8.6.2. Derivations for the description operator

Although, in stating the tautologousness of a single long sentence, the law for the description operator takes a somewhat different form than those we considered for other logical constants, the real novelty in handling this constant lies in the fact that it is used to form terms rather than sentences. This means that what we must account for is not what is said and the role of such a claim as a premise or conclusion. Instead, we need to account for what a definite description refers to.

Conclusions about what a definite description refers to will be relevant to the derivation and the law for the description operator provides a way to draw some. We will implement this law in a rule that amounts to a couple steps in the exploitation of the sentence the law asserts to be a tautology. In particular, our rule will lead us directly to what we would get as the result of using a proof by cases to exploit the disjunctive law and then using proof by choice for its existential first disjunct; the remaining non-atomic sentences are universals so we cannot expect to go further in a single step. We will call this rule ***Securing a Description*** (SD).

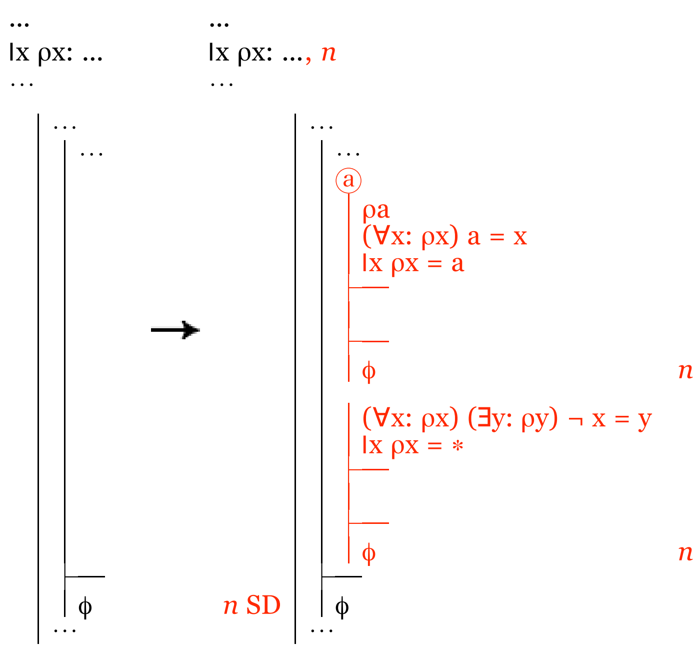


Fig. 8.6.2-1. Developing a derivation at stage *n* by securing a definite description; the parameter a is new to the derivation.

There are really no preconditions for the use of this rule, but it is relevant only when the definite description in question actually appears in the gap being developed. The description is displayed above the derivation (perhaps among a list of other definite descriptions) and the stage number of the development is listed after it to show that it has been handled—we will say ***secured***—at that stage in developing some gap. The description may need to be secured in a number of different gaps at different stages, so this stage is perhaps only the latest of a long list.

As an example of the use of SD, here is a derivation showing that if have the premise *There was at most one winner*, we can conclude *The winner won if anything did*.

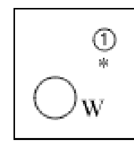
```
℩x Wx: 3
                 | ¬ ∃x (∃y: ¬ y = x) (Wx ∧ Wy)               (11)
                 |——
                 (a)
                 |  | Wa                                       (5), (8)
                 |  |——
                 |  |  (b)
                 |  |  | Wb                                    (4)
                 |  |  | (∀x: Wx) b = x
                 |  |  | ℩x Wx = b                             a, b=℩xWx, *
                 |  |  |——
                 |  |  | •
                 |  |  |——
4 QED=           |  |  | W(℩x Wx)                              3
                 |  |
                 |  |  | (∀x: Wx) (∃y: Wy) ¬ x = y             a:5
                 |  |  | ℩x Wx = *                             a, c, ℩xWx=*
                 |  |  |——
5 SB             |  |  | (∃y: Wy) ¬ a = y                      6
                 |  |  |  (c)
                 |  |  |  | Wc                                 (8)
                 |  |  |  | ¬ a = c                            (9)
                 |  |  |  |——
                 |  |  |  |  | ¬ W(℩x Wx)
                 |  |  |  |  |——
8 Adj            |  |  |  |  | Wc ∧ Wa                         X, (9)
9 REG            |  |  |  |  | (∃y: ¬ y = c) (Wc ∧ Wy)         X, (10)
10 EG            |  |  |  |  | ∃x (∃y: ¬ y = x) (Wx ∧ Wy)      X, (11)
                 |  |  |  |  | •
                 |  |  |  |  |——
11 Nc            |  |  |  |  | ⊥                               7
                 |  |  |  |——
7 IP             |  |  |  | W(℩x Wx)                           6
                 |  |  |——
6 PRCh           |  |  | W(℩x Wx)                              3
                 |  |——
3 SD             |  | W(℩x Wx)                                 2
                 |——
2 CP             | | Wa → W(℩x Wx))                            1
                 |——
1 UG             | ∀y (Wy → W(℩x Wx))
```

The list of alias sets in the first gap includes * even though that term does not appear in either resources or goals of the gap because, when using the description operator, * is part of our logical apparatus and is thus always among the terms.

