8.3.2. Numerical quantifier phrases

So far the only numerical claims we have seen have been ones asserting or denying that a class is empty. We will now move on to a much wider group, considering claims of the three sorts

At least n C*s are such that* … *they* …
At most n C*s are such that* … *they* …
Exactly n C*s are such that* … *they* …,

where *n* may be any positive integer.

To see how to approach these quantificational claims, let us first consider existential claims regarding pairs. We looked at generalizations about pairs in 7.4.1, giving special attention to the example *Not every employer and employee get along*. This is the denial of generalization, so it can be understood to claim the existence of a counterexample, and we can restate it as follows to make this more explicit:

Some employer and employee do not get along.

Now we can analyze this sentence using two existential quantifiers, restricting the second by relation to the first. We would get this:

Some employer and employee do not get along
Something is such that (it and some employee of it do not get along)
∃x x *and some employee of* x *do not get along*
∃x *some employee of* x *is such that (*x *and he or she do not get along)*
∃x (∃y: x *employs* y) x *and* y *do not get along*
∃x (∃y: Exy) ¬ x *and* y *get along*

∃x (∃y: Exy) ¬ Gxy
∃x ∃y (Exy ∧ ¬ Gxy)

[E: λxy (x *employs* y); G: λxy (x *and* y *get along*)]

The English sentence claims the existence of a pair of examples whose members are related in a certain way (as employer and employee). As with generalizations, the limitations of our notation have forced us to treat the two quantifiers asymmetrically in the symbolic form.

Now consider the sentence *At least 2 things are on the agenda*. It can be understood to claim the existence of a pair of examples whose members are specified to be non-identical. Following the pattern we have just used to analyze restricted existential claims concerning pairs, we can express this idea as follows:

At least 2 things are on the agenda
Something is such that (it and something else are on the agenda)
∃x x *and something else are on the agenda*
∃x *something other than* x *is such that (*x *and it are on the agenda)*
∃x (∃y: y *is other than* x) x *and* y *are on the agenda*
∃x (∃y: ¬ y *is* x) (x *is on the agenda* ∧ y *is on the agenda*)

∃x (∃y: ¬ y = x) (Nxa ∧ Nya)
∃x ∃y (¬ y = x ∧ (Nxa ∧ Nya))

[N: λxy (x *is on* y); a: *the agenda*]

The quantifier phrase *something else* has been analyzed here before *and* in order to separate the vocabulary found in the quantifier phrase from that found in the quantified predicate of the original English sentence. We might have analyzed the conjunction before the second quantifier by way of an intermediate form like this:

∃x x *is on the agenda and so is something else*

We would have ended up with the form ∃x (Nxa ∧ (∃y: ¬ y = x) Nya), which is equivalent to the form above by a confinement equivalence discussed in 8.1.4.

This basic idea can be extended to any quantifier phrase of the form *at least n* C*s*. For example, *at least 3* C*s* can be understood to claim the existence of an example, an example different from the first, and an example different from the first two. Let us apply this idea to a case where the restrictions of non-identity are added to other specifications:

At least 3 birds are in the tree
(∃x: x *is a bird*) x *and at least 2 other birds are in the tree*
(∃x: Bx) (∃y: y *is a bird other than* x) x *and* y *and another bird are in the tree*
(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: z *is a bird other than* x *and* y) x *and* y *and* z *are in the tree*
(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: Bz ∧ (¬ z = x ∧ ¬ z = y)) (x *is in the tree* ∧ y *is in the tree* ∧ z *is in the tree*)

(∃x: Bx) (∃y: By ∧ ¬ y = x) (∃z: Bz ∧ (¬ z = x ∧ ¬ z = y)) (Nxt ∧ Nyt ∧ Nzt)

[B: λx (x *is a bird*); N: λxy (x *is in* y); t: *the tree*]

