1.4.6. A general framework

The value of relative exhaustiveness does not lie in capturing some ordinary vocabulary for discussing deductive reasoning but instead in its ability to encompass other ideas that do. For example, entailment is the special case of relative exhaustiveness where a single alternative is rendered exhaustive. And the notation is the same. Given our notational conventions, $\Gamma \Rightarrow \varphi$ means both that φ is entailed by Γ and that the set whose only member is φ is rendered exhaustive by Γ. Inconsistency can be expressed in terms of relative exhaustiveness by way of entailment, but it can be expressed in a more direct way, too. Since there is no way that the empty set could have at least one member true in any possible world, to say that it is exhaustive relative to a set Γ is to say that the members of Γ cannot all be true. So to write $\Gamma \Rightarrow$ (with nothing to the right of the arrow) is to say that Γ is inconsistent.

The following table uses connections with relative exhaustiveness to provide notation for each of the concepts we have seen in this chapter:

Concept	*Notation*
Γ entails φ	$\Gamma \Rightarrow \varphi$
φ is a tautology	$\Rightarrow \varphi$
φ and ψ are equivalent	both $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$ (abbreviated to $\varphi \Leftrightarrow \psi$)
Γ excludes φ	$\Gamma, \varphi \Rightarrow$
Γ is inconsistent	$\Gamma \Rightarrow$
φ and ψ are mutually exclusive	$\varphi, \psi \Rightarrow$
φ is absurd	$\varphi \Rightarrow$
Γ is exhaustive	$\Rightarrow \Gamma$
φ and ψ are jointly exhaustive	$\Rightarrow \varphi, \psi$
φ and ψ are contradictory	both $\varphi, \psi \Rightarrow$ and $\Rightarrow \varphi, \psi$ (abbreviated to $\Leftrightarrow \varphi, \psi$)

The double arrow notation used for equivalence and contradictoriness may suggest the more general idea of sets that render each other exhaustive—i.e., sets Γ and Δ such that $\Gamma \Leftrightarrow \Delta$. But this idea is not of much interest apart from the two special cases. In particular, sets related in this way need not have equivalent roles in deductive reasoning (as can be seen by noting that a pair of contradictory sentences form a set that renders exhaustive and is rendered exhaustive by the empty set).

The properties and relations listed above amount to guarantees that certain patterns of truth values are logically impossible. And the particular patterns are shown by the notation used, for in each case $\Gamma \Rightarrow \Delta$ says that no possible world divides Γ from Δ. It is also useful to have vocabulary for speaking about cases where *no* pattern is ruled out. We will say that a sentence is ***logically contingent*** if both truth values appear for it among possible worlds—if it is both possibly true and possibly false. This is to say that it is neither tautologous nor absurd. We will say that a pair of sentences are ***logically independent*** if each of the four patterns of truth values for the two sentences is exhibited in some possible world. This is the same as saying that the two sentences are neither mutually exclusive nor jointly exhaustive and that neither implies the other. This also implies that each of the two sentences is logically contingent.

Finally, we can extend the idea of logical independence to a set of sentences by saying that the members of a set of sentences are independent when any way of assigning a truth value to each of them is exhibited in at least one possible world. When the sentences in a set are *not* independent, not every way of dividing them into a set of true sentences and a set of false sentences is logically possible—and that is to say that the set contains two disjoint (i.e., non-overlapping) subsets one of which renders the other exhaustive. So the members of a set Γ are logically independent when the relation of relative exhaustiveness never holds between non-overlapping subsets of Γ. (When a pair of sets do share a member, each renders the other exhaustive no matter what the sets are like otherwise.) When a set is logically independent, each member is contingent and any two of its members are logically independent, but contingency of members and independence of pairs does not by itself imply that the set as a whole is logically independent. For example, assuming the sentences *X is fast*, *X is strong*, *X has skill*, and *X has stamina* form an independent set, the sentences

X is fast and strong	*X has skill and stamina*	*X is fast and has stamina*

are each contingent, and any two of them are independent.