Notice that both the premise and the hedge in the conclusion played a role in closing the second gap in the derivation above. Since both are required to insure the existence and uniqueness of a winner, it is to be expected that the absence of either would keep us from ruling out the possibility that the definite description is undefined (which is the possibility explored by the second gap). It may seem odd that *The winner won* is not a tautology.

But on both of the accounts of definite descriptions that we have considered, it entails *Something won* and that is not a tautology. It follows that *The winner didn't win* is not absurd and a derivation showing this provides another example of the use of SD.

```
℩x Wx: 1
          | ¬ W(℩x Wx)                       (2)
          |—
          | (a)
          | | Wa                             (2)
          | | (∀y: Wx) a = x
          | | ℩x Wx = a                      (℩x Wx)—a
          | |—
          | | •
          | |—
2 Nc=     | | ⊥
          |
          | | (∀y: Wx) (∃y: Wy) ¬ x = y     *:3
          | | ℩x Wx = *                      (℩x Wx)=*
          | |—
          | | | | ¬ W*
          | | | |—
          | | | | ○                          ¬ W(℩x Wx), ℩x Wx = *, ¬ W* ⇏ ⊥
          | | | |—
          | | | | ⊥                          4
          | | |—
4 IP      | | | W*                           3
          | | 
          | | | (∃y: Wy) ¬ * = y
          | | |—
          | | | (unfinished)
          | | |—
          | | | ⊥                            3
          | |—
3 MCR     | | ⊥                              1
          |—
1 SD      | ⊥
```



The sentence *The winner didn't win* is consistent also on Russell's analysis provided we interpret it as ¬ *the winner won*, for ¬ ϕ is absurd if and only if ϕ is a tautology. However, on Russell's analysis, an interpretation giving *the winner* widest scope—that is, an interpretation of the sentence as *The winner is such that (he or she didn't win)*—is absurd since it implies *Some winner didn't win* and thus that something has the property of being a winner and not winning.

Our stipulations about the interpretation of definite descriptions insure that any interpretation of the vocabulary in ρ will divide one of the two gaps that result from SD—that's why there is no precondition for its application—so the rule is utterly sound and its addition will not disturb the soundness of our system. It is also clearly safe since the new gaps it introduces differ from their parent only by having added resources. But the argument we had used to establish the completeness of the system of derivations—in particular, the argument used in 7.7.4 to show that any fully developing gap is divided by an interpretation—will no longer apply since this argument assumed that the reference values of all terms could be settled without considering the extensions of predicates, something that is not true in the case of definite descriptions.

However, it is easy to see the completeness of a system of derivations that allows certain uses of the rule LFR. The stipulations we have made concerning the interpretation of the description operator can be imposed on a structure simply by requiring that it make true every sentence of the form:

$$\forall w_1 \ldots \forall w_n \,(\,(\exists z{:}\ \rho z \wedge (\forall y{:}\ \rho y)\ z = y)\ ℩x\ \rho x = z \vee ((\forall x{:}\ \rho x)\ (\exists y{:}\ \rho y)\ \neg\ x = y \wedge ℩x\ \rho x = *)\,)$$

where we follow the form of the law for descriptions but apply a quantifier $\forall w_i$ for each variable w_i that appears unbound in ρ. We will call this sentence a ***meaning postulate*** for the description Ix ρx. Making all these meaning postulates true comes to the same thing as making true all instances of that law for a language expanded by the range of the structure. When assessing the validity of a particular argument, all that is relevant is the interpretation of the definite descriptions actually appearing in the argument, and this can be insured by the truth of the meaning postulates for the descriptions actually appearing in the argument. That is, if Δ includes the meaning postulate for each description an argument Γ / ϕ, this argument is valid given the interpretation of the description operator if and only if the argument Γ, Δ / ϕ is valid even without stipulating the interpretation of definite descriptions.

