7.2.1. The universal quantifier

A ***quantifier*** is an operation that takes predicates as input and yields sentences as output. The quantifiers we will consider all apply only to 1-place predicates, but we will consider them in two forms, one of which is a 2-place operation applying to a pair of 1-place predicates and another that is a 1-place operation applying to a single 1-place predicate. When there is no need to distinguish them we will refer to both as ***universal quantifiers*** and describe the formulas they form also as *universal* (or, less formally, use *universal* as a common noun and refer to them as *universals*).

Although we can make perfectly good sense of the application of quantifiers to unanalyzed predicates, we will almost always apply them to abstracts, using an abstract λx Fx in place of an unanalyzed predicate F. We would need to consider the application of quantifiers to abstracts anyway, and it will simplify things to focus on this case. We can choose alphabetic variants so that any pair of abstracts can be written with the same variable. So, when we speak below of a pair of abstracts λx θx and λx ρx, what we say can be extended to any pair of abstracts whatsoever. Also notice that λx θx—which refers to the property that x has in virtue of θ being true of it—will be true of the same values as θ is. So it will not hurt, when talking about semantics, to think of λx θx and θ as the same predicate (though, strictly speaking, θ may be an unanalyzed predicate while λx θx is complex).

Our 2-place quantifier is the ***restricted universal quantifier*** for which we use the symbol ∀ (the symbol ***for all***). The sentence ∀[λx ρx][λx θx] that results from applying the restricted universal quantifier to abstracts λx ρx and λx θx will be referred to as a *restricted universal*. It says that θ is true of everything that ρ is true of—i.e., that the extension of θ includes the extension of ρ. This makes ∀[λx ρx][λx θx] an affirmative direct generalization whose domain is the extension of ρ and whose attribute is expressed by θ. Since the scope of the generalization is limited to the extension of ρ we will refer to ρ as the ***restricting predicate***, and we will refer to θ, which expresses the property said to hold generally, as the ***quantified predicate***.

The simplest case of a restricted universal is one whose restricting and quantified predicates are unanalyzed. For example, if W is λx (x *walks*) and M is λx (x *moves*), then $\forall[\lambda x\ Wx][\lambda x\ Mx]$ says that anything that walks also moves. More often, the restricting or quantified predicate will have internal structure. For example, if we want to say that anything that walks and talks both moves and communicates, we can do this with the form

$$\forall\ [\lambda x\ (Wx \wedge Tx)]\ [\lambda x\ (Mx \wedge Cx)]$$

where T is λx (x *talks*) and C is λx (x *communicates*).

Since we always are able to write the two abstracts using the same variable, we can use the following more abbreviated notation for the universal sentence:

$$(\forall x\text{: } Wx \wedge Tx)\ (Mx \wedge Cx).$$

The symbolic form $(\forall x\text{: } Wx \wedge Tx)\ (Mx \wedge Cx)$ can be read in something close to English as *Everything, x, such that (x walks and x talks) is such that (x moves and x communicates)*. And, in general, the form

$$(\forall x\text{: } \rho x)\ \theta x$$

can be rendered in English as

`Everything`, x, `such that` ρx `is such that` θx.

Here we can regard ρ and θ as the predicates to which the quantifier applies, with the apparatus of variable binding absorbed into the quantifier.

We may adapt an alternative notation we have used for abstracts to write the form of a restricted universal schematically as

(∀x: ...x...) ---x---

which amounts to

`Everything`, x, `such that` (...x...) `is such that` (---x---)

To extend a grammatical pun used before, this can be read as *Everything, x, such that (x **dots**) is such that (x **dashes**).*

