7.2. Generalizations and quantifiers

7.2.0. Overview

Our symbolic analysis of generalizations is somewhat analogous to our analysis of conditionals: we use a single symbol and distinguish different kinds of generalization by the use of negation.

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7.2.1. The universal quantifier

A ***quantifier*** is an operation that takes predicates as input and yields sentences as output. The quantifiers we will consider all apply only to 1-place predicates, but we will consider them in two forms, one of which is a 2-place operation applying to a pair of 1-place predicates and another that is a 1-place operation applying to a single 1-place predicate. When there is no need to distinguish them we will refer to both as ***universal quantifiers*** and describe the formulas they form also as *universal* (or, less formally, use *universal* as a common noun and refer to them as *universals*).

Although we can make perfectly good sense of the application of quantifiers to unanalyzed predicates, we will almost always apply them to abstracts, using an abstract λx Fx in place of an unanalyzed predicate F. We would need to consider the application of quantifiers to abstracts anyway, and it will simplify things to focus on this case. We can choose alphabetic variants so that any pair of abstracts can be written with the same variable. So, when we speak below of a pair of abstracts λx θx and λx ρx, what we say can be extended to any pair of abstracts whatsoever. Also notice that λx θx—which refers to the property that x has in virtue of θ being true of it—will be true of the same values as θ is. So it will not hurt, when talking about semantics, to think of λx θx and θ as the same predicate (though, strictly speaking, θ may be an unanalyzed predicate while λx θx is complex).

Our 2-place quantifier is the ***restricted universal quantifier*** for which we use the symbol ∀ (the symbol ***for all***). The sentence ∀[λx ρx][λx θx] that results from applying the restricted universal quantifier to abstracts λx ρx and λx θx will be referred to as a *restricted universal*. It says that θ is true of everything that ρ is true of—i.e., that the extension of θ includes the extension of ρ. This makes ∀[λx ρx][λx θx] an affirmative direct generalization whose domain is the extension of ρ and whose attribute is expressed by θ. Since the scope of the generalization is limited to the extension of ρ we will refer to ρ as the ***restricting predicate***, and we will refer to θ, which expresses the property said to hold generally, as the ***quantified predicate***.

The simplest case of a restricted universal is one whose restricting and quantified predicates are unanalyzed. For example, if W is λx (x *walks*) and M is λx (x *moves*), then ∀[λx Wx][λx Mx] says that anything that walks also moves. More often, the restricting or quantified predicate will have internal structure. For example, if we want to say that anything that walks and talks both moves and communicates, we can do this with the form

$$\forall\ [\lambda x\ (Wx \wedge Tx)]\ [\lambda x\ (Mx \wedge Cx)]$$

where T is λx (x *talks*) and C is λx (x *communicates*).

Since we always are able to write the two abstracts using the same variable, we can use the following more abbreviated notation for the universal sentence:

$$(\forall x\!: Wx \wedge Tx)\ (Mx \wedge Cx).$$

The symbolic form (∀x: Wx ∧ Tx) (Mx ∧ Cx) can be read in something close to English as *Everything, x, such that (x walks and x talks) is such that (x moves and x communicates)*. And, in general, the form

$$(\forall x\!: \rho x)\ \theta x$$

can be rendered in English as

`Everything`, x, `such that` ρx `is such that` θx.

Here we can regard ρ and θ as the predicates to which the quantifier applies, with the apparatus of variable binding absorbed into the quantifier.

We may adapt an alternative notation we have used for abstracts to write the form of a restricted universal schematically as

(∀x: ...x...) ---x---

which amounts to

`Everything`, x, `such that` (...x...) `is such that` (---x---)

To extend a grammatical pun used before, this can be read as *Everything, x, such that (x* ***dots****) is such that (x* ***dashes****).*

The components ...x... and ---x--- of this form (or ρx and θx of the other way of writing the general form of a restricted universal) are the bodies of the abstracts to which the operation ∀ is applied. When they are removed from the universal and considered by themselves they will usually contain one or more occurrences of a variable x that is not bound by any abstract. Such a variable is called a ***free variable***; it can be compared to an anaphoric pronoun that is missing an antecedent. Since we refer to sentence-like expressions that may contain free variables as formulas, sentences in the strict sense have no free variables and can be described as ***closed formulas***. We will use the expression *term* in a way analogous to *formula* and apply it to expressions with or without free variables; we can speak of ***open*** and ***closed*** terms depending on whether free variables do or do not occur. Although up to this chapter our symbolic forms have included only closed terms and closed formulas (i.e., sentences), we will now extend the syntactic apparatus of earlier chapters to all terms and formulas. The semantic ideas of earlier chapters apply also with the exception that an open term or open formula has a value on an interpretation of its non-logical vocabulary only when a reference value is assigned to each of its free variables.