Now, any question of validity can be reduced to a question of the validity of a *reductio* argument, so let us limit consideration to such arguments. Given an argument Γ / ⊥, let δ be the conjunction of the meaning postulates for all descriptions appearing in the members of Γ. Now suppose that Γ / ⊥ is valid when we fix the interpretation of definite descriptions. We have seen that Γ, δ / ⊥ will be valid without fixing this interpretation. Therefore, a derivation for Γ, δ / ⊥ will close using only the basic system of previous chapters, so it will certainly close if we add the rules SD and LFR. And the rule SD will enable us to show the meaning postulate for any description is a tautology, so it will certainly enable us to show the validity of Γ / δ. Finally, the rule LFR lets us establish the validity of Γ / ⊥ if we can show both Γ / δ and Γ, δ / ⊥. In short, the system of derivations with SD and LFR is complete because SD enables us to establish any meaning postulate and we can establish the validity of all arguments involving descriptions when we add their meaning postulates as further premises.

Since it introduces a new parameter, the rule SD can prevent gaps from reaching a dead end. It can be modified to search for finite gaps in the way we have done for other rules using parameters, and named following the same pattern as with those rules as ***Securing a Description Supplemented*** (SD+).

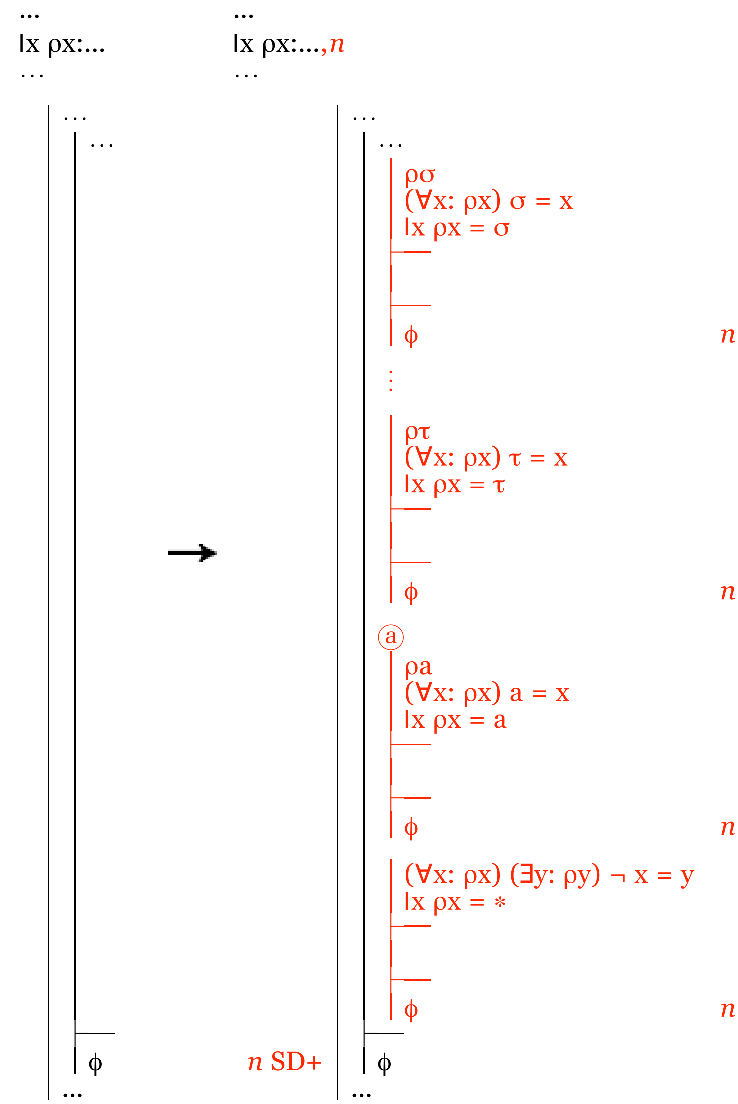


Fig. 8.6.2-2. Developing a derivation at stage *n* by securing a definite description; the parameter a is new to the derivation and the terms σ, …, τ include at least one from each current alias set for the gap.

Here we consider the possibility that one of the already existing alias sets provides names of an object that uniquely satisfies the description. Notice that one of these alias sets will be the one including ∗. And that is to be expected since there are two different ways in which the nil value might end up as the reference of a definite description. This will happen not only when the description fails to be uniquely satisfied but also when the nil value is the one value satisfying it uniquely. Indeed, the reference of any term τ will uniquely satisfy the predicate λx x = τ, so even if λx (x *is a* C) is not uniquely satisfied λx (x = *the thing that is a* C) will be—though, of course, only by the nil value.