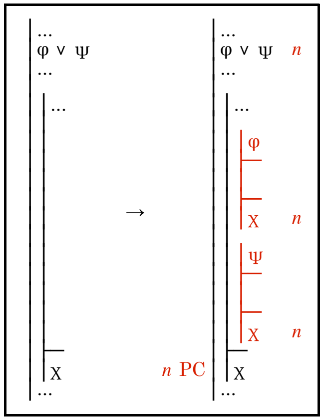
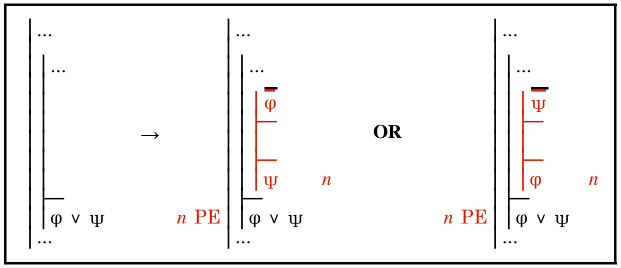
4.2.s. Summary

4.2.1. A disjunction φ ∨ ψ is false only when its disjuncts are both false, and it thus says only what is their shared content. The law for disjunction as a premise tell us that we can establish a conclusion using such a premise by showing that it is entailed by each of the disjuncts (given our other premises). This way of exploiting a disjunction is known as a proof by cases, and it appears in our system of derivations as a rule Proof by Cases (PC) that leads us to divide a gap into two case arguments, each of which takes over the original goal and adds one of the two disjuncts as a supposition.



4.2.2. To show that a disjunction is a valid conclusion, we must show that its disjuncts are rendered jointly exhaustive by the premises. We can do this by showing that one of the disjuncts will follow if we add the contradictory of the other to our premises. In order conveniently refer to a contradictory obtained by either negating or de-negating a sentence, we use the bar notation to indicate a sentence $\overline{\varphi}$ that either is the negation of φ or has φ as its negation. The law for disjunction as a conclusion then tells us that we can conclude a disjunction if we can conclude one disjunct provided we take the barring of the other disjunct as a premise. The rule implementing this idea is Proof of Exhaustion. It enables us to conclude a disjunction from an argument that may be called hypothetical since it draws a conclusion that we may not be prepared to assert categorically by arguing under a supposition in order to establish a relation between the two claims. It does not matter for the soundness or safety of PE which disjunct figures as the goal of this hypothetical argument and which is barred in its supposition.



4.2.3. Derivations, especially those that have a disjunction as a goal as well as a premise can often be developed in a number of different ways. Some of these can be significantly longer than others but the choice between forms of PE will usually have only a limited impact on the length.

4.2.4. Conjunction and disjunction are opposite in the sense of being dual. One manifestation of this relation is in De Morgan's laws, which tell how to restate the denial of a conjunction or disjunction as an assertion of the other form of compound. Another manifestation is a pattern in laws of relative exhaustiveness which allows us to interchange conjunctions and disjunctions if at the same time we interchange ⊤ and ⊥ and also premises and alternatives.

4.2.x. Exercises

1. Use derivations to establish each of the claims of entailment and equivalence shown below. (Remember that claims of equivalence require derivations in both directions.)

- **a.** A ∧ B ⇒ A ∨ B
- **b.** A ∧ B ⇒ B ∨ C
- **c.** A ∨ B, ¬ A ⇒ B
- **d.** A ∨ (A ∧ B) ⇒ A
- **e.** A ∨ B, ¬ (A ∧ C), ¬ (B ∧ C) ⇒ ¬ C
- **f.** A ∧ (B ∨ C) ⇒ (A ∧ B) ∨ C
- **g.** A ∨ B, C ⇒ (A ∧ C) ∨ (B ∧ C)
- **h.** A ∨ B, ¬ A ∨ C ⇒ B ∨ C
- **i.** A ⇔ (A ∧ B) ∨ (A ∧ ¬ B)

2. Use derivations to establish each of the claims of equivalence below.

- **a.** A ∨ A ⇔ A
- **b.** A ∨ B ⇔ B ∨ A
- **c.** A ∨ (B ∨ C) ⇔ (A ∨ B) ∨ C
- **d.** A ∨ (B ∧ ¬ B) ⇔ A
- **e.** ¬ (A ∨ B) ⇔ ¬ A ∧ ¬ B
- **f.** ¬ (A ∧ B) ⇔ ¬ A ∨ ¬ B

3. Use derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample that divides an open gap.

- **a.** A ∨ B, A ⇒ ¬ B
- **b.** A ∨ (B ∧ C) ⇔ (A ∨ B) ∧ C
- **c.** ¬ (A ∨ B) ⇔ ¬ A ∨ ¬ B

Homework assigned Fri 9/30 and due Mon 10/3

Construct a derivation to show: (A ∧ B) ∨ C ⇒ B ∨ (C ∨ D)