(In the preceding paragraph, it was said that the formulas ...x... and ---x--- "usually" contain free variables. That's because the variable x need not appear in the body of an abstract with the lambda operator λx. For example, the abstract λx 2 would be used to express the constant function f defined by f(x) = 2. Such an abstract is said to be ***vacuous***.)

The formula ...x... in (∀x: ...x...) ---x--- (i.e., ρx in (∀x: ρx) θx) says what must be true of x for it to be in the domain of the generalization; we will refer to it as the ***restricting formula***. The formula ---x--- (i.e., θx) says that x has the attribute of the generalization. The generalization says something how many values in the domain will make θx true when they are assigned to x (namely that they all will), so we will refer to θx as the ***quantified formula***. This is a direct extension of our terminology for the component predicates of a generalization: the restricting formula is a predication of the restricting predicate and the quantified formula is a predication of the quantified predicate.

When reading the symbolic notation, we add the variable x as an appositive marked off by commas after the quantifier phrase to indicate that this quantifier phrase serves as the antecedent of the symbolic pronouns x. If we put English pronouns in place of the variables, we have can rely on the conventions of syntax to determine the antecedent and we can drop the appositive to get

Everything such that (...it...) is such that (---it---)

This is a generalization whose class indicator is *thing such that*

(...it...) and whose quantified predicate is λx (x *is such that (---it---))*. Notice that the adjectival phrases *such that (...it...)* and *such that (---it---)* have two different functions in this sentence. The first appears as a modifier of the common noun *thing* while the second is a predicate adjective. Their roles are comparable to those of *scarlet* and *red*, respectively, in *Everything scarlet is red*.

The use of *thing* here also deserves some comment. Consider an English generalization that uses the same form of words as these readings—*Everything such that it walks is such that it moves*, for example. This generalization is direct and affirmative. The class indicator is the phrase *thing such that it walks*; and the predicate λx (x *is such that it moves*) is the quantified predicate. Now if this sentence is to make the same claim as (∀x: x *walks*) x *moves*, the indicated class of the English sentence should be the extension of λx (x *walks*) and the attribute expressed by the English quantified predicate should be the extension of λx (x *moves*). There is certainly no problem in the latter case; λx (x *is such that it moves*) is just a more cumbersome way of expressing λx (x *moves*). But does *thing such that it walks*, or *thing that walks*, really indicate the extension of λx (x *walks*)?

It does if we take the word *thing* to indicate the full range of reference values rather than being limited, say, to inanimate objects. We may say that, in such a use, *thing* is a ***dummy restriction***. It does not itself restrict the domain of the generalization but provides a grammatical anchor for further restrictions. We have been using the word that way as an alternative to *object*, *entity*, and *individual*, but is it used that way ordinarily? This is not the sort of question we can settle here, but notice that if we really want emphasize that our generalization concerns "things" in some specialized sense, we are likely to use the two-word phrase *every* ***thing***, with an emphasis on *thing*, rather than the single word *everything*. This is not to say that *everything* in English is typically used to generalize about all reference values, but more restricted uses can be traced to bounding classes provided by the context. One thing we can do here is to stipulate that, when we use it to read logical forms, *everything* will introduce no bounds narrower than the full referential range.

The second universal quantifier we will consider, the 1-place ***unrestricted universal quantifier***, amounts to a special case of restricted universal quantification where the restricting predicate has the whole range of referential values as its extension. There are a number of predicates that are certain to be ***universal*** in this sense. Since identity is reflexive, the abstract λx x = x is one example. Whenever ρ is a universal predicate, the sentence (∀x: ρx) θx says that the extension of the attribute predicate θ includes the whole of the referential range; that is, it says that θ is also universal. This sort of claim about a predicate θ is important enough that we add a one-place quantifier, enabling us to express it as ∀[λx θx]. The single predicate to which this quantifier applies will be called its ***quantified*** predicate. We will more often use the abbreviated form

∀x θx,

or

`Everything`, x, `is such that` θx

where θx is the ***quantified*** formula.

Similarly, ∀x (...x...) can be read as *Everything, x, is such that (... x...)*. For example, if F is λx (x *is fine*) and D is λx (x *is dandy*), the sentence ∀x (Fx ∧ Dx) can be read as *Everything, x, is such that both x is fine and x is dandy*.