The components ...x... and ---x--- of this form (or ρx and θx of the other way of writing the general form of a restricted universal) are the bodies of the abstracts to which the operation ∀ is applied. When they are removed from the universal and considered by themselves they will usually contain one or more occurrences of a variable x that is not bound by any abstract. Such a variable is called a ***free variable***; it can be compared to an anaphoric pronoun that is missing an antecedent. Since we refer to sentence-like expressions that may contain free variables as formulas, sentences in the strict sense have no free variables and can be described as ***closed formulas***. We will use the expression *term* in a way analogous to *formula* and apply it to expressions with or without free variables; we can speak of ***open*** and ***closed*** terms depending on whether free variables do or do not occur. Although up to this chapter our symbolic forms have included only closed terms and closed formulas (i.e., sentences), we will now extend the syntactic apparatus of earlier chapters to all terms and formulas. The semantic ideas of earlier chapters apply also with the exception that an open term or open formula has a value on an interpretation of its non-logical vocabulary only when a reference value is assigned to each of its free variables.

(In the preceding paragraph, it was said that the formulas …x… and ---x--- "usually" contain free variables. That's because the variable x need not appear in the body of an abstract with the lambda operator λx. For example, the abstract λx 2 would be used to express the constant function f defined by f(x) = 2. Such an abstract is said to be ***vacuous***.)

The formula …x… in (∀x: …x…) ---x--- (i.e., ρx in (∀x: ρx) θx) says what must be true of x for it to be in the domain of the generalization; we will refer to it as the ***restricting formula***. The formula ---x--- (i.e., θx) says that x has the attribute of the generalization. The generalization says something how many values in the domain will make θx true when they are assigned to x (namely that they all will), so we will refer to θx as the ***quantified formula***. This is a direct extension of our terminology for the component predicates of a generalization: the restricting formula is a predication of the restricting predicate and the quantified formula is a predication of the quantified predicate.

When reading the symbolic notation, we add the variable x as an appositive marked off by commas after the quantifier phrase to indicate that this quantifier phrase serves as the antecedent of the symbolic pronouns x. If we put English pronouns in place of the variables, we have can rely on the conventions of syntax to determine the antecedent and we can drop the appositive to get

Everything such that (…it…) is such that (---it---)

This is a generalization whose class indicator is *thing such that*

(...it...) and whose quantified predicate is λx (x *is such that (---it---)*). Notice that the adjectival phrases *such that (...it...)* and *such that (---it---)* have two different functions in this sentence. The first appears as a modifier of the common noun *thing* while the second is a predicate adjective. Their roles are comparable to those of *scarlet* and *red*, respectively, in *Everything scarlet is red*.

The use of *thing* here also deserves some comment. Consider an English generalization that uses the same form of words as these readings—*Everything such that it walks is such that it moves*, for example. This generalization is direct and affirmative. The class indicator is the phrase *thing such that it walks*; and the predicate λx (x *is such that it moves*) is the quantified predicate. Now if this sentence is to make the same claim as (∀x: x *walks*) x *moves*, the indicated class of the English sentence should be the extension of λx (x *walks*) and the attribute expressed by the English quantified predicate should be the extension of λx (x *moves*). There is certainly no problem in the latter case; λx (x *is such that it moves*) is just a more cumbersome way of expressing λx (x *moves*). But does *thing such that it walks*, or *thing that walks*, really indicate the extension of λx (x *walks*)?

It does if we take the word *thing* to indicate the full range of reference values rather than being limited, say, to inanimate objects. We may say that, in such a use, *thing* is a ***dummy restriction***. It does not itself restrict the domain of the generalization but provides a grammatical anchor for further restrictions. We have been using the word that way as an alternative to *object*, *entity*, and *individual*, but is it used that way ordinarily? This is not the sort of question we can settle here, but notice that if we really want emphasize that our generalization concerns "things" in some specialized sense, we are likely to use the two-word phrase *every* ***thing***, with an emphasis on *thing*, rather than the single word *everything*. This is not to say that *everything* in English is typically used to generalize about all reference values, but more restricted uses can be traced to bounding classes provided by the context. One thing we can do here is to stipulate that, when we use it to read logical forms, *everything* will introduce no bounds narrower than the full referential range.

