1.4.4. Absurdity and inconsistency

Just as a tautology is a sentence for which we have an unconditional guarantee of truth, we can define an ***absurd*** sentence as one for which we have an unconditional guarantee of falsity:

φ is absurd	if and only if	there is no possible world in which φ is true
	if and only if	φ is false in every possible world

Like tautologies absurdities are all equivalent and all have the same properties as the representative absurdity ⊥. These properties are the opposite of those of ⊤. In particular, anything can be concluded from ⊥ (and thus from any set of premises containing it). That is,

⊥ ⇒ φ (⊥ ***as a premise***)

for any sentence φ.

We have no law for restating conditions under which a set of sentences has ⊥ as a valid conclusion; the property of entailing ⊥ will be a fundamental deductive concept, an important addition to the range of ideas introduced in 1.2.2. We can, however, define this concept in terms of truth value and possible worlds. Because of the nature of entailment, when a set Γ entails ⊥, we have a conditional guarantee of the truth of ⊥; and, since ⊥ cannot be true, this must be a guarantee whose conditions cannot be met. That is, a set entails ⊥ just in case its members cannot all be true. We will say that such a set is ***inconsistent***, an idea that may be defined more formally as follows:

Γ is inconsistent	if and only if	there is no possible world in which all members of Γ are true
	if and only if	in each possible world, at least one member of Γ is false

Notice that there is no requirement here that any member of Γ be false in all possible worlds, that Γ contain an absurd sentence. There must always be an error of fact somewhere in Γ but its location may change from possible world to possible world.

Notice that an absurd sentence like ⊥ is one that forms an inconsistent set by itself. This means that absurdity is really a special case of inconsistency. Another of moving from absurdity to a more general concept is to think of a conditional guarantee of falsity (in the way that entailment is a conditional guarantee of truth). We will speak of such a concept as ***exclusion*** or ***relative inconsistency***. It is the idea of a sentence being ***excluded by*** a set or being ***inconsistent with*** it:

φ is excluded by (or is inconsistent with) Γ	if and only if	there is no possible world in which φ and the members of Γ are all true
	if and only if	φ is false in every possible world in which all members of Γ are true

The two ways we have used to express this idea reflect the connections with other concepts that are exhibited by the two forms of its definition. On the one hand, a sentence is inconsistent with a set when adding it to the set would produce an inconsistent set. This is the idea behind the negative form of the definition. Notice that any member of an inconsistent set is inconsistent with the set formed of all other members; each member is equally liable to being singled out in this way as a scapegoat for whole set's inconsistency. The situation is symmetric in another way, too: it makes as much sense to say that Γ is inconsistent with φ as to say that φ is inconsistent with Γ since our focus is on the inconsistency of the set formed from the two.

On the other hand, the positive form of the definition describes a conditional guarantee of falsity. This is a negative analogue to entailment and the verb *exclude* provides a corresponding grammatical analogue to *entail*. When applying the earlier unqualified concept of inconsistency to pairs of sentences, we will often speak of members of the pair as ***mutually exclusive*** because, when {φ, ψ} is an inconsistent set, each of φ and ψ excludes the other.

The property of inconsistency and the relation of exclusion or relative inconsistency are tied by entailment by the following basic laws:

Γ is inconsistent if and only if $\Gamma \Rightarrow \bot$ (***basic law for inconsistency***)

Γ excludes (or is inconsistent with) φ if and only if $\Gamma, \varphi \Rightarrow \bot$ (***basic law for exclusion***)

A number of further principles follow directly from the laws stated for entailment by using the laws above. In the case of simple inconsistency, the four features of entailment summarized at the end of 1.4.2 have as direct consequences the following: (i) any set with an absurd member is inconsistent, (ii) any set that entails all members of an inconsistent set is inconsistent, (iii) any sentence may be added to an inconsistent set without destroying its inconsistency, and (iv) a sentence may be dropped from an inconsistent set without destroying inconsistency provide it is entailed by the remaining members. The situation is a little more complex in the case of exclusion, but the four laws for entailment together may used to establish, among other things, the following principles for exclusion: (i) an absurd sentence is excluded by any set, (ii) an inconsistent set will exclude every sentence, (iii) a set excludes anything excluded by a set whose members it entails, (iv) a sentence excluded by a set will also be excluded by any larger set, (v) a sentence may be dropped from an excluding set without destroying the exclusion provided it is entailed by the remaining members, and (vi) a set that both entails and excludes the same sentence is inconsistent.

Entailment and exclusion are opposites in a way analogous to tautologousness and absurdity. And, although they are both deductive concepts, they combine to set bounds for other forms of reasoning. If we are confident about the accuracy of a set of data, any sentence that is entailed by the data is a reasonable conclusion but any that is excluded by the data would be unreasonable. The norms of non-deductive forms of reasoning—whether this be inductive generalization, inference to the best explanation, or non-monotonic reasoning about typical examples—draw a line between the extremes provided by entailment and exclusion. They identify reasonable conclusions that can be added to the ones entailed by the data while avoiding any that are excluded by it. In a picture,

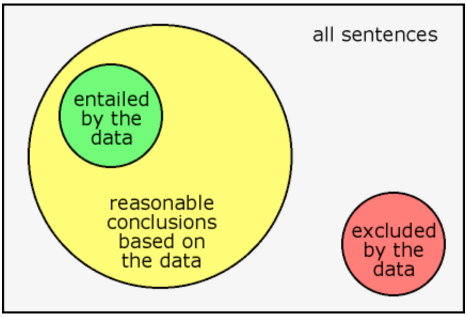

Fig. 1.4.4-1. Reasonable (but not necessarily deductive) conclusions from a body of data in relation to the sentences entailed by or excluded by the data.

Of course, the bounds provided by entailment and exclusion are firm only when we are confident in our data, and calling data into question is itself an important form of reasoning. But here, too, entailment and exclusion are relevant since they indicate ranges of claims that would never or would always lead us to call data into question.

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