We will not often write universals without abbreviation; but the unabbreviated symbolic expressions capture the logical form of universals most clearly, so it would be worth trying, at least once, to read them. A direct symbol-by-symbol reading of the unrestricted universal ∀[λx θx] would be ∀ *holds of* `the property of` x `that` (θ `fits` x), but if we put θ for the abstract, we may use ∀ *holds of* θ. By departing from the order of the symbols we can put the content of the claim made by ∀ into words as

θ *holds universally*.

A symbol-by-symbol reading of the restricted universal ∀[λx ρx][λx θx] would be something like ∀ *holds of* `the property of` x `that` (ρ `fits` x) *and* `the property of` x `that` (θ `fits` x) and, simplifying this a bit, we have ∀ *holds of* ρ *and* θ. Since ∀[λx ρx][λx θx] says that the extension of θ includes the extension of ρ, we can put it into words also as

θ *is (at least) as general as* ρ.

This brings us full circle back to a form that can be used in English. We could restate *Everything that walks moves* as *The property of moving is (at least) as general as the property of walking*. And we can understand the unrestricted quantifier in the same way: to say that θ holds universally is to say that θ is as general as can be.

Thus we have introduced two kinds of claims that might be made about predicates. The unrestricted universal $\forall x\ \theta x$ says that the predicate θ is universal, that it holds of all objects in the referential range. The restricted universal $(\forall x{:}\ \rho x)\ \theta x$ makes a more restricted claim, saying only that θ holds of all objects in the extension of the predicate ρ—i.e., that it is at least as general as ρ. Another way of putting the relation between the two would be to say that $\forall x\ \theta x$ ascribes absolute universality to θ while $(\forall x{:}\ \rho x)\ \theta x$ says only that θ is universal relative to ρ.

We have already seen that we can get the effect of unrestricted universal quantification while using the restricted universal quantifier if we choose a universal predicate $\lambda x\ x = x$ as the restricting predicate. In the other direction, we can get the effect of restricted universal quantification using the unrestricted quantifier by hedging the claim made by the quantified formula. The nature of the hedge that is needed can be found by trying to restate a restricted universal claim in the form *Everything is such that* .... Applying this idea to *Everything that walks moves* we get

Everything is such that (it moves if it walks),

a sentence that says that the predicate λx (x *moves if* x *walks*) is universal. In general, we can get the effect of restricted universal quantification by claiming universality for the result of making the quantified formula conditional on the restricting formula. That is, $(\forall x{:}\ \rho x)\ \theta x$ can be expressed as $\forall x\ (\rho x \rightarrow \theta x)$.

The two sorts of restatements we have been considering are licensed by the following principles of equivalence:

$$\forall x\ \theta x \Leftrightarrow (\forall x{:}\ x = x)\ \theta x$$
$$(\forall x{:}\ \rho x)\ \theta x \Leftrightarrow \forall x\ (\rho x \rightarrow \theta x).$$

We will have reason to make such restatements because the unrestricted universal quantifier is easier to use in stating laws of entailment while the restricted universal quantifier is easier to use in analyzing English sentences. In order to keep the connection between the two in mind, we will often express analyses made using the restricted universal also using the unrestricted quantifier.

7.2.2. Analyzing generalizations

A restricted universal sentence (∀x: ρx) θx is a generalization written symbolically. Its domain is the extension of ρ and its attribute is the property expressed by θ. Since we have already discussed the problem of identifying the domains and attributes of English sentences, we can complete our discussion of analyzing generalizations by saying how to choose restricting and quantified predicates ρ and θ so that the domain and attribute of the generalization (∀x: ρx) θx are what we want them to be. There is little to be said in the case of attributes. The quantified predicate θ of (∀x: ρx) θx should express the attribute, so it should be a symbolic version of the English quantified predicate in cases where the generalization is affirmative and a symbolic version of the denial of that predicate is cases where the generalization is negative. Notice that the quantified predicate appearing in the analysis of a negative generalization will correspond to the *negation* of the quantified predicate of the original English sentence; since symbolic generalizations are always affirmative, negative generalization is expressed by explicit negation in the quantified formula.

There is only a little more to be said in the case of domains. To get from a domain to a restricting predicate, we need a predicate that is true of just the things in the domain. When C is a term picking out the domain, a predicate of the form λx (x *is a* C) will be true of the objects in this class. When the domain is the complement of the class picked out by C, a predicate of the form λx (x *is not a* C) —i.e., λx (¬ x *is a* C)—may be used.

