4.2.4. The duality of conjunction and disjunction

While a conjunction and a disjunction formed from the same components are certainly not contradictories, the two connective are opposites in another sense, the one for which we have used the term *dual*.

This duality can be expressed in one way by saying that when conjunction and disjunction are applied to pairs of sentences whose corresponding components are contradictory, the results are contradictory. For example, let us again take *X was home* and *X was out* to be contradictories. Then note that to get a sentence contradictory to *Ann and Bill were home*, we cannot take *Ann and Bill were out* since both would be false if one of Ann and Bill was home and the other out. To get a contradictory to we need to cover both of those possibilities as well, and *Ann or Bill was out* will do this. That is, *Ann and Bill were home* is contradictory to *Ann or Bill was out* and, similarly, *Ann or Bill was home* is contradictory to *Ann and Bill were out*. And this is to say that ¬ *Ann and Bill were home* ⇔ *Ann or Bill was out* and that ¬ *Ann or Bill was home* ⇔ *Ann and Bill were out*.

When limited to the cases of contradictoriness captured by the bar notation, these patterns of equivalence are know as ***De Morgan's laws***:

$$\neg\,(\varphi \wedge \psi) \Leftrightarrow \overline{\varphi} \vee \overline{\psi}$$
$$\neg\,(\varphi \vee \psi) \Leftrightarrow \overline{\varphi} \wedge \overline{\psi}$$

Although these are named after Augustus De Morgan (1806-1871), they were known well before his time.

Another way to see the duality of conjunction and disjunction is to look at the principles that hold for them with respect to relative exhaustiveness. The table below follows the pattern of the one given for ⊥ and ⊤ in 1.4.6.

	as a premise	as an alternative
Conjunction	$\Gamma, \varphi \wedge \psi \Rightarrow \Delta$ iff $\Gamma, \varphi, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \varphi \wedge \psi, \Delta$ iff both $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma \Rightarrow \psi, \Delta$
Disjunction	$\Gamma, \varphi \vee \psi \Rightarrow \Delta$ iff both $\Gamma, \varphi \Rightarrow \Delta$ and $\Gamma, \psi \Rightarrow \Delta$	$\Gamma \Rightarrow \varphi \vee \psi, \Delta$ iff $\Gamma \Rightarrow \varphi, \psi, \Delta$

(Here *iff* is used as an abbreviation of *if and only if*.) Notice that the analogy between the upper left and lower right and between the lower left and upper right. That is, conjunction behaves as a premise much as disjunction behaves as an alternative and disjunction behaves as premise much as conjunction behaves as an alternative.

Since ⊥ and ⊤ are paired as duals and so are conjunction and disjunction, you might wonder about negation. In fact, it is dual to itself. If we negate each of a pair of contradictory sentences, the results are contradictory; that is, we do not need to apply different operations to the two contradictory sentences in order for the results to be contradictory.

And negations behavior as a premise is analogous to its behavior as an alternative.

$$\Gamma, \neg\,\varphi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \varphi, \Delta$$
$$\Gamma, \varphi \Rightarrow \Delta \text{ iff } \Gamma \Rightarrow \neg\,\varphi, \Delta$$

Having a negated premise or alternative is equivalent to having the unnegated sentence in the opposite role.

The term *duality* in general points to a certain sort of two-for-one principle. In particular, it is used when there is some way of associating vocabulary items as pairs so that replacing one member of a pair by the other throughout any truth will yield another truth. In our case, we have the associations

premise —— alternative

$\bot$ —— $\top$

negation —— negation

conjunction —— disjunction

So, for example (and to deal only with informal statements of the principles), the principle *A conjunction as a premise may be replaced by its components as separate premises* (the upper left in the table above) turns into *A disjunction as an alternative may be replaced by its components as alternatives* (the lower right). And the principle *A negation as a premise may be replaced by its immediate component as an alternative* (the first of the principles for negation above) turns into *A negation as an alternative may be replaced by its immediate component as a premise* (the second of the principles). We will see more examples of such transformations in the next section but we have already seen some further ones: each of the two forms of De Morgan's laws may be transformed into the other by this association.

Since these transformations treat premises and alternatives in a parallel way, not all will apply to entailment, which allows multiple premises but only a single alternative. However, we have also seen that principles for relative exhaustiveness may be transformed still further into principles of entailment by the basic law for relative exhaustiveness (of which the two principles for negation above are special cases) since that law enables us to replace alternatives by premises that are their contradictories.