2.3.s. Summary

2.3.1. When a derivation is constructed for an invalid argument, we eventually reach a point where an open gap has reached a dead end without closing. We mark such a gap with a white circle ○ and write its active resources and goal with the sign ⇏ between to indicate that they do not form a valid argument. We call this argument the proximate argument of the gap to distinguish it from the ultimate argument for which the derivation is constructed. The invalidity of the proximate argument may only be formal is the sense that some intensional interpretation of its unanalyzed components—some way of associating actual sentences with them—yields an invalid argument (though others may yield valid ones). A test of formal validity is whether there is an extensional interpretation of unanalyzed components, an assignment of truth values to them, that makes premises true and conclusion false. We will often be concerned with formal validity, so we extend to assignments of truth values the ideas of dividing premises from a conclusion and of constituting a counterexample to an argument. And we speak of a gap being divided when its proximate argument is. The fact that any dead-end open is divided—that its proximate argument has a counterexample—indicates that our system is sufficient in the sense of having enough rules to close all dead-end gaps whose proximate arguments are valid.

2.3.2. We can be sure that a counterexample to a proximate argument is a counterexample to the derivation's ultimate argument provided all our rules are safe in the sense of never leading us to try to prove a valid argument by completing a proof of an invalid one. When the converse is true, when we our rules never lead us to develop a gap that can be divided by considering only gaps that cannot be divided, they are utterly sound. Since our real interest is in the ultimate argument of derivation, it is really enough to preserve the division of gaps only when all ancestors of the gap have also been divided; rules that do this are minimally sound; when all rules are safe, minimally sound rules are also utterly sound. The idea of minimal soundness enables us to justify the use of available but inactive resources (to, for example, close gaps) even when not all rules are safe. A system whose rules are all safe and minimally sound is conservative.

2.3.3. Since a dead-end open gap is divided by an interpretation is this interpretation is also a counterexample to the ultimate argument of the derivation, we will present such a counterexample as a way of finishing off a derivation that fails.

2.3.4. A system will be decisive (in the sense that a derivation will always come to an end) provided its rules are all progressive (in the sense of always leading us closer to a point where no more can be done). Many rules are progressive because they are direct (in the sense of either closing a gap or replacing a goal or active resource by one or more simpler sentences). A decisive system which is sufficient and conservative (and is therefore correct in the answers it gives) provides a decision procedure for (formal) validity. Not all systems we consider will provide decision procedures but all will be sound in the sense of providing proofs only for valid arguments and complete in the sense of leading us to a proof whenever an argument is formally valid.

2.3.x. Exercise questions

Use the basic system of derivations to check each of the claims below; if a derivation indicates that a claim fails, present a counterexample (that is, give an interpretation that divides an open gap and calculate truth values for the premises and conclusion from it—as is done in the example in 2.3.3):

1. A ⇒ A ∧ B

2. A ∧ B ⇒ A ∧ (B ∧ A)

3. B ∧ E, C ∧ ⊤ ⇒ (A ∧ B) ∧ (C ∧ D)

4. A ∧ B, B ∧ C, C ∧ D ⇒ A ∧ D

5. A, B ∧ A, D ⇒ B ∧ ((C ∧ A) ∧ D)

Homework assigned Fri 9/10 and due Mon 9/13

Construct a derivation to show that the following claim of entailment does not hold and present a counterexample that divides an open gap:

C ∧ (D ∧ E), A ∧ F ⇒ A ∧ (B ∧ C)