There is one complication to this in a case that is special but occurs quite frequently. The quantifier phrases *Everyone* and *No one* have the word *one* as their class indicator. But λx (x *is a one*) is ungrammatical and anyway does little to delimit a domain. So we are forced to treat *everyone* and *no one* as we would the synonymous (or nearly synonymous) *every person* and *no person* and use λx (x *is a person*) as the domain predicate.

Let us apply these ideas to some earlier examples of generalizations, beginning with *Every dog barks*. This is affirmative and direct. So the quantified predicate of the English sentence, λx (x *barks*), expresses the attribute of the generalization and can also give us the attribute predicate of the symbolic form. The domain is the class of dogs, so the domain predicate can be λx (x *is a dog*). Putting the two together we get the following symbolic renderings of the quantifier phrase, using the restricted and unrestricted quantifiers, respectively:

$$(\forall x\colon x\ \textit{is a dog})\ x\ \textit{barks}$$
$$\forall x\,(x\ \textit{is a dog} \rightarrow x\ \textit{barks})$$

These may be read as *Everything, x, such that x is a dog is such that x barks* and *Everything, x, is such that if x is a dog then x barks*.

The example *No dog climbs trees* was also direct but was negative. Thus we may use the same domain predicate but the quantified predicate of the symbolic form should be the denial of the English quantified predicate. This gives us the forms

$$(\forall x\colon x\ \textit{is a dog})\ \neg\ x\ \textit{climbs trees}$$
$$\forall x\,(x\ \textit{is a dog} \rightarrow \neg\ x\ \textit{climbs trees}),$$

which may be read as *Everything, x, such that x is a dog is such that not x climbs trees* and *Everything, x, is such that if x is a dog then not x climbs trees*.

Our first example of a negative and complementary generalization was *Only trucks were advertised*. The attribute here is the property of not having been advertised so the quantified predicate of the symbolic form may be λx (¬ x *was advertised*). The domain is the class of non-trucks. The restricting predicate can then be λx (¬ x *is a truck*) and the symbolic forms are these:

$$(\forall x\colon \neg\ x\ \textit{is a truck})\ \neg\ x\ \textit{was advertised}$$
$$\forall x\,(\neg\ x\ \textit{is a truck} \rightarrow \neg\ x\ \textit{was advertised})$$

These may be read as *Everything, x, such that not x is a truck is such that not x was advertised* and *Everything, x, is such that if not x is a truck then not x was advertised*.

More generally, we can offer the following symbolic versions of the three basic patterns of generalization we identified:

Direct and affirmative: *Every* C *is such that* …*it*…

$$(\forall x\colon x\ \textit{is a}\ C)\ \ldots x\ldots$$
$$\forall x\,(x\ \textit{is a}\ C \rightarrow \ldots x\ldots)$$

Direct and negative: *No* C *is such that* …*it*…

$$(\forall x\colon x\ \textit{is a}\ C)\ \neg\ \ldots x\ldots$$
$$\forall x\,(x\ \textit{is a}\ C \rightarrow \neg\ \ldots x\ldots)$$

Complementary and negative: *Only* C*s are such that* …*they*…

$$(\forall x\colon \neg\ x\ \textit{is a}\ C)\ \neg\ \ldots x\ldots$$
$$\forall x\,(\neg\ x\ \textit{is a}\ C \rightarrow \neg\ \ldots x\ldots)$$

If the domain C of a direct generalization is the whole referential range, the restricting predicate λx (x *is a* C) is not at all restrictive and we may use instead a simpler form with an unrestricted universal quantifier applying to the attribute predicate. So we have the following special cases of the direct forms of generalization:

Unrestricted and affirmative: *Everything is such that* …*it*…

$$\forall x\ \ldots x\ldots$$

Unrestricted and negative: *Nothing is such that* …*it*…

$$\forall x\ \neg\ \ldots x\ldots$$

A similar simplification would apply to complementary forms only if the class indicator was sure to pick out the empty set; you are invited to find an example.

