1.4.s. Summary

1.4.1. Entailment may be defined in two equivalent ways, either as the relation that holds when the conclusion is false in no possible world in which all the premises are true or as the relation which holds when the conclusion is true in all such worlds. The first approach can be stated more briefly by saying that an argument is valid when no world divides the premises from conclusion; a world that does divide premises from conclusion is a counterexample to the claim of entailment or validity.

1.4.2. The idea of entailment can also be understood by way of certain laws governing it. For example, if we limit ourselves to single-premised arguments—i.e., to implication—the relation is reflexive and transitive. The law for premises and the chain law are analogous principles that apply to entailment more generally. Entailment also obeys a principle of monotonicity asserting that a premises may always be added without undermining entailment (something does not hold for many forms of non-deductive inference) and a law for lemmas that tells us that a premise may dropped when it is entailed by other premises.

1.4.3. Other properties and relations besides entailment can be given pairs of negative and positive definitions. This is true for the ideas of logical equivalence and tautologousness introduced in 1.2.2. Sentences are equivalent when they entail each other, and this basic law implies that equivalence is symmetric as well as reflexive and transitive. Moreover, equivalent statements may replace one another either as premises or conclusions of an argument without affecting its validity (unlike the case of entailment which obeys only the weaker laws of conclusion covariance and premise contravariance). The laws governing tautologies are most easily stated by focusing on the particular case of Tautology ⊤. For example, ⊤ is always a valid conclusion, but it never contributes anything as a premise and may be freely added to or dropped from the premises without changing an argument's validity.

1.4.4. The definitions of absurdity are in a way opposite those of tautologousness and having Absurdity ⊥ as a premise, like having a ⊤ as a conclusion, makes an argument valid. When an argument with ⊥ as its conclusion is valid, its premises form an inconsistent set. Inconsistency is the fundamental negative concept of deductive logic and the relative concept of being excluded by or inconsistent with a set is a kind of negative opposite to entailment. As a relation between pairs of sentences relative inconsistency is symmetric and such sentences are said to be mutually exclusive. Although inconsistency is a fundamental deductive property, it is one we will establish by using laws that describe it in terms of entailment.

1.4.5. The negative concepts of inconsistency and exclusiveness are opposed in one way to entailment and in another way to exhaustiveness. Contradictory sentences are ones that are bound to differ in truth value; such sentences can be characterized as both mutually exclusive and jointly exhaustive. Exhaustiveness can be conditional and this is a relation between sets that generalizes entailment to allow a set of alternatives rather than a single conclusion. This relation fails when a possible world divides its premises from its alternatives by making the former all true and the latter all false. Relative exhaustiveness obeys cut law which are analogous to, but more symmetric than, the principles governing entailment.

1.4.6. Relative exhaustiveness has an important role in unifying the concepts of deductive logic. All the ones we have seen can be described as special cases of it. We can also use it to describe the absence of deductive properties and relations, whether this is the logical contingency of individual sentences or the logical independence of pairs or larger sets. Laws governing relative exhaustiveness in its own right tend to be symmetric in form. Relative exhaustiveness can be connected with entailment by law employing the idea of contradictoriness. This law exhibits a kind of symmetry that is found also in the laws for ⊤ and ⊥ stated in terms of relative exhaustiveness. Their symmetry can also be seen as one instance of a relation of duality that we will encounter in other cases as well.

1.4.x. Exercise questions

1. Restate each of the following claims about logical properties and relations, putting into symbolic notation those stated in English and into English those stated in symbolic notation:

- **a.** $\phi, \psi \Rightarrow \chi$
- **b.** ϕ is entailed by ψ
- **c.** $\phi \Leftrightarrow \phi$
- **d.** $\psi \Rightarrow$
- **e.** ϕ is inconsistent with Γ
- **f.** ϕ is entailed by the members of Γ together with ψ

2. The following steps lead you to construct a proof of the law for lemmas

if $\Gamma,\phi \Rightarrow \psi$ and $\Gamma \Rightarrow \phi$, then $\Gamma \Rightarrow \psi$

Begin by supposing that $\Gamma, \phi \Rightarrow \psi$ and $\Gamma \Rightarrow \phi$ are both true. We want to show that, under this supposition, $\Gamma \Rightarrow \psi$ is also true. To do that, we consider any possible world w in which all members of Γ are true and try to show that ψ is true in w.

- **a.** Our supposition that $\Gamma, \phi \Rightarrow \psi$ and $\Gamma \Rightarrow \phi$ are both true combined with what we know about w enables us to conclude that ϕ is true. Why?
- **b.** Adding the information that ϕ is true in Γ to what we already knew, we can conclude that ψ is true. Why?

So, knowing about w only that all members of Γ were true, we are able to conclude that ψ is true. And that shows us that ψ is true in every world in which all members of Γ are true, which means that $\Gamma \Rightarrow \psi$.

Another approach to proving the law is to show that $\Gamma \Rightarrow \psi$ fails only if at least one of $\Gamma, \phi \Rightarrow \psi$ and $\Gamma \Rightarrow \phi$ fails. The following three steps show this:

- **c.** Suppose that w is a counterexample to $\Gamma \Rightarrow \psi$. What truth values do ψ and the members of Γ have in w?
- **d.** What truth values are needed to have a counterexample to $\Gamma \Rightarrow \phi$? To have a counterexample to $\Gamma, \phi \Rightarrow \psi$?
- **e.** The world w from **c** will be a counterexample to either $\Gamma, \phi \Rightarrow \psi$ or $\Gamma \Rightarrow \phi$. Why?

Homework assigned Fri 9/3 due Mon 9/6

Define entailment in terms of truth values and possible worlds for the specific case of two premises--that is, complete the following:

$$\phi, \psi \Rightarrow \chi \text{ if and only if } \ldots$$

Give the definition in both positive and negative form (so complete the pattern above twice).