4.3.1. Detachment rules

When we exploit a disjunction using a proof by cases, we divide the parent gap into two children. Something like this is essential in any rule that allows us to exploit a disjunction by way of reasoning about its disjuncts, for the truth of a disjunction does not settle the truth values of its disjuncts. However, if we add to the disjunction information about the truth value of one disjunct, it can be possible to conclude something about the other.

In particular, if we know both that a disjunction is true and that one of its disjuncts is false, we can conclude that the other disjunct is true. This idea appears in a pattern of argument recognized long enough to have acquired a Latin name: ***modus tollendo ponens***

$$\text{MTP}\ \frac{\varphi \vee \psi \quad \overline{\varphi}}{\psi} \qquad \text{MTP}\ \frac{\varphi \vee \psi \quad \overline{\psi}}{\varphi}$$

The name refers to what the second premise and conclusion say about the two disjuncts. It can be translated, very roughly, as *way, by taking, of putting*. That is, the argument enables you to put forth one component as the conclusion if you take away the other component by asserting a premise that bars it.

The use of this idea in derivations will be based on a somewhat stronger pair of principles for which we will also use the name ***modus tollendo ponens***.

$\Gamma, \varphi \vee \psi, \overline{\varphi} \Rightarrow \chi$ if and only if $\Gamma, \psi, \overline{\varphi} \Rightarrow \chi$

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These tell that, in the presence of a sentence barring of one component of a disjunction, having the disjunction as a premise comes to the same thing as having its other component as a premise. These principles depend on the validity of the arguments above and also on the fact that a disjunction is entailed by each of this components individually.

The *modus tollendo ponens* principles describe grounds under which we can drop a disjunction from our active resources (and replace it by one of its disjuncts), so they justify a rule ***Modus Tollendo Ponens*** (MTP) that provides an added way of exploiting a disjunction.

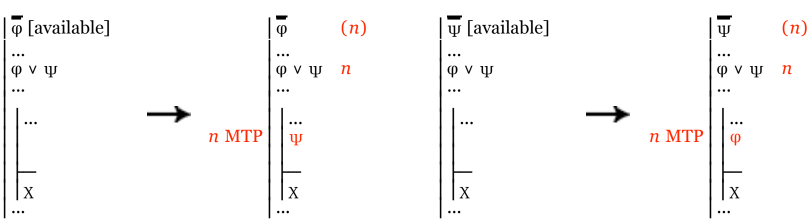

Fig. 4.3.1-1. Developing a derivation at stage *n* by exploiting a disjunction when a sentence barring one component is also an active resource.

Notice that the barred component is not exploited, so the stage number to its right is enclosed in parentheses. And, since we are not exploiting this resource, there is no need for it to be active. As is the case with the resources required by adjunction rules or rules for closing gaps, it is enough that this resource be available. On the other hand, the disjunction itself is exploited, so it must be active and the stage number added at its right is not parenthesized.

This is only the first of a number of rules that will enable us to exploit weak compounds in the presence of information about a component. We will label as ***detachment rules*** these rules along with others that enable us to exploit resources given certain further information. The resource that is exploited by such a rule will be called the ***main resource*** while the resource that must be available but is not exploited will be called the ***auxiliary resource***. In the case of MTP, the disjunction is the main resource and the sentence barring one of its disjuncts is the auxiliary resource.

The second detachment rule we will add concerns the *not-both* form. De Morgan's laws tell us that the form $\neg (\varphi \wedge \psi)$ is equivalent to the disjunction $\overline{\varphi} \vee \overline{\psi}$, so we should expect some appropriate modification of *modus tollendo ponens* to be valid. The proper form is this:

$$\text{MPT}\ \frac{\neg (\varphi \wedge \psi) \quad \varphi}{\overline{\psi}} \qquad\qquad \text{MPT}\ \frac{\neg (\varphi \wedge \psi) \quad \psi}{\overline{\varphi}}$$

These arguments are called ***modus ponendo tollens***: they are a way of, by putting, taking. That is, if we know that φ and ψ are not both true, adding the information that one of them is true (i.e., putting it forth), enables us to conclude that the other is not true (i.e., we can take it away). The corresponding principles called *modus ponendo tollens* are these:

$\Gamma, \neg (\varphi \wedge \psi), \varphi \Rightarrow \chi$ if and only if $\Gamma, \overline{\psi}, \varphi \Rightarrow \chi$

$\Gamma, \neg (\varphi \wedge \psi), \psi \Rightarrow \chi$ if and only if $\Gamma, \overline{\varphi}, \psi \Rightarrow \chi$

They are based on the *modus ponendo tollens* arguments and also on the fact that a *not-both* form $\neg (\varphi \wedge \psi)$ is entailed by a sentence barring either φ or ψ. That is, in the presence of a premise asserting φ or ψ, the *not-both* $\neg (\varphi \wedge \psi)$ can be replaced by a sentence that bars the other component.

The rule ***Modus Ponendo Tollens*** (MPT) is this:

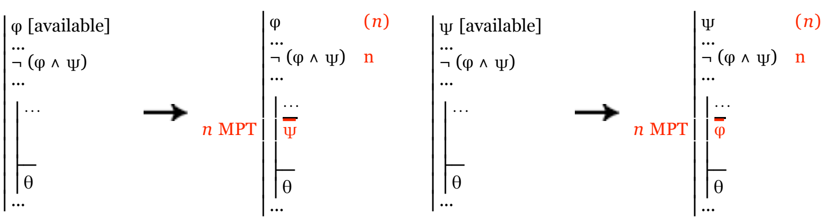

Fig. 4.3.1-2. Developing a derivation at stage *n* by exploiting a negated conjunction when a conjunct is also an active resource.

As with MTP, one resource, the main resource, is exploited (and should be active) while the other, auxiliary resource, is not exploited and need only be available.

As an example of these new rules, here is an alternative version of the derivation at the end of 4.2.1 :

	¬ (P ∧ ¬ G)	2
	¬ (C ∧ ¬ G)	4
	P ∨ C	3
	│ ¬ G	(2),(5)
2 MPT	│ ¬ P	(3)
3 MTP	│ C	(4)
4 MPT	│ G	(5)
	│ ●	
5 Nc	│ ⊥	1
1 IP	G	

This is far from the only way of using the new rules to complete the derivation. To choose only the most minor variation on the one above, notice that in the second use of MPT either G or ¬ ¬ G could be concluded (since both can be described as $\overline{\neg\ G}$). And either could be used along with ¬ G to conclude ⊥ by Nc.

Notice that the supposition ¬ G (*Sam didn't grant the proposal's significance*) enables us to conclude first that ¬ P (*Sam didn't praise the proposal*), then C (*Sam condemned the proposal*), and finally G itself. An argument by which a claim is shown to follow from its own denial is traditionally called a ***consequentia mirabilis*** (an amazing consequence) and has been a standard form of philosophical argumentation since antiquity. (For example, a common way of arguing against a skeptic who denies the existence of knowledge is show that this claim, that there is no knowledge, in fact implies that there is knowledge, which leads to the conclusion that knowledge must exist. Any reply to this argument must question the moves by which one is supposed to get from the claim that there is no knowledge to the consequence that there is knowledge because, if this transition is valid, an indirect proof will show that knowledge does exist.)