These symbolic representations show us something about the relation between the English forms *All* C*s are such that* …*they*… and *Only* C*s are such that* …*they*…. If we represent these symbolically by applying unrestricted quantifiers to conditionals, we have the following (which are given with possible English readings below):

All C*s are such that* …*they*…	∀x (x *is a* C → …x…) *Everything, x, is such that (…x… if x is a C)*
Only C*s are such that* …*they*…	∀x (¬ x *is a* C → ¬ …x…) *Everything, x, is such that (…x… only if x is a C)*

This gives us a reason for saying that *all* is to *only* as *if* is to *only if*. And we can compare the fact that an *all*-generalization implicates an *only*-generalization to the fact that an *if*-conditional implicates an *only if*-conditional. Just as biconditionals expressing conjunctions of *if*-conditionals and *only-if*-conditionals can be stated using the compound conjunction *if and only if*, conjunctions of the corresponding sorts of generalizations can be expressed using the compound quantifier term *all and only*. The effect of the latter phrase is to claim that the indicated class is identical with the extension of the quantified predicate, and this claim can be expressed symbolically either as a conjunction of generalizations or by an unrestricted universal applying to a biconditional predicate. For example, *All and only winners of the first round are entitled to advance* might be analyzed by either of the following:

$$(\forall x\colon Wxf)\ Ex \land (\forall x\colon \neg\, Wxf)\ \neg\, Ex$$
$$\forall x\ ((Wxf \rightarrow Ex) \land (\neg\, Wxf \rightarrow \neg\, Ex))$$

[E: λx (x *is entitled to advance*); W; λxy (x *is a winner of* y); f: *the first round*]

The second can be read as *Everything, x, is such that (x is entitled to advance if and only if x is a winner of the first round)*.

Figure 7.2.4-1 below provides an overview of the process of analyzing generalizations.

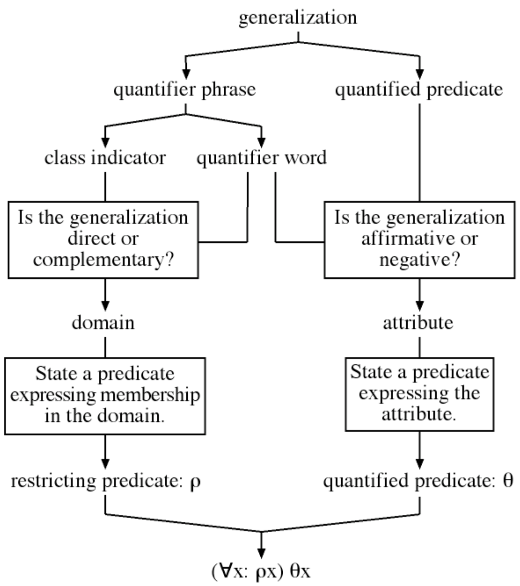

Fig. 7.2.4-1. The process of analyzing a generalization.

There are essentially four stages to the process:

(i) analyze the generalization into a quantifier phrase and quantified predicate (by restating it with the quantifier phrase as subject followed by *is such that*) and analyze the quantifier phrase into a quantifier word and class indicator;

(ii) find the domain and attribute of the generalization given the class indicator and quantified predicate, using the quantifier word to determine whether the generalization is direct or complementary and affirmative or negative;

(iii) state restricting and quantified predicates, which express membership in the domain and possession of the attribute, respectively;

(iv) combine the restricting and quantified predicates to state the generalization in symbolic form.

The restricting and quantified formulas, ρx and θx should be stated as English sentences containing the variable x so that they themselves can then be subjected to analysis.

7.2.3. Compound restrictions

Connectives may appear within generalizations when we analyze their restricting and quantified predicates. What we really analyze in such cases are the bodies of the lambda abstracts to which the quantifiers are applied. The analysis of such formulas and the problems that arise are not much different from those of truth-functional logic though the frequency with which various kinds of problems occur is different.

Since a restricting formula takes the form x *is a* C where C is a common noun together with modifiers, an analysis of it as a truth-functional compound will not be guided initially by English words marking connectives (apart from cases like λx (x *is a boy or girl*) or λx (x *is a non-smoker*) where the noun phrase itself is compounded using them). Indeed, the analysis of restricting formulas will usually be a matter of separating a common noun from its modifiers. As we saw in 2.1.3, considerable care must be taken in separating attributive adjectives from a common noun. The other modifiers we may find with common nouns—prepositional phrases and relative clauses—are less of a problem in this regard. The word *large* in x *is a flea that is large* acquires some of its significance from the word *flea* and should be restated more expansively when we analyze the open sentence to give something like x *is a flea* ∧ x *is large relative to fleas*. Other problems with attributive adjectives are absent or less pressing with relative clauses. While the open sentence x *is a good thief* is ambiguous (referring either to skill as a thief or to some compensating virtue that makes the thief a good person), x *is a thief who is good* probably speaks of compensating virtue and we would tend to use x *is a thief who is good at it* to speak of skill in thievery. The open sentence x *is an alleged murderer*, which does not admit any analysis as a conjunction, does not admit restatement with a relative clause either; x *is a murderer who is alleged to be one* means something different. The latter formula carries the implication x *is a murderer* and may be analyzed as a conjunction.

