured by a pair of conjoined generalizations while *or* gives rise to conjoined rather than disjoined ones. For example, *No plane landed in either Detroit or Windsor* amounts to *No plane landed in Detroit* ∧ *no plane landed in Windsor*; but it would be better to analyze it more directly as a single generalization whose quantified formula is a disjunction.

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