The second universal quantifier we will consider, the 1-place

unrestricted universal quantifier, amounts to a special case of restricted universal quantification where the restricting predicate has the whole range of referential values as its extension. There are a number of predicates that are certain to be ***universal*** in this sense. Since identity is reflexive, the abstract λx x = x is one example. Whenever ρ is a universal predicate, the sentence (∀x: ρx) θx says that the extension of the attribute predicate θ includes the whole of the referential range; that is, it says that θ is also universal. This sort of claim about a predicate θ is important enough that we add a one-place quantifier, enabling us to express it as ∀[λx θx]. The single predicate to which this quantifier applies will be called its ***quantified*** predicate. We will more often use the abbreviated form

∀x θx,

or

`Everything`, x, `is such that` θx

where θx is the ***quantified*** formula.

Similarly, ∀x (...x...) can be read as *Everything, x, is such that (... x...)*. For example, if F is λx (x *is fine*) and D is λx (x *is dandy*), the sentence ∀x (Fx ∧ Dx) can be read as *Everything, x, is such that both x is fine and x is dandy*.

We will not often write universals without abbreviation; but the unabbreviated symbolic expressions capture the logical form of universals most clearly, so it would be worth trying, at least once, to read them. A direct symbol-by-symbol reading of the unrestricted universal ∀[λx θx] would be ∀ *holds of* `the property of` x `that` (θ `fits` x), but if we put θ for the abstract, we may use ∀ *holds of* θ. By departing from the order of the symbols we can put the content of the claim made by ∀ into words as

θ *holds universally*.

A symbol-by-symbol reading of the restricted universal ∀[λx ρx][λx θx] would be something like ∀ *holds of* `the property of` x `that` (ρ `fits` x) *and* `the property of` x `that` (θ `fits` x) and, simplifying this a bit, we have ∀ *holds of* ρ *and* θ. Since ∀[λx ρx][λx θx] says that the extension of θ includes the extension of ρ, we can put it into words also as θ *is (at least) as general as* ρ.

This brings us full circle back to a form that can be used in English. We could restate *Everything that walks moves* as *The property of moving is (at least) as general as the property of walking*. And we can understand the unrestricted quantifier in the same way: to say that θ holds universally is to say that θ is as general as can be.

Thus we have introduced two kinds of claims that might be made about predicates. The unrestricted universal ∀x θx says that the predicate θ is universal, that it holds of all objects in the referential range. The restricted universal (∀x: ρx) θx makes a more restricted claim, saying only that θ holds of all objects in the extension of the predicate ρ—i.e., that it is at least as general as ρ. Another way of putting the relation between the two would be to say that ∀x θx ascribes absolute universality to θ while (∀x: ρx) θx says only that θ is universal relative to ρ.

We have already seen that we can get the effect of unrestricted universal quantification while using the restricted universal quantifier if we choose a universal predicate λx x = x as the restricting predicate. In the other direction, we can get the effect of restricted universal quantification using the unrestricted quantifier by hedging the claim made by the quantified formula. The nature of the hedge that is needed can be found by trying to restate a restricted universal claim in the form *Everything is such that* .... Applying this idea to *Everything that walks moves* we get

Everything is such that (it moves if it walks),

a sentence that says that the predicate λx (x *moves if* x *walks*) is universal. In general, we can get the effect of restricted universal quantification by claiming universality for the result of making the quantified formula conditional on the restricting formula. That is, (∀x: ρx) θx can be expressed as ∀x (ρx → θx).

The two sorts of restatements we have been considering are licensed by the following principles of equivalence:

∀x θx ⇔ (∀x: x = x) θx
(∀x: ρx) θx ⇔ ∀x (ρx → θx).

We will have reason to make such restatements because the unrestricted universal quantifier is easier to use in stating laws of entailment while the restricted universal quantifier is easier to use in analyzing English sentences. In order to keep the connection between the two in mind, we will often express analyses made using the restricted universal also using the unrestricted quantifier.