Once modifiers are separated from the common noun of a class indicator, a whole range of further logical structure may be open to logical analysis. Relative clauses, in particular, can be rich stores of truth-functional structure. For example, *The officer stopped every car that was either speeding or moving slowly and erratically* may be analyzed as follows:

Every car that was either speeding or moving slowly and erratically is such that (the officer stopped it)
(∀x: x *is a car that was either speeding or moving slowly and erratically*) (*the officer stopped* x)
(∀x: x *is a car* ∧ x *was either speeding or moving slowly and erratically*) Tox
(∀x: Cx ∧ (x *was speeding* ∨ x *was moving slowly and erratically*)) Tox
(∀x: Cx ∧ (Sx ∨ (x *was moving slowly* ∧ x *was moving erratically*))) Tox

$$(\forall x\colon Cx \wedge (Sx \vee (Lx \wedge Ex)))\ Tpx$$
$$\forall x\ (\ (Cx \wedge (Sx \vee (Lx \wedge Ex))) \to Tox)$$

[C: λx (x *is a car*); E: λx (x *was moving erratically*); L: λx (x *was moving slowly*); S: λx (x *was speeding*); T: λxy (x *stopped* y); o: *the officer*]

There is no special problem in finding the correct truth-functional analysis is this sort of case.

In some cases where we might expect a truth-functional analysis, we do not find one. This happens when a relative clause modifies the dummy class indicator *thing*. We would analyze the open sentence x *is a thing that is red* as we would x *is red*. And, in general, x *is a thing that* ... can be treated as ... x ... where the variable x may appear in any of a number of different positions when we put this into English; x *is a thing that Jack built* amounts to *Jack built* x and x *is a thing Dave sold to Ed* becomes *Dave sold* x *to Ed*. Of course, we can expect *thing* to drop out only when it appears as a dummy restriction (see the discussion of *everything vs. every* ***thing*** in 7.2.1).

Bounds and exceptions are another source of logical complexity in the restricting formula. To see how to represent them symbolically, let us return to the example that led us to these ideas. The generalization *Among members of the House, all Republicans except Midwesterners supported the bill* is affirmative so its attribute is expressed by its quantified predicate λx (x *supported the bill*) without use of negation; this will serve as the quantified predicate of the symbolic generalization. We found the domain to be the class of members of the House who are Republicans but not Midwesterners. Membership in this domain is expressed by the predicate λx (x *is a House member* ∧ x *is a Republican* ∧ ¬ x *is a Midwesterner*); this is the restricting predicate. Putting the two predicates together, we have the following:

(∀x: x *is a House member* ∧ x *is a Republican* ∧ ¬ x *is a Midwesterner*) x *supported the bill*

∀x ((x *is a House member* ∧ x *is a Republican* ∧ ¬ x *is a Midwesterner*) → x *supported the bill*)

(Parenthetical grouping of the conjuncts is neglected here only to make the result easier to read.)

The general pattern for an direct affirmative generalization with both bounds and exceptions is as follows:

Among B*s, all* C*s except* E*s are such that* ...*they*...

(∀x: x *is a* B ∧ x *is a* C ∧ ¬ x *is an* E) ...x...
∀x ((x *is a* B ∧ x *is a* C ∧ ¬ x *is an* E) → ...x...)

That is, to handle a bounding class picked out by B, we need to conjoin the formula x *is a* B to what we have otherwise. And, to handle a class of exceptions picked out by a term E, we need to conjoin the formula ¬ x *is an* E. The restricting formula of a direct negative generalization would be handled in the same way since the only difference from a corresponding affirmative generalization lies in the quantified formula.

The effect of bounds on complementary generalizations is analogous; the general pattern is this:

Among Bs, only Cs are such that ...they...

(∀x: x *is a* B ∧ ¬ x *is a* C) ¬ ...x...
∀x ((x *is a* B ∧ ¬ *x is a* C) → ¬ ...x...)

While the restricting formula of an unbounded complementary generalization is a negation, here the restricting formula is a *but-not* form.