This can be restated in a number of different ways by using unrestricted quantifiers and applying confinement principles. The following may help in thinking about the net result of the three quantifier phrases above:

$$\exists x\ \exists y\ \exists z\ ((\neg\ y = x \wedge \neg\ z = x \wedge \neg\ z = y) \wedge (Bx \wedge By \wedge Bz) \wedge (Nxt \wedge Nyt \wedge Nzt))$$

That is, we assert the existence of a triple with three properties: (i) no two of its members are the same, (ii) each member is a bird, and (iii) each member is in the tree. The sentence *Heinz produces at least 57 varieties* could be handled (in principle if not in practice) by extending the same ideas to assert the existence of a series of 57 things no two of which are the same and each of which is both a variety and produced by Heinz. If you are mathematically minded, you might try calculating the number of denied equations you would need in that case.

In the other direction, if the scopes of quantifier phrases are confined to parts of the sentence in which they bind variables, we would have instead

$$(\exists x\!: Bx)\ (Nxt \wedge (\exists y\!: By \wedge \neg\ y = x)\ (Nyt \wedge (\exists z\!: Bz \wedge (\neg\ z = x \wedge \neg\ z = y))\ Nzt))$$

which might be expressed in English as *Some bird is such that it is in the tree and some bird other than it is such that it, too, is in the tree and some bird different from both of the them is in the tree also*.

As a general pattern for *At least* n *things are such that* … *they* …, we might use either of the following:

$$\exists x_1\ (\exists x_2\!: \neg\ x_2 = x_1) \ldots (\exists x_n\!: \neg\ x_n = x_1 \wedge \neg\ x_n = x_2 \wedge \ldots \wedge \neg\ x_n = x_{n-1})\ (\theta x_1 \wedge \theta x_2 \wedge \ldots \wedge \theta x_n)$$

$$\exists x_1\ \exists x_2 \ldots \exists x_n\ ((\neg\ x_2 = x_1) \wedge \ldots \wedge (\neg\ x_n = x_1 \wedge \neg\ x_n = x_2 \wedge \ldots \wedge \neg\ x_n = x_{n-1}) \wedge (\theta x_1 \wedge \theta x_2 \wedge \ldots \wedge \theta x_n))$$

where $\theta\tau$ abbreviates … τ …. These logical forms differ in whether the denied equations appear as restrictions on quantifiers or as conjuncts of the formula to which the quantifiers are applied. In either case, the list of denied equations should include $\neg\ x_i = x_j$ for each $i > j$ where $i, j \leq n$.

At least n C*s are such that* … *they* … can be captured by adding the formulas x_i *is a* C, for each $i \leq n$, either as restrictions on the relevant quantifiers or as further conjuncts of the quantified formula.

The corresponding pattern with the quantifiers confined would be:

$\exists x_1 (\theta x_1 \wedge (\exists x_2 : \neg x_2 = x_1) (\theta x_2 \wedge \ldots (\exists x_n : \neg x_n = x_1 \wedge \neg x_n = x_2 \wedge \ldots \wedge \neg x_n = x_{n-1}) \theta x_n \ldots))$

This says roughly, *Something is such that ...it... and so is something else ... and so is something else*. In spite of appearances, this English sentence is not a conjunction because each use of *else* refers implicitly to all of the previous uses of *something* and cannot be separated from them in an independent component.

We are also now in a position to analyze the other two sorts of numerical quantifier phrases mentioned earlier, for claims made using them can be restated as truth-functional compounds of claims made using *at least n*.

At most n C*s are such that* ... *they* ...

may be paraphrased as

¬ *at least* n+1 C*s are such that* ... *they* ...

and

Exactly n C*s are such that* ... *they* ...

may be paraphrased as

At least n C*s are such that* ... *they* ...
∧ *at most n* C*s are such that* ... *they* ...

For example, to claim that there was at most one winner is to deny that there were at least two, and to claim that there was exactly one is to say both there was at least one and that there was at most one—i.e., it is to say that there was at least one and deny that there were at least two.