1.4.3. Equivalence and tautologousness

Recall that the relation of equivalence applies to sentences that have the same informational content—for example,

Neither the shoulders nor the median are finished
The shoulders and the median are both unfinished

In each possible world, such sentences must have the same truth value as each other, which is the same as saying that neither can be false when the other is true, that each entails the other.

We could define equivalence as mutual entailment; but it will be useful to define it directly using definitions similar to those we have given for entailment. The key idea is that logical equivalence amounts to the necessary identity of truth values. Formally, we can describe the conditions under which a pair of sentences φ and ψ are ***(logically) equivalent*** as follows:

φ ⇔ ψ	if and only if	there is no possible world in which φ and ψ have different truth values
	if and only if	φ and ψ have the same truth value as each other in every possible world

Notice that the second form does not say that the truth values of these sentences do not vary from possibility to possibility, only that, if they vary, they vary in the same way.

The connection between equivalence and entailment can then be stated as the law:

φ ⇔ ψ if and only if both φ ⇒ ψ and ψ ⇒ φ

for any sentences ψ and φ. You may think of this as the ***basic law for equivalence*** because the properties of equivalence can be derived from those of entailment by using it.

The key properties are stated in the following group of laws, which hold for any sentences φ, ψ, and χ and any set Γ of sentences:

φ ⇔ φ (***reflexivity***)
φ ⇔ ψ if and only if ψ ⇔ φ (***symmetry***)
if φ ⇔ ψ and ψ ⇔ χ, then φ ⇔ χ (***transitivity***)
if Γ ⇒ φ and either φ ⇔ ψ or ψ ⇔ φ, then Γ ⇒ ψ
(***conclusion replacement***)

if Γ, $\varphi \Rightarrow \chi$ and either $\varphi \Leftrightarrow \psi$ or $\psi \Leftrightarrow \varphi$, then Γ, $\psi \Rightarrow \chi$ (***premise replacement***)

We saw in 1.4.2 that laws of reflexivity and transitivity hold for implication. But implication is not symmetric; it is one consequence of equivalence amounting to *mutual* entailment or implication. The last two laws tell us that equivalence sentences play the same role as conclusions and as premises; each of two equivalent sentences may be replaced by the other as either a conclusion or a premise without destroying validity.

Given the symmetry of equivalence the alternative *or* $\psi \Leftrightarrow \phi$ in the replacement laws is redundant. It is stated to emphasize that the direction of the replacement does not matter—unlike the following laws for entailment alone (which hold for any set Γ and any sentences φ and ψ):

if $\Gamma \Rightarrow \varphi$ and $\varphi \Rightarrow \psi$, then $\Gamma \Rightarrow \psi$ (***conclusion covariance***)

if Γ, $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \varphi$, then Γ, $\psi \Rightarrow \chi$ (***premise contravariance***)

These are more general versions of a couple of ways of stating the transitivity of implication that were noted in 1.4.2. They tell us that we can replace a conclusion by something it implies and replace a premise by something that it is implied by. The terms *covariance* and *contravariance* refer to the fact that in one case the direction of replacement is same as the direction of the implication and in the other case has the opposite direction. Equivalence in either direction between a pair of sentences licenses replacement in both directions because equivalent sentences both entail and are entailed by each other. We will see later that equivalence licenses further sorts of replacement—not only of whole sentences but of their components—and this is to be expected because equivalent sentences are identical with regard to the aspects of meaning that are of concern to deductive logic.

The two forms of the definition of a ***tautology*** are as follows:

φ is a tautology	if and only if	there is no possible world in which φ is false
	if and only if	φ is true in every possible world

That is, because a tautology says nothing, it cannot be false and we have an unconditional guarantee of its truth.

Recall that, because the truth value of a tautology is fixed for every possible world, any two tautologies are equivalent. That means that laws for the specific tautology $\top$ apply also to other tautologies. The following two laws concerning $\top$ hold for every set Γ of sentences and every sentence φ:

$\Rightarrow \top$ ($\top$ ***as a conclusion***)

if $\Gamma, \top \Rightarrow \varphi$, then $\Gamma \Rightarrow \varphi$ ($\top$ ***as a premise***)

The law for $\top$ as a conclusion says that $\top$ is a valid conclusion from the empty set of premises (which we represent by leaving the left side of $\Rightarrow$ empty). It follows from the monotonicity of entailment that $\top$ is a valid conclusion from any set of premises. The law for $\top$ as a premise reflects another consequence of the fact that a tautology conveys no information: since $\top$ contributes nothing as a premise, it can be dropped from any list of premises without destroying the validity of an argument.

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