7.2.s. Summary

Generalizations will be expressed symbolically using quantifiers, operations that take predicates as input and yield sentences as output. More specifically, we will use two universal quantifiers both written using the symbol ∀ (for all). The sentences form using these quantifiers will be called universals. The two quantifiers are the restricted universal quantifier, which applies to a pair of predicates to form a sentence, and the unrestricted universal quantifier, which applies to a single predicate. We will apply quantifiers only to abstracts. Since any pair of abstracts can be written in the form λx (...x...) and λx (---x---) using the same variable, we can abbreviate universal sentences as (∀x:...x...) ---x--- and ∀x ---x---. These may be put into English notation as `Everything`, x, `such that` ρx `is such that` θx `and Everything`, x, `is such that` θx. (Here the word *thing* is used as a dummy restriction that merely provides a hook for the relative clause.) The component expressions ...x... and ---x---, the restricting and quantified formulas of the universal, will not ordinarily be sentences in the strictest sense because they will contain free occurrences of the variable x. (The exceptions are the bodies of vacuous abstracts expressing predicates with a constant value.) Such expressions are included in the broader class of formulas, among which sentences are distinguished as closed formulas. Terms, too, can be classified as open or closed. A restricted universal says that the extension of the first predicate to which it is applied, the restricting predicate, is included in the extension of the second, the quantified predicate —i.e., it says that the second expresses a property that is at least as general as that expressed by the first. The unrestricted quantifier says that the quantified predicate to which it applies is universal, that it is a predicate that expresses a fully general property. An unrestricted universal sentence can be restated as a restricted universal whose domain predicate is universal, and a restricted universal can be restated as an unrestricted universal provided we make the attribute predicate conditional on the domain predicate.

An English generalization may be analyzed symbolically by using restricting and quantified predicates that capture its domain and attribute. If its domain consists of all reference values, an unrestricted universal may be used and we need only capture its attribute. In an affirmative generalization, the attribute predicate will be the quantified predicate of the English generalization while in a negative generalization it will be the negation of the quantified predicate. A formula applying the restricting predicate can be formed from the class indicator C by using the form x *is a* C, adding negation if the generalization is complementary. (However, we start with x *is a person* in the case of *everyone* and *no one*.) Bounds and exceptions may be captured by conjoining predicates of the same form, negated in the case of exceptions. The phrase *all and only* is used to express a conjunction of affirmative direct and negative complementary generalizations; but a generalization of this sort can be analyzed also by an unrestricted universal applying to a biconditional predicate because the two generalizations it implies can be expressed using an *if*-conditional and an *only-if*-conditional, respectively.

7.2.x. Exercise questions

1. Restate, with unrestricted quantifiers, the generalizations below that employ restricted quantifiers—and vice versa. Write out English readings for the results.

- **a.** (∀x: Fx) Gx
- **b.** ∀x (Fx → ¬ Gx)
- **c.** (∀x: Fx ∧ ¬ Gx) Hx
- **d.** ∀x ((Px ∧ ¬ Rxx) → Rxa)
- **e.** (∀x: Rxa ∧ ¬ Rbx) ¬ (Fx ∨ Gx)
- **f.** ∀x ((Fx ∨ Gx) → (Hx ∧ ¬ Kx))

2. Analyze the following in as much detail as possible, stating the resulting form using both restricted and unrestricted quantifiers:

- **a.** *Everyone had heard about the accident*.
- **b.** *Every relative of Sam agreed with him about the issue*.
- **c.** *Edna took pleasure in none of her possessions*.
- **d.** *Tom found only empty boxes*
- **e.** *The survey was sent to all members of the organization except its officers*.
- **f.** *Only countries bordering the Pacific will prosper*.

3. State in idiomatic English the generalizations that could be represented symbolically by the following:

- **a.** (∀x: x *is a dog*) x *chases cats*.
- **b.** (∀x: x *is a hole*) *Holly patched* x.
- **c.** (∀x: x *is a person*) ¬ x *volunteered*.
- **d.** (∀x: ¬ x *is a cockroach*) ¬ x *will survive*.
- **e.** ∀x ¬ x *seemed right*.
- **f.** (∀x: x *was a reviewer* ∧ ¬ x *was a friend of the director*) x *panned the movie*.
- **g.** (∀x: x *is a bird* ∧ ¬ x *is an early bird*) ¬ x *gets a worm*.
- **h.** (∀x: ¬ x *is a small child*) *the movie bored* x.

