4.2.1. Proofs by cases

The validity of the argument

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam either praised or condemned the proposal
Sam granted the proposal's significance.

can be accounted for by the validity of the following two arguments:

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam praised the proposal
Sam granted the proposal's significance

Sam didn't praise the proposal without granting its significance
Sam didn't condemn the proposal without granting its significance
Sam condemned the proposal
Sam granted the proposal's significance.

Each replaces the disjunctive third premise of the original argument by one of its two components. This way of establishing an entailment is sometimes called a ***proof by cases***. In this example, the two cases are Sam having praised the proposal and Sam having condemned it. Since the disjunction says all and only what is common to these two claims, what follows from the disjunction in isolation or in addition to other premises is what follows from each of these claims under similar circumstances.

More formally, the idea behind proofs by cases is captured by a ***law for disjunction as a premise***:

$\Gamma, \varphi \vee \psi \Rightarrow \chi$ if and only if both $\Gamma, \varphi \Rightarrow \chi$ and $\Gamma, \psi \Rightarrow \chi$

To see why this law is true note that to divide the members of Γ and $\varphi \vee \psi$ on the one hand from χ on the other, a possible world must make $\varphi \vee \psi$ and all members of Γ true while making χ false. To do this it must make at least one of φ and ψ true, so it must divide at least one of the arguments $\Gamma, \varphi \,/\, \chi$ and $\Gamma, \psi \,/\, \chi$. So, to say that the original argument is valid is to say that neither of these latter arguments can have its premises and alternatives divided—that is, that both are valid.

This idea appears in derivations by way of a rule we will call ***Proof by Cases*** (PC); it is shown in Figure 4.2.1-1.

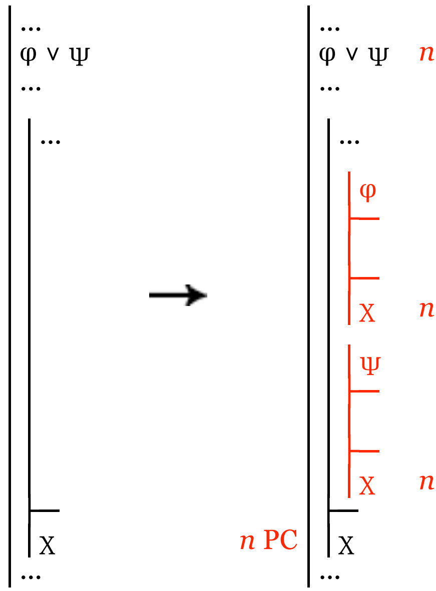

Fig. 4.2.1-1. Developing a derivation by exploiting a disjunction at stage *n*.

PC divides a gap into two new gaps. Each is a ***case argument*** that retains the original goal but adds one of the components of the disjunction as a supposition. The function of each supposition is to specify one of the two sorts of case in which the original disjunction is true. A supposition is required because, although our premises tell us that at least one of the disjuncts is true, we do not know which that is and the one that is true will vary among the possible worlds in which the premises are all true.

Here is a derivation which uses this rule to provide a proof for example with which we began.

	│ $\neg (P \wedge \neg G)$	(4)
	│ $\neg (C \wedge \neg G)$	(7)
	│ $P \vee C$	1
	│ │ P	(3)
	│ │ │ $\neg G$	(3)
3 Adj	│ │ │ $P \wedge \neg G$	X,(4)
	│ │ │ ●	
4 Nc	│ │ │ $\bot$	2
2 IP	│ │ G	1
	│ │ C	(6)
	│ │ │ $\neg G$	(6)
6 Adj	│ │ │ $C \wedge \neg G$	X,(7)
	│ │ │ ●	
7 Nc	│ │ │ $\bot$	5
5 IP	│ │ G	1
1 PC	│ G	

[C: *Sam condemned the proposal*; G: *Sam granted the proposal's significance*; P: *Sam praised the proposal*]

In the two case arguments, we suppose first that Sam praised the proposal and then that he condemned it and, in each case, we show that he granted the proposal's significance (by showing that he could not have failed to grant it). Since at least one of these two cases must be true whenever the premises are all true, we know that the conclusion must be true also.

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