However, the first two taken together entail the third, so these three more complex sentences do not form an independent set.

The idea of relative exhaustiveness thus provides the resources for both surveying other deductive properties and relations and for speaking about cases where none of them hold. But the kind of symmetry built into the idea (and exhibited in the laws for it seen at the end of 1.4.5) also provides a way of describing connections among various deductive principles.

This description will also employ the idea of contradictoriness. If a pair of sentences are contradictory, then each will be true in a possible world if and only if the other is false. Since in assessing relative exhaustiveness, we consider, for each possible world, the truth of premises and the falsity of alternatives, having one of a pair of contradictory sentences as a premise comes to the same thing as having the other as an alternative. So we can remove a sentence from one side of the arrow if we add a contradictory sentence on the other side. This is stated more formally in the following, which will serve as our ***basic law for relative exhaustiveness***

if $\Leftrightarrow \varphi, \varphi'$, then $\Gamma, \varphi \Rightarrow \Delta$ if and only if $\Gamma \Rightarrow \varphi', \Delta$

We can use this principle (read from right to left) to replace alternatives by contradictory premises. If we begin with a finite set, we can eventually transform a claim of exhaustiveness into a claim of entailment. Our chief application of this will be one its simplest cases: if $\Leftrightarrow \varphi, \varphi'$, then $\Gamma \Rightarrow \varphi, \psi$ if and only if $\Gamma, \varphi' \Rightarrow \psi$. That is, a pair of sentences φ and ψ are rendered exhaustive by a set Γ of premises if and only one of the pair, ψ, is entailed by Γ together with a sentence φ' that is contradictory to the other member of the pair. Since the order of a list of alternatives does not matter (so saying that $\Gamma \Rightarrow \varphi, \psi$ is the same as saying that $\Gamma \Rightarrow \varphi, \psi$), this law tells us that we can drop either of a pair of alternatives if we add to the premises a sentence contradictory to the alternative we drop.

The properties of $\top$ and $\bot$ take a particularly symmetric form when stated in terms of relative exhaustiveness.

	as a premise	*as an alternative*
Tautology	if $\Gamma, \top \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$	$\Rightarrow \top$
Absurdity	$\bot \Rightarrow$	if $\Gamma \Rightarrow \bot, \Delta$, then $\Gamma \Rightarrow \Delta$

That is, while $\top$ contributes nothing as a premise and may be dropped, it is sufficient by itself as the only alternative (no matter how small our set of premises). And while $\bot$ is sufficient by itself as a premise (no matter how small the set of alternatives is), it contributes nothing as an alternative and may be dropped. The symmetry here might be traced to that of relative exhaustiveness: since $\top$ and $\bot$ are contradictory, the principles on each diagonal are connected by the basic law for relative exhaustiveness.

However, there is a more general idea behind this symmetry. To take the simplest case above, we might state the lower left and upper right as follows:

$$\bot \Rightarrow$$
$$\top \Leftarrow$$

(where an arrow running right to left is understood to have its alternatives on its left and its premises on its right). That is, the difference lies in interchanging Absurdity and Tautology and reversing the direction of the arrow—or, what comes to the same thing, interchanging premises and alternatives. If we apply the same transition to the principle at the upper left we get

$$\text{if } \Gamma, \bot \Leftarrow \Delta, \text{ then } \Gamma \Leftarrow \Delta$$

or, rewriting so the arrows run left to right (without change of premises and alternatives),

$$\text{if } \Delta \Rightarrow \bot, \Gamma, \text{ then } \Delta \Rightarrow \Gamma$$

The latter differs from the principle for Absurdity as an alternative on the lower right above only in the interchange of Γ and Δ; and, since each could be any set, their interchange does not change the content of the principle. The possibility of this sort of transformation can be expressed by saying that $\top$ and $\bot$ on the one hand and *premise* and *alternative* on the other constitute pairs of ***dual*** terms. We will run into other pairs of dual terms later.