7.2.xa. Exercise answers

1. **a.** ∀x (Fx → Gx)
Everything, x, is such that if Fx [*or:* F fits x] then Gx

b. (∀x: Fx) ¬ Gx
Everything, x, such that F x is such that not Gx

c. ∀x ((Fx ∧ ¬ Gx) → Hx)
Everything, x, is such that if both Fx and not Gx then Hx

d. (∀x: Px ∧ ¬ Rxx) Rxa
Everything, x, such that both Px and not Rxx is such that Rxa

e. ∀x ((Rxa ∧ ¬ Rbx) → ¬ (Fx ∨ Gx))
Everything, x, is such that if both Rxa and not Rbx then not either Fx or Gx

f. (∀x: Fx ∨ Gx) (Hx ∧ ¬ Kx)
Everything, x, such that either Fx or Gx is such that both Hx and not Kx

2. **a.** *Everyone had heard about the accident*
Everyone is such that (he or she had heard about the accident)
(∀x: x *is a person*) x *had heard about the accident*

(∀x: Px) Hxa
∀x (Px → Hxa)

[H: λxy (x *had heard about* y); P: λx (x *is a person*); a: *the accident*]

b. *Every relative of Sam agreed with him about the issue*
Every relative of Sam is such that (he or she agreed with Sam about the issue)
(∀x: x *is a relative of Sam*) x *agreed with Sam about the issue*

(∀x: Rxs) Axsi
∀x (Rxs → Axsi)

[A: λxyz (x *agreed with* y *about* z); R: λxy (x *is a relative of* y); i: *the issue*; s: *Sam*]

c. *Edna took pleasure in none of her possessions*
No possession of Edna is such that (Edna took pleasure in it)
(∀x: x *is a possession of Edna*) ¬ *Edna took pleasure in* x

(∀x: Pxe) ¬ Lex
∀x (Pxe → ¬ Lex)

[L: λxy (x *took pleasure in* y); P: λxy (x *is a possession of* y); e: *Edna*]

d. *Tom found only empty boxes*
Among boxes, only empty ones are such that (Tom found them)
(∀x: x *is a box* ∧ ¬ x *is empty*) ¬ *Tom found* x

(∀x: Bx ∧ ¬ Ex) ¬ Ftx
∀x ((Bx ∧ ¬ Ex) → ¬ Ftx)

[B: λx (x *is a box*); E: λx (x *is empty*); F: λx (x *found* y); t: *Tom*]

e. *The survey was sent to all members of the organization except its officers*
All members of the organization except the organization's officers are such that (the survey was sent to them)
(∀x: x *is a member of the organization* ∧ ¬ x *is an officer of the organization*) *the survey was sent to* x

(∀x: Mxo ∧ ¬ Oxo) Ssx
∀x ((Mxo ∧ ¬ Oxo) → Ssx)

[M: λxy (x *is a member of* y); O: λxy (x *is an officer of* y); S: λxy (x *was sent to* y); o: *the organization*; s: *the survey*]

f. *Only countries bordering the Pacific will prosper*
Among countries, only those bordering the Pacific are such that (they will prosper)
(∀x: x *is a country* ∧ ¬ x *borders the Pacific*) ¬ x *will prosper*

(∀x: Cx ∧ ¬ Bxp) ¬ Px
∀x ((Cx ∧ ¬ Bxp) → ¬ Px)

[B: λxy (x *borders* y); C: λx (x *is a country*); P: λx (x *will prosper*); p: *the Pacific*]

3. **a.** (∀x: x *is a dog*) x *chases cats*
Every dog is such that (it chases cats)
Every dog chases cats (*or: All dogs chase cats*; *or: Dogs chase cats*)

b. (∀x: x *is a hole*) *Holly patched* x
Every hole is such that (Holly patched it)
Holly patched every hole (*or: Holly patched each hole*)

c. (∀x: x *is a person*) ¬ x *volunteered*
No one is such that (he or she volunteered)
No one volunteered.

d. (∀x: ¬ x *is a cockroach*) ¬ x *will survive*
Only cockroaches are such that (they will survive)
Only cockroaches will survive

e. ∀x ¬ x *seemed right*
Nothing is such that (it seemed right)
Nothing seemed right

f. (∀x: x *was a reviewer* ∧ ¬ x *was a friend of the director*) x *panned the movie*
All reviewers except friends of the director are such that (they panned the movie)
All reviewers except friends of the director panned the movie
(*or: Among reviewers, all but friends of the director panned the movie*)

g. (∀x: x *is a bird* ∧ ¬ x *is an early bird*) ¬ x *gets a worm*
Among birds, only early ones are such that (they get worms)
Only early birds get worms (*or: No birds except early ones get worms*)

h. (∀x: ¬ x *is a small child*) *the movie bored* x
All things except small children are such that (the movie bored them)
The movie bored all but small children